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On Generalized Tetranacci Numbers: Closed Form Formulas of the Sum $\sum_{k=0}^{n} W_k^2$ of the Squares of Terms

Research Article

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Abstract: In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} W_k^2$ for the squares of generalized Tetranacci numbers are presented. We also present the sum formulas $\sum_{k=0}^{n} W_{k+1}W_k$, $\sum_{k=0}^{n} W_{k+2}W_k$, and $\sum_{k=0}^{n} W_{k+3}W_k$. As special cases, we give summation formulas of Tetranacci, Tetranacci-Lucas and some other fourth-order linear recurrence sequences.

MSC: 11B39 • 11B83

Keywords: Sum of squares • Fourth order recurrence • Tetranacci numbers • Tetranacci-Lucas numbers

1. Introduction

There have been many studies of the sequences of numbers in the literature which are defined recursively (see for example [10],[11],[12],[13],[8], [7]). Two of these types of sequences are the sequences of Tetranacci and Tetranacci-Lucas which are special cases of generalized Tetranacci numbers. A generalized Tetranacci sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \quad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \ \ n \geq 4 \tag{1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [9],[16],[17],[22],[31],[32].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1) holds for all integer n.

In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t, u and initial values.

Here, OEIS stands for On-line Encyclopedia of Integer Sequences. In the rest of the paper, for easy writing, we drop the superscripts and write P_n, Q_n and E_n for $P_n^{(4)}, Q_n^{(4)}$ and $E_n^{(4)}$, respectively. For generalized fourth order Pell numbers and generalized 4-primes numbers see [23] and [29], respectively.

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Table 1. A few special case of generalized Tetranacci sequences.

Sequences (Numbers)	Notation	OEIS [30]
Tetranacci	$\{M_n\} = \{W_n(0,1,1,2;1,1,1,1)\}$	A000078
Tetranacc i-Lucas	$\{R_n\} = \{W_n(4,1,3,7;1,1,1,1)\}$	A073817
fourth order Pell	$\{P_n^{(4)}\} = \{W_n(0,1,2,5;2,1,1,1)\}$	A103142
fourth order Pell-Lucas	$\{Q_n^{(4)}\} = \{W_n(4,2,6,17;2,1,1,1)\}$	A331413
modified fourth order Pell	$\{E_n^{(4)}\} = \{W_n(0,1,1,3;2,1,1,1)\}$	A190139
4-primes	$\{G_n\} = \{W_n(0,0,1,2;2,3,5,7)\}$	
Lucas 4-primes	$\{H_n\} = \{W_n(4,2,10,41;2,3,5,7)\}$	
modified 4-primes	$\{E_n\} = \{W_n(0,0,1,1;2,3,5,7)\}$	

The evaluation of sums of powers of these sequences is a challenging issue. Two interesting examples are $\sum_{k=0}^{n} R_k^2 = \frac{1}{3}(-R_{n+4}^2 - 3R_{n+3}^2 - 4R_{n+2}^2 - 4R_{n+1}^2 + 3R_{n+4}R_{n+3} + 2R_{n+4}R_{n+2} + R_{n+4}R_{n+1} - R_{n+2}R_{n+1} + 43)$ and $\sum_{k=0}^{n} P_k^2 = \frac{1}{56}(-9P_{n+4}^2 - 57P_{n+3}^2 - 68P_{n+2}^2 - 65P_{n+1}^2 + 42P_{n+4}P_{n+3} + 20P_{n+4}P_{n+2} + 6P_{n+4}P_{n+1} - 4P_{n+3}P_{n+2} + 2P_{n+3}P_{n+1} - 12P_{n+2}P_{n+1} + 9)$.

In [21], the author showed that

$$\sum_{k=0}^{n} M_k^2 = \frac{1}{3}(1 + 3M_nM_{n+1} - (M_{n+1} - M_{n-1})^2 + M_nM_{n-2} + M_{n-2}M_{n-3}) \tag{2}$$

by using induction with a (long) proof. In this paper, as an easy corollary to our main result, we find that

$$\sum_{k=0}^{n} M_k^2 = \frac{1}{3}(-M_{n+4}^2 - 3M_{n+3}^2 - 4M_{n+2}^2 - 4M_{n+1}^2 + 3M_{n+4}M_{n+3} + 2M_{n+4}M_{n+2} + M_{n+4}M_{n+1} - M_{n+2}M_{n+1} + 1). \tag{3}$$

By using the Binet's formula of Tetranacci numbers (or the fourth-order recurrence relations (1)) it can be seen that the rights sides of formulas (2) and (3) are equal.

In this work, we derive expressions for sums of second powers of generalized Tetranacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. Several special studies on sum formulas of second, third and arbitrary powers.

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[1],[2],[6],[14],[15],[24]	[5],[25],[27],[28],[33]	[3],[4],[18]
Generalized Tribonacci	[20],[26]		
Generalized Tetranacci	[19],[21]		

2. Main Result

Let

$$\Delta = (r - s + t - u + 1)(s + u + r^2u + su^2 + rt + 2su - t^2 - u^2 - u^3 - rtu + 1)(r + s + t + u - 1).$$

Theorem 2.1.
If $\Delta \neq 0$ *then*

(a)

$$\sum_{k=0}^{n} W_k^2 = \frac{\Delta_1}{\Delta},$$

(b)

$$\sum_{k=0}^{n} W_{k+1}W_k = \frac{\Delta_2}{\Delta},$$

(c)

$$\sum_{k=0}^{n} W_{k+2}W_k = \frac{\Delta_3}{\Delta},$$

(d)

$$\sum_{k=0}^{n} W_{k+3}W_k = \frac{\Delta_4}{\Delta},$$

where

$$\Delta_1 = \sum_{k=1}^{20} \Gamma_k,\ \Delta_2 = \sum_{k=1}^{20} \Theta_k,\ \Delta_3 = \sum_{k=1}^{20} \Phi_k,\ \Delta_4 = \sum_{k=1}^{20} \Psi_k$$

with

$\Gamma_1 = -(s+u+r^2u-su^2+rt+t^2+u^2-u^3+rtu-1)W_{n+4}^2,$

$\Gamma_2 = -(s+u+r^2s+r^3t-su^2+r^4u+r^2t^2+r^2u^2-r^2u^3+rt+r^2+t^2+u^2-u^3+2r^2su-2rtu^2+r^3tu-r^2su^2+2rst+3rtu-1)W_{n+3}^2,$

$\Gamma_3 = -(r^4u+r^3tu+r^3t-r^2s^2u-r^2su^2+4r^2su+r^2s+r^2t^2-r^2u^3+r^2u^2+r^2+rs^2tu-rs^2t+2rstu^2+4rst-2rtu^2+3rtu+rt-s^3u^2-2s^3u-s^3+s^2t^2-s^2u^3-s^2u^2+s^2u+s^2+2st^2u-su^2+s+t^2-u^3+u^2+u-1)W_{n+2}^2,$

$\Gamma_4 = -(r^4u+r^3tu+r^3t-r^2s^2u-r^2su^2+4r^2su+r^2s-r^2t^2u+r^2t^2-r^2u^3+r^2u^2+r^2+rs^2tu-rs^2t+4rstu^2+4rst-rt^3u-rt^3-4rtu^2+5rtu+rt-s^3u^2-2s^3u-s^3+s^2t^2-s^2u^3-s^2u^2+s^2u+s^2+st^2u^2+4st^2u-st^2-su^2+s-t^4-t^2u^3+t^2u^2-t^2u+2t^2-u^3+u^2+u-1)W_{n+1}^2,$

$\Gamma_5 = 2(rt^2+r^2t+ru^2-ru^3+r^3u-tu^2+rs+st+tu-rsu^2+r^2tu+rsu)W_{n+4}W_{n+3},$

$\Gamma_6 = 2(r^2u+rstu+rtu^2+rt-s^2u^2-s^2u+st^2-su^3+su+t^2u)W_{n+4}W_{n+2},$

$\Gamma_7 = 2u(r-tu^2-ru+st+tu+rsu)W_{n+4}W_{n+1},$

$\Gamma_8 = -2(r^2stu+r^2tu^2-r^2tu-rs^2u^2+rst^2-rsu^3+rsu^2-rsu+rt^2u-rt^2+ru^3-ru^2+s^2t-stu^2+stu-st+tu^2-tu)W_{n+3}W_{n+2},$

$\Gamma_9 = -2u(r^2su-r^2u+rst-rtu^2-rt+s^2u+s^2+su^2-s-t^2u)W_{n+3}W_{n+1},$

$\Gamma_{10} = -2u(rt^2-t+ru^2-ru+st+tu+t^3-rsu^2+rt^2u+r^2tu-stu^2-stu)W_{n+2}W_{n+1},$

$\Gamma_{11} = (s+u+r^2u-su^2+rt+t^2+u^2-u^3+rtu-1)W_3^2,$

$\Gamma_{12} = (s+u+r^2s+r^3t-su^2+r^4u+r^2t^2+r^2u^2-r^2u^3+rt+r^2+t^2+u^2-u^3+2r^2su-2rtu^2+r^3tu-r^2su^2+2rst+3rtu-1)W_2^2,$

$\Gamma_{13} = (r^4u+r^3tu+r^3t-r^2s^2u-r^2su^2+4r^2su+r^2s+r^2t^2-r^2u^3+r^2u^2+r^2+rs^2tu-rs^2t+2rstu^2+4rst-2rtu^2+3rtu+rt-s^3u^2-2s^3u-s^3+s^2t^2-s^2u^3-s^2u^2+s^2u+s^2+2st^2u-su^2+s+t^2-u^3+u^2+u-1)W_1^2,$

$\Gamma_{14} = (r^4u+r^3tu+r^3t-r^2s^2u-r^2su^2+4r^2su+r^2s-r^2t^2u+r^2t^2-r^2u^3+r^2u^2+r^2+rs^2tu-rs^2t+4rstu^2+4rst-rt^3u-rt^3-4rtu^2+5rtu+rt-s^3u^2-2s^3u-s^3+s^2t^2-s^2u^3-s^2u^2+s^2u+s^2+st^2u^2+4st^2u-st^2-su^2+s-t^4-t^2u^3+t^2u^2-t^2u+2t^2-u^3+u^2+u-1)W_0^2,$

$\Gamma_{15} = -2(rt^2+r^2t+ru^2-ru^3+r^3u-tu^2+rs+st+tu-rsu^2+r^2tu+rsu)W_3W_2,$

$\Gamma_{16} = -2(r^2u+rstu+rtu^2+rt-s^2u^2-s^2u+st^2-su^3+su+t^2u)W_3W_1,$

$\Gamma_{17} = 2(r^2stu+r^2tu^2-r^2tu-rs^2u^2+rst^2-rsu^3+rsu^2-rsu+rt^2u-rt^2+ru^3-ru^2+s^2t-stu^2+stu-st+tu^2-tu)W_2W_1,$

$\Gamma_{18} = -2u(r-tu^2-ru+st+tu+rsu)W_3W_0,$

$\Gamma_{19} = 2u(r^2su-r^2u+rst-rtu^2-rt+s^2u+s^2+su^2-s-t^2u)W_2W_0,$

$\Gamma_{20} = 2u(rt^2-t+ru^2-ru+st+tu+t^3-rsu^2+rt^2u+r^2tu-stu^2-stu)W_1W_0,$

and

$\Theta_1 = (r-tu^2-ru+st+tu+rsu)W_{n+4}^2,$

$\Theta_2 = (r^3su+r^2st-r^2tu^2+r^2tu+rs^2u+rs^2+rsu-rt^2u+rt^2-ru^3+ru^2+st-tu^2+tu)W_{n+3}^2,$

$\Theta_3 = (-r^2tu^2+r^2tu+rst^2u+rsu^3-rt^2u+rt^2-ru^3+ru^2-s^2tu^2-s^2tu+st^3+stu^2-tu^2+tu)W_{n+2}^2,$

$\Theta_4 = u^2(r-tu^2-ru+st+tu+rsu)W_{n+1}^2,$

$\Theta_5 = -(2r^2su-r^2u+r^2+2rst-2rtu^2+2rtu+s^2u+s^2-t^2u+t^2-u^3+u^2+u-1)W_{n+4}W_{n+3},$

$\Theta_6 = (r^3u+r^2t-rs^2u+2rsu-rt^2u-ru^3+ru-s^2t+2stu^2-t^3-tu^2+t)W_{n+4}W_{n+2},$

$\Theta_7 = u(-r^2u + r^2 - s^2u - s^2 + t^2u - t^2 + u^3 - u^2 - u + 1)W_{n+4}W_{n+1}$,

$\Theta_8 = -(r^4u + r^3t - r^2s^2u + 3r^2su + r^2s - r^2t^2u - r^2u^3 + r^2 - rs^2t + 2rstu^2 + 2rst - rt^3 - 3rtu^2 + 2rtu + rt - s^3u - s^3 + s^2u + s^2 + st^2u - st^2 + su^3 - su^2 - su + s - t^2u + t^2 - u^3 + u^2 + u - 1)W_{n+3}W_{n+2}$,

$\Theta_9 = u(r^3u + r^2t + rs^2u + 2rs - rt^2u - ru^3 + ru + s^2t + 2stu - t^3 - tu^2 + t)W_{n+3}W_{n+1}$,

$\Theta_{10} = -(r^4u + r^3tu + r^3t - r^2s^2u - r^2su^2 + 4r^2su + r^2s - r^2t^2u + r^2t^2 - r^2u^3 + r^2 + rs^2tu - rs^2t + 2rstu^2 + 4rst - rt^3u - rt^3 - rtu^3 - 3rtu^2 + 3rtu + rt - s^3u^2 - 2s^3u - s^3 + s^2t^2 + s^2u + s^2 + st^2u^2 + 2st^2u - st^2 + su^4 - 2su^2 + s - t^4 - t^2u^2 - t^2u + 2t^2 - u^3 + u^2 + u - 1)W_{n+2}W_{n+1}$,

$\Theta_{11} = -(r - tu^2 - ru + st + tu + rsu)W_3^2$,

$\Theta_{12} = -(r^3su + r^2st - r^2tu^2 + r^2tu + rs^2u + rs^2 + rsu - rt^2u + rt^2 - ru^3 + ru^2 + st - tu^2 + tu)W_2^2$,

$\Theta_{13} = (r^2tu^2 - r^2tu - rst^2u - rsu^3 + rt^2u - rt^2 + ru^3 - ru^2 + s^2tu^2 + s^2tu - st^3 - stu^2 + tu^2 - tu)W_1^2$,

$\Theta_{14} = -u^2(r - tu^2 - ru + st + tu + rsu)W_0^2$,

$\Theta_{15} = (2r^2su - r^2u + r^2 + 2rst - 2rtu^2 + 2rtu + s^2u + s^2 - t^2u + t^2 - u^3 + u^2 + u - 1)W_3W_2$,

$\Theta_{16} = -(r^3u + r^2t - rs^2u + 2rsu - rt^2u - ru^3 + ru - s^2t + 2stu^2 - t^3 - tu^2 + t)W_3W_1$,

$\Theta_{17} = (r^4u + r^3t - r^2s^2u + 3r^2su + r^2s - r^2t^2u - r^2u^3 + r^2 - rs^2t + 2rstu^2 + 2rst - rt^3 - 3rtu^2 + 2rtu + rt - s^3u - s^3 + s^2u + s^2 + st^2u - st^2 + su^3 - su^2 - su + s - t^2u + t^2 - u^3 + u^2 + u - 1)W_2W_1$,

$\Theta_{18} = u(r^2u - r^2 + s^2u + s^2 - t^2u + t^2 - u^3 + u^2 + u - 1)W_3W_0$,

$\Theta_{19} = -u(r^3u + r^2t + rs^2u + 2rs - rt^2u - ru^3 + ru + s^2t + 2stu - t^3 - tu^2 + t)W_2W_0$,

$\Theta_{20} = (r^4u + r^3tu + r^3t - r^2s^2u - r^2su^2 + 4r^2su + r^2s - r^2t^2u + r^2t^2 - r^2u^3 + r^2 + rs^2tu - rs^2t + 2rstu^2 + 4rst - rt^3u - rt^3 - rtu^3 - 3rtu^2 + 3rtu + rt - s^3u^2 - 2s^3u - s^3 + s^2t^2 + s^2u + s^2 + st^2u^2 + 2st^2u - st^2 + su^4 - 2su^2 + s - t^4 - t^2u^2 - t^2u + 2t^2 - u^3 + u^2 + u - 1)W_1W_0$,

and

$\Phi_1 = (r^2 + rtu + rt - s^2u - s^2 - su^2 + s + t^2u)W_{n+4}^2$,

$\Phi_2 = (r^3tu - r^2s^2u - r^2su^2 - r^2s + r^2t^2u + r^2t^2 + r^2u^2 - rs^2t - 2rstu + rt^3 + rtu^2 + rtu - s^2u - s^2 - su^2 + s + t^2u)$ W_{n+3}^2,

$\Phi_3 = (r^3tu + r^2t^2u + r^2t^2 + r^2u^2 - rs^2tu + 2rstu + rt^3 + rtu^2 + rtu - s^2t^2 - s^2u^3 - s^2u^2 + st^2u^2 + st^2 - su^4 + su^2 + t^2u)$ W_{n+2}^2,

$\Phi_4 = u^2(r^2 + rtu + rt - s^2u - s^2 - su^2 + s + t^2u)W_{n+1}^2$,

$\Phi_5 = (-r^3 - 2r^2tu - r^2t + 2rs^2u + rs^2 + 2rsu^2 - 2rt^2u - rt^2 - ru^2 + r + s^2t + 2stu - t^3 - tu^2 + t)W_{n+4}W_{n+3}$,

$\Phi_6 = -(r^2su + r^2s + r^2u^2 + r^2 + 2rst + 4rtu - s^3u - s^3 - s^2u^2 + s^2 + st^2u - st^2 - su^3 - su^2 + su + s + t^2u^2 + t^2 - u^4 + 2u^2 - 1)W_{n+4}W_{n+2}$,

$\Phi_7 = u(r^3 + r^2t - rs^2 + 2rs - rt^2 - ru^2 + r + s^2t + 2stu - t^3 - tu^2 + t)W_{n+4}W_{n+1}$,

$\Phi_8 = (r^3su + r^3u^2 + r^2st + 2r^2tu - r^2t - rs^3u - rs^2u^2 + 2rs^2u + rst^2u - rsu^3 + 2rsu^2 + rsu + rt^2u^2 - 2rt^2u - ru^4 + ru^2 - s^3t - 2s^2tu + s^2t + st^3 + stu^2 + 2stu - st - t^3 - tu^2 + t)W_{n+3}W_{n+2}$,

$\Phi_9 = -(r^4u + r^3tu + r^3t - r^2s^2u + 3r^2su + r^2s - r^2t^2u + r^2t^2 - r^2u^3 + r^2u^2 + r^2u + r^2 + rs^2tu - rs^2t + 2rstu^2 + 4rst - rt^3u - rt^3 - rtu^3 - rtu^2 + 5rtu + rt - s^3u - s^3 + s^2t^2 - s^2u^2 + s^2 + 3st^2u - st^2 - su^3 - su^2 + su + s - t^4 + 2t^2 - u^4 + 2u^2 - 1)W_{n+3}W_{n+1}$,

$\Phi_{10} = u(r^3u + r^2tu + 2r^2t - rs^2u + 2rsu + rt^2u + 2rt^2 - ru^3 + ru - s^2tu - 2s^2t + 2st + t^3u - tu^3 + tu)W_{n+2}W_{n+1}$,

$\Phi_{11} = -(r^2 + rtu + rt - s^2u - s^2 - su^2 + s + t^2u)W_3^2$,

$\Phi_{12} = -(r^3tu - r^2s^2u - r^2su^2 - r^2s + r^2t^2u + r^2t^2 + r^2u^2 - rs^2t - 2rstu + rt^3 + rtu^2 + rtu - s^2u - s^2 - su^2 + s + t^2u)$ W_2^2,

$\Phi_{13} = -(r^3tu + r^2t^2u + r^2t^2 + r^2u^2 - rs^2tu + 2rstu + rt^3 + rtu^2 + rtu - s^2t^2 - s^2u^3 - s^2u^2 + st^2u^2 + st^2 - su^4 + su^2 + t^2u)$ W_1^2,

$\Phi_{14} = -u^2(r^2 + rtu + rt - s^2u - s^2 - su^2 + s + t^2u)W_0^2$,

$\Phi_{15} = (r^3 + 2r^2tu + r^2t - 2rs^2u - rs^2 - 2rsu^2 + 2rt^2u + rt^2 + ru^2 - r - s^2t - 2stu + t^3 + tu^2 - t)W_3W_2$,

$\Phi_{16} = (r^2su + r^2s + r^2u^2 + r^2 + 2rst + 4rtu - s^3u - s^3 - s^2u^2 + s^2 + st^2u - st^2 - su^3 - su^2 + su + s + t^2u^2 + t^2 - u^4 + 2u^2 - 1)W_3W_1$,

$\Phi_{17} = -(r^3su + r^3u^2 + r^2st + 2r^2tu - r^2t - rs^3u - rs^2u^2 + 2rs^2u + rst^2u - rsu^3 + 2rsu^2 + rsu + rt^2u^2 - 2rt^2u - ru^4 + ru^2 - s^3t - 2s^2tu + s^2t + st^3 + stu^2 + 2stu - st - t^3 - tu^2 + t)W_2W_1$,

$\Phi_{18} = -u(r^3 + r^2t - rs^2 + 2rs - rt^2 - ru^2 + r + s^2t + 2stu - t^3 - tu^2 + t)W_3W_0$,

$\Phi_{19} = (r^4u+r^3tu+r^3t-r^2s^2u+3r^2su+r^2s-r^2t^2u+r^2t^2-r^2u^3+r^2u^2+r^2u+r^2+rs^2tu-rs^2t+2rstu^2+4rst-rt^3u-rt^3-rtu^3-rtu^2+5rtu+rt-s^3u-s^3+s^2t^2-s^2u^2+s^2+3st^2u-st^2-su^3-su^2+su+s-t^4+2t^2-u^4+2u^2-1)W_2W_0,$

$\Phi_{20} = -u(r^3u+r^2tu+2r^2t-rs^2u+2rsu+rt^2u+2rt^2-ru^3+ru-s^2tu-2s^2t+2st+t^3u-tu^3+tu)W_1W_0,$

and

$\Psi_1 = (r^3+r^2t-rs^2-rsu+2rs-rt^2-ru^2+ru+s^2t+2stu-st-t^3-tu+t)W_{n+4}^2,$

$\Psi_2 = (-r^3s+r^3u^2-2r^2stu-r^2st+r^2tu^2+r^2tu-r^2t+rs^3u+rs^3+rs^2u^2+rs^2u-rs^2-rst^2u-rst^2-rsu^3+rsu^2+rs+rt^2u^2-rt^2u-rt^2-ru^4+ru^3+s^2t+2stu-st-t^3-tu+t)W_{n+3}^2,$

$\Psi_3 = (r^3u^2-r^2stu-r^2st+r^2tu^2+r^2tu-r^2t-rs^2u^2-rst^2u-2rst^2-rsu^3+2rsu^2+rt^2u^2-rt^2u-rt^2-ru^4+ru^3+s^3tu+s^3t+s^2tu^2-s^2tu-s^2t-st^3u+stu^3+stu-st-t^3-tu+t)W_{n+2}^2,$

$\Psi_4 = u^2(r^3+r^2t-rs^2-rsu+2rs-rt^2-ru^2+ru+s^2t+2stu-st-t^3-tu+t)W_{n+1}^2,$

$\Psi_5 = (-r^4-r^3t+r^2s^2+r^2su-r^2s+r^2t^2-r^2u+r^2-rs^2t+2rst+rt^3-rtu^2+rt-s^3u-s^3-s^2u^2-s^2u+st^2u+st^2+su^3-su^2-su+s-t^2u^2+t^2u+u^4-u^3-u^2+u)W_{n+4}W_{n+3},$

$\Psi_6 = (-r^3s-r^3-r^2st-r^2tu+rs^3+2rs^2u-rs^2+rst^2+3rsu^2-rs+rt^2-ru^2+r-s^3t-3s^2tu+st^3-stu^2+2stu+st+t^3u-tu^3+tu)W_{n+4}W_{n+2},$

$\Psi_7 = -(r^3t+r^2su+r^2s+r^2t^2-r^2u+r^2-rs^2t+4rst-rt^3-3rtu^2+2rtu+rt-s^3u-s^3+s^2t^2+s^2u+s^2+3st^2u-st^2+su^3-su^2-su+s-t^4-t^2u^2-t^2u+2t^2-u^3+u^2+u-1)W_{n+4}W_{n+1},$

$\Psi_8 = (r^3tu-r^3t-r^2s^2u-r^2s^2-r^2su^2-r^2s+r^2u^2-r^2u-rs^2tu-rs^2t+2rstu^2-2rst-rt^3u+rt^3+rtu^3-rtu^2-rtu+rt+s^4u+s^4+s^3u^2-s^3-s^2t^2u-s^2t^2-s^2u^3-s^2+st^2u^2+st^2-su^4+2su^3-2su+s-t^2u^2+t^2u+u^4-u^3-u^2+u)W_{n+3}W_{n+2},$

$\Psi_9 = u(r^3s-r^3+r^2st-r^2tu-rs^3+3rs^2-rst^2+rsu^2+2rsu-rs+rt^2-ru^2+r+s^3t+s^2tu-2s^2t-st^3-stu^2+3st+t^3u-tu^3+tu)W_{n+3}W_{n+1},$

$\Psi_{10} = u(r^3t-r^2su-r^2s+r^2t^2+r^2u-r^2-rs^2t-2rstu-rt^3+rtu^2-rt+s^3u+s^3+s^2t^2-s^2u-s^2+st^2u-st^2-su^3+su^2+su-s-t^4+t^2u^2-t^2u+u^3-u^2-u+1)W_{n+2}W_{n+1},$

$\Psi_{11} = -(r^3+r^2t-rs^2-rsu+2rs-rt^2-ru^2+ru+s^2t+2stu-st-t^3-tu+t)W_3^2,$

$\Psi_{12} = (r^3s-r^3u^2+2r^2stu+r^2st-r^2tu^2-r^2tu+r^2t-rs^3u-rs^3-rs^2u^2-rs^2u+rs^2+rst^2u+rst^2+rsu^3-rsu^2-rs-rt^2u^2+rt^2u+rt^2+ru^4-ru^3-s^2t-2stu+st+t^3+tu-t)W_2^2,$

$\Psi_{13} = -(r^3u^2-r^2stu-r^2st+r^2tu^2+r^2tu-r^2t-rs^2u^2-rst^2u-2rst^2-rsu^3+2rsu^2+rt^2u^2-rt^2u-rt^2-ru^4+ru^3+s^3tu+s^3t+s^2tu^2-s^2tu-s^2t-st^3u+stu^3+stu-st-t^3-tu+t)W_1^2,$

$\Psi_{14} = -u^2(r^3+r^2t-rs^2-rsu+2rs-rt^2-ru^2+ru+s^2t+2stu-st-t^3-tu+t)W_0^2,$

$\Psi_{15} = (r^4+r^3t-r^2s^2-r^2su+r^2s-r^2t^2+r^2u-r^2+rs^2t-2rst-rt^3+rtu^2-rt+s^3u+s^3+s^2u^2+s^2u-st^2u-st^2-su^3+su^2+su-s+t^2u^2-t^2u-u^4+u^3+u^2-u)W_3W_2,$

$\Psi_{16} = (r^3s+r^3+r^2st+r^2tu-rs^3-2rs^2u+rs^2-rst^2-3rsu^2+rs-rt^2+ru^2-r+s^3t+3s^2tu-st^3+stu^2-2stu-st-t^3u+tu^3-tu)W_3W_1,$

$\Psi_{17} = -(r^3tu-r^3t-r^2s^2u-r^2s^2-r^2su^2-r^2s+r^2u^2-r^2u-rs^2tu-rs^2t+2rstu^2-2rst-rt^3u+rt^3+rtu^3-rtu^2-rtu+rt+s^4u+s^4+s^3u^2-s^3-s^2t^2u-s^2t^2-s^2u^3-s^2+st^2u^2+st^2-su^4+2su^3-2su+s-t^2u^2+t^2u+u^4-u^3-u^2+u)W_2W_1,$

$\Psi_{18} = (r^3t+r^2su+r^2s+r^2t^2-r^2u+r^2-rs^2t+4rst-rt^3-3rtu^2+2rtu+rt-s^3u-s^3+s^2t^2+s^2u+s^2+3st^2u-st^2+su^3-su^2-su+s-t^4-t^2u^2-t^2u+2t^2-u^3+u^2+u-1)W_3W_0,$

$\Psi_{19} = -u(r^3s-r^3+r^2st-r^2tu-rs^3+3rs^2-rst^2+rsu^2+2rsu-rs+rt^2-ru^2+r+s^3t+s^2tu-2s^2t-st^3-stu^2+3st+t^3u-tu^3+tu)W_2W_0,$

$\Psi_{20} = u(-r^3t+r^2su+r^2s-r^2t^2-r^2u+r^2+rs^2t+2rstu+rt^3-rtu^2+rt-s^3u-s^3-s^2t^2+s^2u+s^2-st^2u+st^2+su^3-su^2-su+s+t^4-t^2u^2+t^2u-u^3+u^2+u-1)W_1W_0.$

Proof. First, we obtain $\sum_{k=0}^{n} W_k^2$. Using the recurrence relation

$$W_{n+4} = rW_{n+3} + sW_{n+2} + tW_{n+1} + uW_n$$

or

$$uW_n = W_{n+4} - rW_{n+3} - sW_{n+2} - tW_{n+1}$$

we obtain

$$\begin{array}{rcl} u^2W_n^2 & = & W_{n+4}^2+r^2W_{n+3}^2+s^2W_{n+2}^2+t^2W_{n+1}^2-2rW_{n+4}W_{n+3}-2sW_{n+4}W_{n+2} \\ & & -2tW_{n+4}W_{n+1}+2rsW_{n+3}W_{n+2}+2rtW_{n+3}W_{n+1}+2stW_{n+2}W_{n+1} \\ u^2W_{n-1}^2 & = & W_{n+3}^2+r^2W_{n+2}^2+s^2W_{n+1}^2+t^2W_n^2-2rW_{n+3}W_{n+2}-2sW_{n+3}W_{n+1} \\ & & -2tW_{n+3}W_n+2rsW_{n+2}W_{n+1}+2rtW_{n+2}W_n+2stW_{n+1}W_n \\ & & \vdots \\ u^2W_1^2 & = & W_5^2+r^2W_4^2+s^2W_3^2+t^2W_2^2-2rW_5W_4-2sW_5W_3 \\ & & -2tW_5W_2+2rsW_4W_3+2rtW_4W_2+2stW_3W_2 \\ u^2W_0^2 & = & W_4^2+r^2W_3^2+s^2W_2^2+t^2W_1^2-2rW_4W_3-2sW_4W_2 \\ & & -2tW_4W_1+2rsW_3W_2+2rtW_3W_1+2stW_2W_1 \end{array}$$

If we add the equations side by side, we get

$$\begin{array}{rcl} u^2\sum_{k=0}^{n} W_k^2 & = & (r^2+s^2+t^2+1)\sum_{k=0}^{n} W_k^2+2(-r+rs+st)\sum_{k=0}^{n} W_{k+1}W_k \\ & & +2(-s+rt)\sum_{k=0}^{n} W_{k+2}W_k-2t\sum_{k=0}^{n} W_{k+3}W_k+W_{n+4}^2+(r^2+1)W_{n+3}^2 \\ & & +(r^2+s^2+1)W_{n+2}^2+(r^2+s^2+t^2+1)W_{n+1}^2-2rW_{n+4}W_{n+3}-2sW_{n+4}W_{n+2} \\ & & -2tW_{n+4}W_{n+1}+2r(s-1)W_{n+3}W_{n+2}+2(-s+rt)W_{n+3}W_{n+1} \\ & & +2(-r+rs+st)W_{n+2}W_{n+1}-W_3^2-(r^2+1)W_2^2-(r^2+s^2+1)W_1^2 \\ & & -(r^2+s^2+t^2+1)W_0^2+2rW_3W_2+2sW_3W_1-2r(s-1)W_2W_1 \\ & & +2tW_3W_0-2(-s+rt)W_2W_0-2(-r+rs+st)W_1W_0. \end{array} \tag{4}$$

Next we obtain $\sum_{k=0}^{n} W_{k+1}W_k$. Multiplying both sides of the recurrence relation

$$uW_n = W_{n+4}-rW_{n+3}-sW_{n+2}-tW_{n+1}$$

by W_{n+1} we get

$$uW_{n+1}W_n = W_{n+4}W_{n+1}-rW_{n+3}W_{n+1}-sW_{n+2}W_{n+1}-tW_{n+1}^2$$

Then using last recurrence relation, we obtain

$$\begin{array}{rcl} uW_{n+1}W_n & = & W_{n+4}W_{n+1}-rW_{n+3}W_{n+1}-sW_{n+2}W_{n+1}-tW_{n+1}^2 \\ uW_nW_{n-1} & = & W_{n+3}W_n-rW_{n+2}W_n-sW_{n+1}W_n-tW_n^2 \\ uW_{n-1}W_{n-2} & = & W_{n+2}W_{n-1}-rW_{n+1}W_{n-1}-sW_nW_{n-1}-tW_{n-1}^2 \\ & & \vdots \\ uW_3W_2 & = & W_6W_3-rW_5W_3-sW_4W_3-tW_3^2 \\ uW_2W_1 & = & W_5W_2-rW_4W_2-sW_3W_2-tW_2^2 \\ uW_1W_0 & = & W_4W_1-rW_3W_1-sW_2W_1-tW_1^2 \end{array}$$

If we add the equations side by side, we get

$$\begin{array}{rcl} u\sum_{k=0}^{n} W_{k+1}W_k & = & (W_{n+4}W_{n+1}-W_3W_0+\sum_{k=0}^{n} W_{k+3}W_k) \\ & & -r(W_{n+3}W_{n+1}-W_2W_0+\sum_{k=0}^{n} W_{k+2}W_k) \\ & & -s(W_{n+2}W_{n+1}-W_1W_0+\sum_{k=0}^{n} W_{k+1}W_k) \\ & & -t(W_{n+1}^2-W_0^2+\sum_{k=0}^{n} W_k^2). \end{array} \tag{5}$$

Next we obtain $\sum_{k=0}^{n} W_{k+2}W_k$. Multiplying both sides of the recurrence relation

$$uW_n = W_{n+4}-rW_{n+3}-sW_{n+2}-tW_{n+1}$$

by W_{n+2} we get

$$uW_{n+2}W_n = W_{n+4}W_{n+2} - rW_{n+3}W_{n+2} - sW_{n+2}^2 - tW_{n+2}W_{n+1}$$

Then using last recurrence relation, we obtain

$$\begin{aligned}
uW_{n+2}W_n &= W_{n+4}W_{n+2} - rW_{n+3}W_{n+2} - sW_{n+2}^2 - tW_{n+2}W_{n+1} \\
uW_{n+1}W_{n-1} &= W_{n+3}W_{n+1} - rW_{n+2}W_{n+1} - sW_{n+1}^2 - tW_{n+1}W_n \\
uW_nW_{n-2} &= W_{n+2}W_n - rW_{n+1}W_n - sW_n^2 - tW_nW_{n-1} \\
&\vdots \\
uW_3W_1 &= W_5W_3 - rW_4W_3 - sW_3^2 - tW_3W_2 \\
uW_2W_0 &= W_4W_2 - rW_3W_2 - sW_2^2 - tW_2W_1
\end{aligned}$$

If we add the equations side by side, we get

$$\begin{aligned}
u\sum_{k=0}^{n} W_{k+2}W_k &= (W_{n+4}W_{n+2} + W_{n+3}W_{n+1} - W_3W_1 - W_2W_0 + \sum_{k=0}^{n} W_{k+2}W_k) \\
&\quad -r(W_{n+3}W_{n+2} + W_{n+2}W_{n+1} - W_2W_1 - W_1W_0 + \sum_{k=0}^{n} W_{k+1}W_k) \\
&\quad -s(W_{n+2}^2 + W_{n+1}^2 - W_1^2 - W_0^2 + \sum_{k=0}^{n} W_k^2) \\
&\quad -t(W_{n+2}W_{n+1} - W_1W_0 + \sum_{k=0}^{n} W_{k+1}W_k).
\end{aligned} \tag{6}$$

Next we obtain $\sum_{k=0}^{n} W_{k+3}W_k$. Multiplying both sides of the recurrence relation

$$uW_n = W_{n+4} - rW_{n+3} - sW_{n+2} - tW_{n+1}$$

by W_{n+3} we get

$$uW_{n+3}W_n = W_{n+4}W_{n+3} - rW_{n+3}^2 - sW_{n+3}W_{n+2} - tW_{n+3}W_{n+1}$$

Then using last recurrence relation, we obtain

$$\begin{aligned}
uW_{n+3}W_n &= W_{n+4}W_{n+3} - rW_{n+3}^2 - sW_{n+3}W_{n+2} - tW_{n+3}W_{n+1} \\
uW_{n+2}W_{n-1} &= W_{n+3}W_{n+2} - rW_{n+2}^2 - sW_{n+2}W_{n+1} - tW_{n+2}W_n \\
uW_{n+1}W_{n-2} &= W_{n+2}W_{n+1} - rW_{n+1}^2 - sW_{n+1}W_n - tW_{n+1}W_{n-1} \\
&\vdots \\
uW_5W_2 &= W_6W_5 - rW_5^2 - sW_5W_4 - tW_5W_3 \\
uW_4W_1 &= W_5W_4 - rW_4^2 - sW_4W_3 - tW_4W_2 \\
uW_3W_0 &= W_4W_3 - rW_3^2 - sW_3W_2 - tW_3W_1
\end{aligned}$$

If we add the equations side by side, we get

$$\begin{aligned}
u\sum_{k=0}^{n} W_{k+3}W_k &= (W_{n+4}W_{n+3} + W_{n+3}W_{n+2} + W_{n+2}W_{n+1} - W_3W_2 - W_2W_1 - W_1W_0 + \sum_{k=0}^{n} W_{k+1}W_k) \\
&\quad -r(W_{n+3}^2 + W_{n+2}^2 + W_{n+1}^2 - W_2^2 - W_1^2 - W_0^2 + \sum_{k=0}^{n} W_k^2) \\
&\quad -s(W_{n+3}W_{n+2} + W_{n+2}W_{n+1} - W_2W_1 - W_1W_0 + \sum_{k=0}^{n} W_{k+1}W_k) \\
&\quad -t(W_{n+3}W_{n+1} - W_2W_0 + \sum_{k=0}^{n} W_{k+2}W_k).
\end{aligned} \tag{7}$$

Solving the system (4)-(5)-(6)-(7), the results in (a), (b), (c) and (d) follow.

3. Specific Cases

In this section, we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n} W_k^2$, $\sum_{k=0}^{n} W_{k+1}W_k$, $\sum_{k=0}^{n} W_{k+2}W_k$ and $\sum_{k=0}^{n} W_{k+3}W_k$ for the specific case of sequence $\{W_n\}$.

Taking $r = s = t = u = 1$ in Theorem 2.1, we obtain the following proposition.

Proposition 3.1.
If $r = s = t = u = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{3}(-W_{n+4}^2 - 3W_{n+3}^2 - 4W_{n+2}^2 - 4W_{n+1}^2 + 3W_{n+4}W_{n+3} + 2W_{n+4}W_{n+2} + W_{n+4}W_{n+1} - W_{n+2}W_{n+1} + W_3^2 + 3W_2^2 + 4W_1^2 + 4W_0^2 - 3W_3W_2 - 2W_3W_1 - W_3W_0 + W_1W_0)$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{6}(W_{n+4}^2 + 3W_{n+3}^2 + W_{n+2}^2 + W_{n+1}^2 - 3W_{n+4}W_{n+3} + W_{n+4}W_{n+2} - W_{n+4}W_{n+1} - 3W_{n+3}W_{n+2} + 3W_{n+3}W_{n+1} - 5W_{n+2}W_{n+1} - W_3^2 - 3W_2^2 - W_1^2 - W_0^2 + 3W_3W_2 - W_3W_1 + W_3W_0 + 3W_2W_1 - 3W_2W_0 + 5W_1W_0)$.

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{6}(W_{n+4}^2 + 4W_{n+2}^2 + W_{n+1}^2 - 5W_{n+4}W_{n+2} + 2W_{n+4}W_{n+1} + 3W_{n+3}W_{n+2} - 9W_{n+3}W_{n+1} + 4W_{n+2}W_{n+1} - W_3^2 - 4W_1^2 - W_0^2 + 5W_3W_1 - 2W_3W_0 - 3W_2W_1 + 9W_2W_0 - 4W_1W_0)$.

(d) $\sum_{k=0}^{n} W_{k+3}W_k = \frac{1}{6}(W_{n+4}^2 - 2W_{n+2}^2 + W_{n+1}^2 + W_{n+4}W_{n+2} - 4W_{n+4}W_{n+1} - 3W_{n+3}W_{n+2} + 3W_{n+3}W_{n+1} - 2W_{n+2}W_{n+1} - W_3^2 + 2W_1^2 - W_0^2 - W_3W_1 + 4W_3W_0 + 3W_2W_1 - 3W_2W_0 + 2W_1W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Tetranacci numbers (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2$).

Corollary 3.1.
For $n \geq 0$, Tetranacci numbers have the following properties:

(a) $\sum_{k=0}^{n} M_k^2 = \frac{1}{3}(-M_{n+4}^2 - 3M_{n+3}^2 - 4M_{n+2}^2 - 4M_{n+1}^2 + 3M_{n+4}M_{n+3} + 2M_{n+4}M_{n+2} + M_{n+4}M_{n+1} - M_{n+2}M_{n+1} + 1)$.

(b) $\sum_{k=0}^{n} M_{k+1}M_k = \frac{1}{6}(M_{n+4}^2 + 3M_{n+3}^2 + M_{n+2}^2 + M_{n+1}^2 - 3M_{n+4}M_{n+3} + M_{n+4}M_{n+2} - M_{n+4}M_{n+1} - 3M_{n+3}M_{n+2} + 3M_{n+3}M_{n+1} - 5M_{n+2}M_{n+1} - 1)$.

(c) $\sum_{k=0}^{n} M_{k+2}M_k = \frac{1}{6}(M_{n+4}^2 + 4M_{n+2}^2 + M_{n+1}^2 - 5M_{n+4}M_{n+2} + 2M_{n+4}M_{n+1} + 3M_{n+3}M_{n+2} - 9M_{n+3}M_{n+1} + 4M_{n+2}M_{n+1} - 1)$.

(d) $\sum_{k=0}^{n} M_{k+3}M_k = \frac{1}{6}(M_{n+4}^2 - 2M_{n+2}^2 + M_{n+1}^2 + M_{n+4}M_{n+2} - 4M_{n+4}M_{n+1} - 3M_{n+3}M_{n+2} + 3M_{n+3}M_{n+1} - 2M_{n+2}M_{n+1} - 1)$.

Taking $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ in the above proposition, we have the following corollary which presents sum formulas of Tetranacci-Lucas numbers.

Corollary 3.2.
For $n \geq 0$, Tetranacci-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} R_k^2 = \frac{1}{3}(-R_{n+4}^2 - 3R_{n+3}^2 - 4R_{n+2}^2 - 4R_{n+1}^2 + 3R_{n+4}R_{n+3} + 2R_{n+4}R_{n+2} + R_{n+4}R_{n+1} - R_{n+2}R_{n+1} + 43)$.

(b) $\sum_{k=0}^{n} R_{k+1}R_k = \frac{1}{6}(R_{n+4}^2 + 3R_{n+3}^2 + R_{n+2}^2 + R_{n+1}^2 - 3R_{n+4}R_{n+3} + R_{n+4}R_{n+2} - R_{n+4}R_{n+1} - 3R_{n+3}R_{n+2} + 3R_{n+3}R_{n+1} - 5 R_{n+2}R_{n+1} - 16)$.

(c) $\sum_{k=0}^{n} R_{k+2}R_k = \frac{1}{6}(R_{n+4}^2 + 4R_{n+2}^2 + R_{n+1}^2 - 5R_{n+4}R_{n+2} + 2R_{n+4}R_{n+1} + 3R_{n+3}R_{n+2} - 9R_{n+3}R_{n+1} + 4R_{n+2}R_{n+1} - 7)$.

(d) $\sum_{k=0}^{n} R_{k+3}R_k = \frac{1}{6}(R_{n+4}^2 - 2R_{n+2}^2 + R_{n+1}^2 + R_{n+4}R_{n+2} - 4R_{n+4}R_{n+1} - 3R_{n+3}R_{n+2} + 3R_{n+3}R_{n+1} - 2R_{n+2}R_{n+1} + 23)$.

Taking $r = 2, s = 1, t = 1, u = 1$ in Theorem 2.1, we obtain the following proposition.

Proposition 3.2.
If $r = 2, s = 1, t = 1, u = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{56}(-9W_{n+4}^2 - 57W_{n+3}^2 - 68W_{n+2}^2 - 65W_{n+1}^2 + 42W_{n+4}W_{n+3} + 20W_{n+4}W_{n+2} + 6W_{n+4}W_{n+1} - 4W_{n+3}W_{n+2} + 2W_{n+3}W_{n+1} - 12W_{n+2}W_{n+1} + 9W_3^2 + 57W_2^2 + 68W_1^2 + 65W_0^2 - 42W_3W_2 - 20W_3W_1 - 6W_3W_0 + 4W_2W_1 - 2W_2W_0 + 12W_1W_0)$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{56}(3W_{n+4}^2+19W_{n+3}^2+4W_{n+2}^2+3W_{n+1}^2-14W_{n+4}W_{n+3}+12W_{n+4}W_{n+2}-2W_{n+4}W_{n+1}-36W_{n+3}W_{n+2}+18W_{n+3}W_{n+1}-52W_{n+2}W_{n+1}-3W_3^2-19W_2^2-4W_1^2-3W_0^2+14W_3W_2-12W_3W_1+2W_3W_0+36W_2W_1-18W_2W_0+52W_1W_0)$.

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{8}(W_{n+4}^2+W_{n+3}^2+4W_{n+2}^2+W_{n+1}^2-2W_{n+4}W_{n+3}-4W_{n+4}W_{n+2}+2W_{n+4}W_{n+1}+4W_{n+3}W_{n+2}-10W_{n+3}W_{n+1}+4W_{n+2}W_{n+1}-W_3^2-W_2^2-4W_1^2-W_0^2+2W_3W_2+4W_3W_1-2W_3W_0-4W_2W_1+10W_2W_0-4W_1W_0)$.

(d) $\sum_{k=0}^{n} W_{k+3}W_k = \frac{1}{56}(11W_{n+4}^2-5W_{n+3}^2-4W_{n+2}^2+11W_{n+1}^2-14W_{n+4}W_{n+3}-12W_{n+4}W_{n+2}-26W_{n+4}W_{n+1}-20W_{n+3}W_{n+2}+10W_{n+3}W_{n+1}-4W_{n+2}W_{n+1}-11W_3^2+5W_2^2+4W_1^2-11W_0^2+14W_3W_2+12W_3W_1+26W_3W_0+20W_2W_1-10W_2W_0+4W_1W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of fourth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5$).

Corollary 3.3.
For $n \geq 0$, fourth-order Pell numbers have the following properties:

(a) $\sum_{k=0}^{n} P_k^2 = \frac{1}{56}(-9P_{n+4}^2-57P_{n+3}^2-68P_{n+2}^2-65P_{n+1}^2+42P_{n+4}P_{n+3}+20P_{n+4}P_{n+2}+6P_{n+4}P_{n+1}-4P_{n+3}P_{n+2}+2P_{n+3}P_{n+1}-12P_{n+2}P_{n+1}+9)$.

(b) $\sum_{k=0}^{n} P_{k+1}P_k = \frac{1}{56}(3P_{n+4}^2+19P_{n+3}^2+4P_{n+2}^2+3P_{n+1}^2-14P_{n+4}P_{n+3}+12P_{n+4}P_{n+2}-2P_{n+4}P_{n+1}-36P_{n+3}P_{n+2}+18P_{n+3}P_{n+1}-52P_{n+2}P_{n+1}-3)$.

(c) $\sum_{k=0}^{n} P_{k+2}P_k = \frac{1}{8}(P_{n+4}^2+P_{n+3}^2+4P_{n+2}^2+P_{n+1}^2-2P_{n+4}P_{n+3}-4P_{n+4}P_{n+2}+2P_{n+4}P_{n+1}+4P_{n+3}P_{n+2}-10P_{n+3}P_{n+1}+4P_{n+2}P_{n+1}-1)$.

(d) $\sum_{k=0}^{n} P_{k+3}P_k = \frac{1}{56}(11P_{n+4}^2-5P_{n+3}^2-4P_{n+2}^2+11P_{n+1}^2-14P_{n+4}P_{n+3}-12P_{n+4}P_{n+2}-26P_{n+4}P_{n+1}-20P_{n+3}P_{n+2}+10P_{n+3}P_{n+1}-4P_{n+2}P_{n+1}-11)$.

Taking $W_n = Q_n$ with $Q_0 = 4, Q_1 = 2, Q_2 = 6, Q_3 = 17$ in the last proposition, we have the following corollary which presents sum formulas of fourth-order Pell-Lucas numbers.

Corollary 3.4.
For $n \geq 0$, fourth-order Pell-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} Q_k^2 = \frac{1}{56}(-9Q_{n+4}^2-57Q_{n+3}^2-68Q_{n+2}^2-65Q_{n+1}^2+42Q_{n+4}Q_{n+3}+20Q_{n+4}Q_{n+2}+6Q_{n+4}Q_{n+1}-4Q_{n+3}Q_{n+2}+2Q_{n+3}Q_{n+1}-12Q_{n+2}Q_{n+1}+689)$.

(b) $\sum_{k=0}^{n} Q_{k+1}Q_k = \frac{1}{56}(3Q_{n+4}^2+19Q_{n+3}^2+4Q_{n+2}^2+3Q_{n+1}^2-14Q_{n+4}Q_{n+3}+12Q_{n+4}Q_{n+2}-2Q_{n+4}Q_{n+1}-36Q_{n+3}Q_{n+2}+18Q_{n+3}Q_{n+1}-52Q_{n+2}Q_{n+1}-43)$.

(c) $\sum_{k=0}^{n} Q_{k+2}Q_k = \frac{1}{8}(Q_{n+4}^2+Q_{n+3}^2+4Q_{n+2}^2+Q_{n+1}^2-2Q_{n+4}Q_{n+3}-4Q_{n+4}Q_{n+2}+2Q_{n+4}Q_{n+1}+4Q_{n+3}Q_{n+2}-10Q_{n+3}Q_{n+1}+4Q_{n+2}Q_{n+1}+7)$.

(d) $\sum_{k=0}^{n} Q_{k+3}Q_k = \frac{1}{56}(11Q_{n+4}^2-5Q_{n+3}^2-4Q_{n+2}^2+11Q_{n+1}^2-14Q_{n+4}Q_{n+3}-12Q_{n+4}Q_{n+2}-26Q_{n+4}Q_{n+1}-20Q_{n+3}Q_{n+2}+10Q_{n+3}Q_{n+1}-4Q_{n+2}Q_{n+1}+477)$.

From the last proposition, we have the following corollary which gives sum formulas of modified fourth-order Pell numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1, E_3 = 3$).

Corollary 3.5.
For $n \geq 0$, modified fourth-order Pell numbers have the following properties:

(a) $\sum_{k=0}^{n} E_k^2 = \frac{1}{56}(-9E_{n+4}^2-57E_{n+3}^2-68E_{n+2}^2-65E_{n+1}^2+42E_{n+4}E_{n+3}+20E_{n+4}E_{n+2}+6E_{n+4}E_{n+1}-4E_{n+3}E_{n+2}+2E_{n+3}E_{n+1}-12E_{n+2}E_{n+1}+24)$.

(b) $\sum_{k=0}^{n} E_{k+1}E_k = \frac{1}{56}(3E_{n+4}^2+19E_{n+3}^2+4E_{n+2}^2+3E_{n+1}^2-14E_{n+4}E_{n+3}+12E_{n+4}E_{n+2}-2E_{n+4}E_{n+1}-36E_{n+3}E_{n+2}+18E_{n+3}E_{n+1}-52E_{n+2}E_{n+1}-8)$.

(c) $\sum_{k=0}^{n} E_{k+2}E_k = \frac{1}{8}(E_{n+4}^2+E_{n+3}^2+4E_{n+2}^2+E_{n+1}^2-2E_{n+4}E_{n+3}-4E_{n+4}E_{n+2}+2E_{n+4}E_{n+1}+4E_{n+3}E_{n+2}-10E_{n+3}E_{n+1}+4E_{n+2}E_{n+1})$.

(d) $\sum_{k=0}^{n} E_{k+3}E_k = \frac{1}{56}(11E_{n+4}^2 - 5E_{n+3}^2 - 4E_{n+2}^2 + 11E_{n+1}^2 - 14E_{n+4}E_{n+3} - 12E_{n+4}E_{n+2} - 26E_{n+4}E_{n+1} - 20E_{n+3}E_{n+2} + 10E_{n+3}E_{n+1} - 4E_{n+2}E_{n+1} + 8)$.

Taking $r = 2, s = 3, t = 5$ in Theorem 2.1, we obtain the following Proposition.

Proposition 3.3.
If $r = 2, s = 3, t = 5$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{7968}(299W_{n+4}^2 + 2155W_{n+3}^2 + 2608W_{n+2}^2 + 6683W_{n+1}^2 - 1526W_{n+4}W_{n+3} - 1048W_{n+4}W_{n+2} - 2310W_{n+4}W_{n+1} + 1560W_{n+3}W_{n+2} + 5222W_{n+3}W_{n+1} + 4760W_{n+2}W_{n+1} - 299W_3^2 - 2155W_2^2 - 2608W_1^2 - 6683W_0^2 + 1526W_3W_2 + 1048W_3W_1 + 2310W_3W_0 - 1560W_2W_1 - 5222W_2W_0 - 4760W_1W_0)$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{2656}(-55W_{n+4}^2 - 503W_{n+3}^2 - 80W_{n+2}^2 - 2695W_{n+1}^2 + 334W_{n+4}W_{n+3} + 24W_{n+4}W_{n+2} + 798W_{n+4}W_{n+1} - 56W_{n+3}W_{n+2} - 2142W_{n+3}W_{n+1} - 920W_{n+2}W_{n+1} + 55W_3^2 + 503W_2^2 + 80W_1^2 + 2695W_0^2 - 334W_3W_2 - 24W_3W_1 - 798W_3W_0 + 56W_2W_1 + 2142W_2W_0 + 920W_1W_0)$.

(c) $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{7968}(43W_{n+4}^2 + 683W_{n+3}^2 - 5008W_{n+2}^2 + 2107W_{n+1}^2 - 406W_{n+4}W_{n+3} + 1768W_{n+4}W_{n+2} - 1638W_{n+4}W_{n+1} - 3240W_{n+3}W_{n+2} + 6214W_{n+3}W_{n+1} - 8456W_{n+2}W_{n+1} - 43W_3^2 - 683W_2^2 + 5008W_1^2 - 2107W_0^2 + 406W_3W_2 - 1768W_3W_1 + 1638W_3W_0 + 3240W_2W_1 - 6214W_2W_0 + 8456W_1W_0)$.

(d) $\sum_{k=0}^{n} W_{k+3}W_k = \frac{1}{2656}(-23W_{n+4}^2 - 983W_{n+3}^2 + 208W_{n+2}^2 - 1127W_{n+1}^2 + 526W_{n+4}W_{n+3} - 328W_{n+4}W_{n+2} + 382W_{n+4}W_{n+1} - 120W_{n+3}W_{n+2} - 3262W_{n+3}W_{n+1} + 1064W_{n+2}W_{n+1} + 23W_3^2 + 983W_2^2 - 208W_1^2 + 1127W_0^2 - 526W_3W_2 + 328W_3W_1 - 382W_3W_0 + 120W_2W_1 + 3262W_2W_0 - 1064W_1W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of 4-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 1, G_3 = 2$).

Corollary 3.6.
For $n \geq 0$, 4-primes numbers have the following properties:

(a) $\sum_{k=0}^{n} G_k^2 = \frac{1}{7968}(299G_{n+4}^2 + 2155G_{n+3}^2 + 2608G_{n+2}^2 + 6683G_{n+1}^2 - 1526G_{n+4}G_{n+3} - 1048G_{n+4}G_{n+2} - 2310G_{n+4}G_{n+1} + 1560G_{n+3}G_{n+2} + 5222G_{n+3}G_{n+1} + 4760G_{n+2}G_{n+1} - 299)$.

(b) $\sum_{k=0}^{n} G_{k+1}G_k = \frac{1}{2656}(-55G_{n+4}^2 - 503G_{n+3}^2 - 80G_{n+2}^2 - 2695G_{n+1}^2 + 334G_{n+4}G_{n+3} + 24G_{n+4}G_{n+2} + 798G_{n+4}G_{n+1} - 56G_{n+3}G_{n+2} - 2142G_{n+3}G_{n+1} - 920G_{n+2}G_{n+1} + 55)$.

(d) $\sum_{k=0}^{n} G_{k+2}G_k = \frac{1}{7968}(43G_{n+4}^2 + 683G_{n+3}^2 - 5008G_{n+2}^2 + 2107G_{n+1}^2 - 406G_{n+4}G_{n+3} + 1768G_{n+4}G_{n+2} - 1638G_{n+4}G_{n+1} - 3240G_{n+3}G_{n+2} + 6214G_{n+3}G_{n+1} - 8456G_{n+2}G_{n+1} - 43)$.

(c) $\sum_{k=0}^{n} G_{k+3}G_k = \frac{1}{2656}(-23G_{n+4}^2 - 983G_{n+3}^2 + 208G_{n+2}^2 - 1127G_{n+1}^2 + 526G_{n+4}G_{n+3} - 328G_{n+4}G_{n+2} + 382G_{n+4}G_{n+1} - 120G_{n+3}G_{n+2} - 3262G_{n+3}G_{n+1} + 1064G_{n+2}G_{n+1} + 23)$.

Taking $W_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 10, H_3 = 41$ in the last proposition, we have the following corollary which presents sum formulas of Lucas 4-primes numbers.

Corollary 3.7.
For $n \geq 0$, Lucas 4-primes numbers have the following properties:

(a) $\sum_{k=0}^{n} H_k^2 = \frac{1}{7968}(299H_{n+4}^2 + 2155H_{n+3}^2 + 2608H_{n+2}^2 + 6683H_{n+1}^2 - 1526H_{n+4}H_{n+3} - 1048H_{n+4}H_{n+2} - 2310H_{n+4}H_{n+1} + 1560H_{n+3}H_{n+2} + 5222H_{n+3}H_{n+1} + 4760H_{n+2}H_{n+1} - 23203)$.

(b) $\sum_{k=0}^{n} H_{k+1}H_k = \frac{1}{2656}(-55H_{n+4}^2 - 503H_{n+3}^2 - 80H_{n+2}^2 - 2695H_{n+1}^2 + 334H_{n+4}H_{n+3} + 24H_{n+4}H_{n+2} + 798H_{n+4}H_{n+1} - 56H_{n+3}H_{n+2} - 2142H_{n+3}H_{n+1} - 920H_{n+2}H_{n+1} + 10575)$.

(c) $\sum_{k=0}^{n} H_{k+2}H_k = \frac{1}{7968}(43H_{n+4}^2 + 683H_{n+3}^2 - 5008H_{n+2}^2 + 2107H_{n+1}^2 - 406H_{n+4}H_{n+3} + 1768H_{n+4}H_{n+2} - 1638H_{n+4}H_{n+1} - 3240H_{n+3}H_{n+2} + 6214H_{n+3}H_{n+1} - 8456H_{n+2}H_{n+1} + 19741)$.

(d) $\sum_{k=0}^{n} H_{k+3}H_k = \frac{1}{2656}(-23H_{n+4}^2 - 983H_{n+3}^2 + 208H_{n+2}^2 - 1127H_{n+1}^2 + 526H_{n+4}H_{n+3} - 328H_{n+4}H_{n+2} + 382H_{n+4}H_{n+1} - 120H_{n+3}H_{n+2} - 3262H_{n+3}H_{n+1} + 1064H_{n+2}H_{n+1} + 27119)$.

From the last proposition, we have the following corollary which gives sum formulas of modified 4-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 1, E_3 = 1$).

Corollary 3.8.
For $n \geq 0$, modified 4-primes numbers have the following properties:

(a) $\sum_{k=0}^{n} E_k^2 = \frac{1}{7968}(299E_{n+4}^2 + 2155E_{n+3}^2 + 2608E_{n+2}^2 + 6683E_{n+1}^2 - 1526E_{n+4}E_{n+3} - 1048E_{n+4}E_{n+2} - 2310E_{n+4}E_{n+1} + 1560E_{n+3}E_{n+2} + 5222E_{n+3}E_{n+1} + 4760E_{n+2}E_{n+1} - 928)$.

(b) $\sum_{k=0}^{n} E_{k+1}E_k = \frac{1}{2656}(-55E_{n+4}^2 - 503E_{n+3}^2 - 80E_{n+2}^2 - 2695E_{n+1}^2 + 334E_{n+4}E_{n+3} + 24E_{n+4}E_{n+2} + 798E_{n+4}E_{n+1} - 56E_{n+3}E_{n+2} - 2142E_{n+3}E_{n+1} - 920E_{n+2}E_{n+1} + 224)$.

(c) $\sum_{k=0}^{n} E_{k+2}E_k = \frac{1}{7968}(43E_{n+4}^2 + 683E_{n+3}^2 - 5008E_{n+2}^2 + 2107E_{n+1}^2 - 406E_{n+4}E_{n+3} + 1768E_{n+4}E_{n+2} - 1638E_{n+4}E_{n+1} - 3240E_{n+3}E_{n+2} + 6214E_{n+3}E_{n+1} - 8456E_{n+2}E_{n+1} - 320)$.

(d) $\sum_{k=0}^{n} E_{k+3}E_k = \frac{1}{2656}(-23E_{n+4}^2 - 983E_{n+3}^2 + 208E_{n+2}^2 - 1127E_{n+1}^2 + 526E_{n+4}E_{n+3} - 328E_{n+4}E_{n+2} + 382E_{n+4}E_{n+1} - 120E_{n+3}E_{n+2} - 3262E_{n+3}E_{n+1} + 1064E_{n+2}E_{n+1} + 480)$.

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