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# Lower dimensional approximation of eigenvalue problem for thin elastic shells with nonuniform thickness

**Research Article**

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**Abstract:** In this paper we consider the eigenvalue problem for thin elastic shells with nonuniform thickness and show that as the thickness goes to zero the eigensolutions of the three dimensional problem converge to the eigensolutions of a two dimensional eigenvalue problem.

**MSC:** 74K25

**Keywords:** Shallow shells • Flexural shells • Eigenvalue problem • Nonuniform thickness



## 1. Introduction

In the study of modeling of elastic or piezoelectric bodies often one of the dimensions, say thickness, is small compared to other dimensions and in most of the cases the thickness is nonuniform. In such cases lower dimensional approximation of the three dimensional models are preferred as they are more suitable for numerical computations.

In this connection lower dimensional approximation of thin elastic and piezoelectric plates, shells and rods with uniform thickness has been studied in static cases(cf: [7], [9], [10], [11], [13], [16], [14]) and the corresponding eigenvalue problems for uniform thickness has been studied (cf.[17], [18], [19], [23], [24], [20]). Contact problem for elastic and piezoelectric materials has been studied (cf. [1], [25], [26]). Junctions in plates and rods as been studied (cf. [3], [4]) and thin elastic plates supported over small areas has been studied (cf. [5], [6], [15]).

In the case of nonuniform thickness lower dimensional approximation of static problem for elastic membrane and flexural shells has been studied (cf. [2]) and for elastic shallow shells and piezoelectric shells has been studied in (cf. [21], [22]). The eigenvalue problem for rods with nonuniform thickness has been studied in (cf. [12]) and in this paper we consider the eigenvalue problem for thin elastic shells with nonuniform thickness and study their limiting behaviour.

We consider a bounded domain, $\Omega^\epsilon = \omega \times (-\epsilon, \epsilon)$, $\omega \subset \mathbb{R}^2$ and let $x^\epsilon = (x_1, x_2, x_3^\epsilon)$ be a generic point on $\Omega^\epsilon$. Let $\phi^\epsilon$ be an injective mapping and $a^3(x_1, x_2)$ denotes unit normal vector to the surface $\phi^\epsilon(\omega)$. For each $\epsilon > 0$, we define the mapping $\Phi^\epsilon : \Omega^\epsilon \to \mathbb{R}^3$ by

$$\Phi^\epsilon(x^\epsilon) = \phi^\epsilon(x_1, x_2) + x_3^\epsilon e(x_1, x_2) a^3(x_1, x_2) \text{ for all } x^\epsilon \in \Omega^\epsilon.$$

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where $e \in W^{2,\infty}(\omega),\ 0 < e_0 < e(x_1,x_2)$ denotes the nonuniform thickness function, and $\Phi^\epsilon(\Omega^\epsilon)$ denotes the reference configuration of the shell. Note that when $e(x_1,x_2) = 1$ then we get shell with uniform thickness $\epsilon$.

We first consider thin elastic shallow shells with variable thickness and show that the eigenvalues of the three dimensional problem are $O(\epsilon^2)$ and the corresponding scaled eigensolutions converge to the eigensolutions of the a two dimensional model. In the limit problem we note that there is an additional term (compared to the uniform thickness case) contributed by the nonuniform thickness function $e(x_1,x_2)$. We then consider flexural shells with variable thickness and show that if the space of inextensional displacement is infinite dimensional then the eigenvalues are $O(\epsilon^2)$ and the corresponding eigensolutions converge to the eigensolutions of two dimensional flexural shell model. Here again we note that the linearized change of curvature tensor $(\rho_{\alpha\beta})$ depends on the function $e(x_1,x_2)$ and is different from the one in the case of uniform thickness.

This paper is organised as follows. In section 2 we describe the three dimensional eigenvalue problem. In section 3 we consider the case of shallow shells, in section 4 we derive the a priori estimate for eigenvalues and in section 5 we study the limiting problem for shallow shells. In section 6 we consider flexural shells, in section 7 we derive a priori estimates for eigenvalues for flexural shells and in section 8 we study the limiting eigenvalue problem for flexural shells.

## 2. The Three-dimensional Problem

Throughout this paper, Latin indices vary over the set $\{1,2,3\}$ and Greek indices over the set $\{1,2\}$ for the components of vectors and tensors. The summation over repeated indices will be used.

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz continuous boundary $\gamma$ and let $\omega$ lie locally on one side of $\gamma$. For each $\epsilon > 0$, we define the sets

$$\Omega^\epsilon = \omega \times (-\epsilon,\epsilon),\ \ \Gamma^{\pm,\epsilon} = \omega \times \{\pm\epsilon\},\ \ \Gamma^\epsilon = \gamma \times (-\epsilon,\epsilon).$$

Let $x^\epsilon = (x_1,x_2,x_3^\epsilon)$ be a generic point on $\Omega^\epsilon$ and let $\partial_\alpha = \partial_\alpha^\epsilon = \frac{\partial}{\partial x_\alpha}$ and $\partial_3^\epsilon = \frac{\partial}{\partial x_3^\epsilon}$.

Let $\phi^\epsilon$ be an injective mapping of class $C^3(\omega)$ such that the two vectors

$$a_\alpha(y) = \partial_\alpha \phi^\epsilon$$

are linearly independent for all $y \in \omega$. We define

$$a^3(y) = a_3(y) = \frac{a_1 \times a_2}{|a_1 \times a_2|},$$

and

$$\left.\begin{array}{ll} a_{\alpha\beta} := a_\alpha . a_\beta & a^{\alpha\beta} := a^\alpha . a^\beta \\ b_{\alpha\beta} := a^3 . \partial_\beta a_\alpha & b_\alpha^\beta := a^{\beta\sigma} b_{\sigma\alpha} \\ \Gamma_{\alpha\beta}^\sigma := a^\sigma . \partial_\beta a_\alpha & \end{array}\right\}. \tag{2.1}$$

These verify the usual symmetry relations. We also define

$$b_\beta^\sigma|_\alpha = \partial_\alpha b_\beta^\sigma - \Gamma_{\alpha\tau}^\sigma b_\beta^\tau - \Gamma_{\beta\alpha}^\tau b_\tau^\sigma, \tag{2.2}$$

$$c_{\alpha\beta} = b_\alpha^\sigma b_{\sigma\beta}. \tag{2.3}$$

The area element along S is $\sqrt{a}dy$, where

$$a := det(a_{\alpha\beta}). \tag{2.4}$$

By the continuity of the functions defined above, there exists $a_0 > 0$ such that

$$0 < a_0 \le a(y) \text{ for all } y \in \overline{\omega}.$$

For each $\epsilon > 0$, we define the mapping $\Phi^\epsilon : \Omega^\epsilon \to \mathbb{R}^3$ by

$$\Phi^\epsilon(x^\epsilon) = \phi^\epsilon(x_1,x_2) + x_3^\epsilon e(x_1,x_2) a^3(x_1,x_2) \text{ for all } x^\epsilon \in \Omega^\epsilon, \tag{2.5}$$

where $e \in W^{2,\infty}(\omega),\ 0 < e_0 < e(x_1,x_2)$.

We define vectors $g_i^\epsilon$ and $g^{i,\epsilon}$ by the relations

$$g_i^\epsilon = \partial_i^\epsilon \Phi^\epsilon \text{ and } g^{j,\epsilon} . g_i^\epsilon = \delta_i^j,$$

which form the covariant and contravariant basis respectively of the tangent plane of the reference configuration $\Phi^\epsilon(\Omega^\epsilon)$ of the shell at $\Phi^\epsilon(x^\epsilon)$. The covariant and contravariant metric tensors are given respectively by

$$g_{ij}^\epsilon = g_i^\epsilon . g_j^\epsilon \text{ and } g^{ij,\epsilon} = g^{i,\epsilon}.g^{j,\epsilon}.$$

The Christoffel symbols are defined by

$$\Gamma_{ij}^{p,\epsilon} = g^{p,\epsilon}.\partial_j^\epsilon g_i^\epsilon.$$

The volume element is given by $\sqrt{g^\epsilon}dx^\epsilon$ where

$$g^\epsilon = det(g_{ij}^\epsilon).$$

The elasticity tensor is defined by

$$A^{ijkl,\epsilon} = \lambda g^{ij,\epsilon} g^{kl,\epsilon} + \mu(g^{ik,\epsilon} g^{kl,\epsilon} + g^{il,\epsilon} g^{jk,\epsilon}).$$

where $\lambda$ and $\mu$ are Lamé constants. This tensor satisfy the coersivity condition

$$A^{ijkl,\epsilon} t_{ij} t_{kl} \geq C|t_{ij}|^2 \ \forall \ symmetric \ tensors \ (t_{ij}). \tag{2.6}$$

We define the space

$$V(\Omega^\epsilon) = \{v^\epsilon \in (H^1(\Omega^\epsilon))^3; v^\epsilon = 0 \text{ on } \Gamma^\epsilon\}.$$

For $v^\epsilon \in V(\Omega^\epsilon)$, define

$$e_{i||j}^\epsilon(v^\epsilon) = \frac{1}{2}(\partial_i^\epsilon v_j^\epsilon + \partial_j^\epsilon v_i^\epsilon) - \Gamma_{ij}^{p,\epsilon} v_p^\epsilon. \tag{2.7}$$

Then the eigenvalue problem posed over the domain $\Omega^\epsilon$ consists of finding pairs $(\xi^\epsilon, u^\epsilon) \in R \times V(\Omega^\epsilon)$ such that

$$\int_{\Omega^\epsilon} A^{ijkl,\epsilon} e_{k||l}^\epsilon(u^\epsilon) e_{i||j}^\epsilon(v^\epsilon)\sqrt{g^\epsilon}dx^\epsilon = \xi^\epsilon \int_{\Omega^\epsilon} u^{i,\epsilon} v_i^\epsilon \sqrt{g^\epsilon} dx^\epsilon \quad \forall v^\epsilon \in V(\Omega^\epsilon). \tag{2.8}$$

By classical arguments (cf: [18]), it follows that there exists a sequence of eigenvalues

$$0 < \xi^{\epsilon,1} \leq \xi^{\epsilon,2} \leq \ldots \leq \xi^{\epsilon,l} \leq \ldots \infty \tag{2.9}$$

and a sequence of corresponding eigenfunctions $\{u^{\epsilon,l}\}$ such that

$$\int_{\Omega^\epsilon} u_i^{\epsilon,l} u_i^{\epsilon,m} \sqrt{g} dx^\epsilon = \delta_{lm}. \tag{2.10}$$

The sequence forms an orthonormal basis in the weighted space

$$L^2(g_\epsilon, \Omega^\epsilon) = \{u^\epsilon | \int_{\Omega^\epsilon} u_i^\epsilon u_i^\epsilon \sqrt{g} dx^\epsilon < \infty\}. \tag{2.11}$$

## 3. Shallow shells

**Assumption:** We first assume that the shell is a shallow shell; i.e. there exists a function $\phi \in C^3(\omega)$ such that $\phi^\epsilon = \epsilon\phi$

$$i.e; \quad \phi^\epsilon(x_1, x_2) = (x_1, x_2, \epsilon\phi(x_1, x_2)), \text{ for all } (x_1, x_2) \in \omega. \tag{3.1}$$

We set $\Omega = \omega \times (-1, 1)$, a domain independent of $\epsilon$,. With $x^\epsilon = (x_i^\epsilon) \in \Omega^\epsilon$, we associate $x \in \Omega$ by $x_\alpha = x_\alpha^\epsilon, \ \ x_3 = \frac{1}{\epsilon} x_3^\epsilon$.

Define $\Gamma^\pm = \omega \times \{\pm 1\}, \ \ \Gamma = \gamma \times (-1, 1)$ and

$$V(\Omega) = \{v \in (H^1(\Omega))^3 : v = 0 \ \ on \ \ \Gamma\}. \tag{3.2}$$

With $v^\epsilon \in V(\Omega^\epsilon)$ we associate the vector $v \in V(\Omega)$ by

$$v_\alpha(x^\epsilon) = \epsilon^2 v_\alpha(x), \ \ v_3(x^\epsilon) = \epsilon v_3(x). \tag{3.3}$$

With the eigenvalues $\xi^{\epsilon}$, we associate the eigenvalues $\xi(\epsilon)$ through the relation

$$\xi^{\epsilon,l} = \epsilon^2 \xi^l(\epsilon). \tag{3.4}$$

With the tensors $e^{\epsilon}_{i||j}$, we associate the tensors $e_{i||j}(\epsilon)$ through the relation

$$e^{\epsilon}_{i||j}(v^{\epsilon})(x^{\epsilon}) = \epsilon^2 e_{i||j}(\epsilon; v)(x). \tag{3.5}$$

Also with the tensors $A^{ijkl,\epsilon}(x^{\epsilon})$ we associate the tensors $A^{ijkl}(\epsilon)(x)$ and with the volume element $g^{\epsilon}(x^{\epsilon})$ we associate $g(\epsilon)(x)$. Then the scaled eigensolution $(\xi^l(\epsilon), u^l(\epsilon))$ satisfies

$$\int_{\Omega} A^{ijkl}(\epsilon) e_{k||l}(\epsilon; u^l(\epsilon)) e_{i||j}(\epsilon; v)) \sqrt{g(\epsilon)}\, dx = \xi^l(\epsilon) \int_{\Omega} [\epsilon^2 u^l_{\alpha}(\epsilon).v_{\alpha} + u^l_3(\epsilon) v_3] \sqrt{g(\epsilon)}\, dx \;\; \forall \;\; v \in V, \tag{3.6}$$

$$\int_{\Omega} [\epsilon^2 u^l_{\alpha}(\epsilon) u^m_{\alpha}(\epsilon) + u^l_3(\epsilon) u^m_3(\epsilon)] \sqrt{g(\epsilon)}\, dx = \delta_{lm}. \tag{3.7}$$

We henceforth denote by $C$ a generic constant which is independent of both $l$ and $\epsilon$ but whose value vary from place to place.

Based on the above scalings, it can be shown that (cf. [21])

$$\left.\begin{aligned} e_{\alpha||\beta}(\epsilon; v) &= \tilde{e}_{\alpha\beta}(v) + \epsilon^2 e^{\sharp}_{\alpha||\beta}(\epsilon; v), \\ e_{\alpha||3}(\epsilon; v) &= \tfrac{1}{\epsilon}\{\tilde{e}_{\alpha 3}(v) + \epsilon^2 e^{\sharp}_{\alpha||3}(\epsilon; v)\}, \\ e_{3||3}(\epsilon; v) &= \tfrac{1}{\epsilon^2} \tilde{e}_{33}(v) \end{aligned}\right\} \tag{3.8}$$

where

$$\left.\begin{aligned} \tilde{e}_{\alpha\beta}(v) &= \tfrac{1}{2}(\partial_{\alpha} v_{\beta} + \partial_{\beta} v_{\alpha}) - \tfrac{v_3}{e}(\partial_{\alpha\beta}\phi + x_3 \partial_{\alpha\beta} e) \\ \tilde{e}_{\alpha 3}(v) &= \tfrac{1}{2}(\partial_{\alpha} v_3 + \partial_3 v_{\alpha}), \\ \tilde{e}_{33}(v) &= \partial_3 v_3 \end{aligned}\right\} \tag{3.9}$$

Also there exists constant $C$ such that

$$\left.\begin{aligned} \sup_{0<\epsilon\le\epsilon_0} \max_{\alpha,j} ||e^{\sharp}_{\alpha,j}(\epsilon; v)||_{0,\Omega} &\le C||v||_{1,\Omega} \text{ for all } v \in V, \\ \sup_{0<\epsilon\le\epsilon_0} \max_{x\in\Omega} |g(x) - e^2| &\le C\epsilon^2, \\ \sup_{0<\epsilon\le\epsilon_0} \max_{x\in\Omega} |A^{ijkl}(\epsilon) - A^{ijkl}_0| &\le C\epsilon^2 \end{aligned}\right\} \tag{3.10}$$

where

$$\left.\begin{aligned} A^{\alpha\beta\gamma\tau}_0 &= \lambda\delta^{\alpha\beta}\delta^{\gamma\tau} + \mu(\delta^{\alpha\gamma}\delta^{\beta\tau} + \delta^{\alpha\tau}\delta^{\beta\gamma}) \\ A^{\alpha\beta\gamma 3}(0) = 0, \quad A^{\alpha\beta 33}(0) &= \tfrac{1}{e^2}\lambda\delta^{\alpha\beta}, \quad A^{\alpha 3\gamma 3}(0) = \tfrac{1}{e^2}\mu\delta^{\alpha\gamma} \\ A^{\alpha 333}(0) = 0, \quad A^{3333}(0) &= \tfrac{1}{e^4}(\lambda + 2\mu) \end{aligned}\right\} \tag{3.11}$$

$$A^{ijkl}(\epsilon) t_{kl} t_{ij} \ge C t_{ij} t_{ij} \tag{3.12}$$

for $0 < \epsilon \le \epsilon_0$ and for all symmetric tensors $(t_{ij})$,

$$||v||^2_{1,\Omega} \le C\{\Sigma_{i=1,j} ||\tilde{e}_{ij}(v)||^2_{0,\Omega}\} \;\; for \;\; all \;\; v \in V(\Omega). \tag{3.13}$$

## 4. A priori estimates

In this section we show that the sequence $\xi^l(\epsilon)$ is bounded for each positive integer $l$.

Define the Rayleigh quotient $R(\epsilon)(v)$ for $v \in V(\Omega)$ by

$$R(\epsilon)(v) = \frac{\int_\Omega A^{ijkl}(\epsilon)e_{i||j}(\epsilon,v)e_{k||l}(\epsilon,v)\sqrt{g(\epsilon)}dx}{\int_\Omega [\epsilon^2 v_\alpha^2 + v_3^2]\sqrt{g(\epsilon)}dx} \tag{4.1}$$

Then

$$\xi^l(\epsilon) = \min_{W\in V_l} \max_{v\in W-\{0\}} R(\epsilon)(v) \tag{4.2}$$

where $V_l$ is the collection of all $l$-dimensional subspaces of $V(\Omega)$.

Define

$$V_{KL}(\Omega) = \{v \in V(\Omega) : v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3,\ \ v_3 = \eta_3, \eta_i \in (H_0^1(\omega))^2 \times H^2(\omega)\}. \tag{4.3}$$

**Theorem 4.1.**
For each positive integer l, there exists a constant $C(l) > 0$ such that

$$\xi^l(\epsilon) \le C(l) \tag{4.4}$$

*Proof.* Note that

$$\xi^l(\epsilon) = \min_{W\in V_l} \max_{v\in W-\{0\}} R(\epsilon)(v). \tag{4.5}$$

Let $W_l$ denote the collection of all $l-$ dimensional subspaces of $H_0^2(\omega)$.

Let $W \in W_l$. For $\varphi \in W$, define

$$v_\varphi = (-x_1\partial_2\varphi, -x_2\partial_3\varphi, \varphi),$$

and

$$U = \{v_\varphi : \varphi \in W\}. \tag{4.6}$$

It follows that $U \in V_l$. Hence

$$\zeta^l(\epsilon) \le \min_{U\in V_l} \max_{\varphi\in W-\{0\}} R(\epsilon)(v_\varphi). \tag{4.7}$$

From the definition of $A^{ijkl}(\epsilon)$ we have

$$\int_\Omega A^{ijkl}(\epsilon)e_{i||j}(\epsilon,v_\varphi)e_{k||l}(\epsilon,v_\varphi)\sqrt{g(\epsilon)}dx \le C\Sigma_{i,j}||e_{i||j}(\epsilon,v_\varphi)||^2_{0,\Omega}. \tag{4.8}$$

Using (3.8)-(3.9) we have

$$\begin{aligned} ||e_{\alpha||\beta}(\epsilon,v_\varphi)||^2_{0,\Omega} &\le C||\tilde{e}_{\alpha\beta}(v_\varphi)||^2 + \le C\epsilon^2||e^\sharp_{\alpha||\beta}(\epsilon;v_\varphi)||^2_{0,\Omega} \\ &\le C||\triangle\varphi||^2_{0,\omega}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} ||e_{\alpha||3}(\epsilon,v_\varphi)||^2_{0,\Omega} &\le C||\tilde{e}_{\alpha 3}(v_\varphi)||^2 + \le C\epsilon^2||e^\sharp_{\alpha||3}(\epsilon;v_\varphi)||^2_{0,\Omega} \\ &\le C||\triangle\varphi||^2_{0,\omega}, \end{aligned} \tag{4.10}$$

$$||e_{3||3}(\epsilon,v_\varphi)||^2_{0,\Omega} = 0. \tag{4.11}$$

Also it follows from (3.10) and the definition of $v_\varphi$ that there exists a constant C such that

$$\int_\Omega [\epsilon^2(v_\varphi)^2_\alpha + (v_\varphi)^2_3]\sqrt{g(\epsilon)}dx \ge C\int_\Omega \varphi^2 dx. \tag{4.12}$$

Hence

$$\begin{aligned} \xi^l(\epsilon) &\le C \min_{U\in W_l} \max_{\varphi\in W-\{0\}} \frac{\int_\omega |\triangle\varphi|^2 d\omega}{\int_\omega \varphi^2 d\omega} \\ &\le C\mu^l \end{aligned} \tag{4.13}$$

where the $\mu^l$ is $l$-th eigenvalue of the two dimensional eigenvalue problem

$$\left.\begin{aligned} \triangle^2 u = \lambda u \ \ in \ \ \omega \\ u = \partial_\nu u = 0 \ \ on \ \ \partial\omega \end{aligned}\right\} \tag{4.14}$$

This completes the proof by setting $C(l) = C\mu^l$.

□

## 5. Limit Problem

**Theorem 5.1.**
For each positive integer $l$ there exists a subsequence (still indexed by $\epsilon$ for convenience) such that $(\xi^l(\epsilon), u^l(\epsilon)) \to (\xi^l, u^l)$ in $R \times V(\Omega)$. Also the limit function $u^l = \{u^l_\alpha, u^l_3\}$ is a Kirchhoff-Love displacement, that is

$$u^l_\alpha = \zeta^l_\alpha - x_3\partial_\alpha\zeta^l_3, \;\; u^l_3 = \zeta^l_3, \;\; \zeta^l_i \text{ is independent of } x_3. \tag{5.1}$$

and $\boldsymbol{\zeta}^{\mathbf{l}} = (\zeta^l_\alpha, \zeta^l_3)$ satisfies

$$\begin{aligned} &-\int_\omega m_{\alpha\beta}(\zeta^l_3)\partial_{\alpha\beta}\eta_3 e d\omega - \int_\omega [n_{\alpha\beta}(\zeta^l)\partial_{\alpha\beta}\phi + m_{\alpha\beta}(\zeta^l_3)\partial_{\alpha\beta}e]\eta_3 e d\omega \\ &+ \int_\omega n_{\alpha\beta}(\zeta^l)\partial_\alpha\eta_\beta e d\omega \;= \xi^l \int_\omega \zeta^l_3\eta_3 e d\omega \quad \forall(\eta_\alpha, \eta_3) \in (H^1_0(\omega))^2 \times H^2_0(\omega), \end{aligned} \tag{5.2}$$

where

$$m_{\alpha\beta}(\zeta_3) = -\frac{2\lambda\mu}{3(\lambda+2\mu)}(\Delta\zeta_3 + \zeta_3\frac{\Delta e}{e})\delta_{\alpha\beta} + \frac{4\mu}{3}(\partial_{\alpha\beta}\zeta_3 + \zeta_3\frac{\partial_{\alpha\beta}e}{e}), \tag{5.3}$$

$$n_{\alpha\beta}(\zeta) = \frac{2\lambda\mu}{\lambda+2\mu}\hat{e}_{\Sigma\sigma}(\zeta)\delta_{\alpha\beta} + 2\mu\hat{e}_{\alpha\beta}(\zeta) \tag{5.4}$$

where

$$\hat{e}_{\alpha\beta}(\zeta) = \frac{1}{2}(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha) - \zeta_3\frac{\partial_{\alpha\beta}\phi}{e} = \frac{1}{2}\int_{-1}^{1}\tilde{e}_{\alpha\beta}(\zeta)dx_3 \tag{5.5}$$

*Proof.* Taking $v = u^l(\epsilon)(x)$ in (3.6), we get

$$\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon; u^l(\epsilon))e_{i||j}(\epsilon; u^l(\epsilon))\sqrt{g(\epsilon)}dx = \zeta^l(\epsilon)\int_\Omega [\epsilon^2(u^l_\alpha(\epsilon))^2 + (u^l_3(\epsilon))^2]\sqrt{g(\epsilon)}dx \tag{5.6}$$

Using the Korn's inequality

$$\left\{\sum_i ||v_i||^2_{1,\Omega}\right\} \le C\left\{\sum_{i,j}||e_{i||j}(\epsilon; v)||^2_{0,\Omega}\right\} \quad \forall v \in V(\Omega), \tag{5.7}$$

the boundedness of the eigenvalues and the relations (3.7)-(3.13) we have

$$\begin{aligned} \sum_i ||u^l_i(\epsilon)||^2_{1,\Omega} &\le C\sum_{i,j}||e_{i||j}(u^l(\epsilon))||^2_{0,\Omega} \\ &\le C\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon; u^l(\epsilon))e_{i||j}(\epsilon; u^l(\epsilon))\sqrt{g(\epsilon)}dx \\ &\le C(l). \end{aligned} \tag{5.8}$$

Hence there exists a subsequence $(u^l(\epsilon))$ (still indexed by $\epsilon$)and a function $u^l \in V(\Omega)$ such that

$$u^l(\epsilon) \rightharpoonup \;u^l \;\; weak \;\; in \;\; V(\Omega), \tag{5.9}$$

$$e_{i||j}(\epsilon; u^l(\epsilon)) \rightharpoonup e_{i||j}(u^l) \;\; weak \;\; in \;\; L^2(\Omega). \tag{5.10}$$

This implies that there exists $(\zeta^l_\alpha, \zeta^l_3)$ (cf: [21]) such that

$$u^l_\alpha = \zeta^l_\alpha - x_3\partial_\alpha\zeta^l_3, \quad u^l_3 = \zeta^l_3. \tag{5.11}$$

Define

$$K_{\alpha\beta}(\epsilon) = \tilde{e}_{\alpha\beta}(u^l(\epsilon)), \;\; K_{\alpha3}(\epsilon) = \frac{1}{\epsilon}\tilde{e}_{\alpha3}(u^l(\epsilon)), \;\; K_{33}(\epsilon) = \frac{1}{\epsilon^2}\tilde{e}_{33}(u^l(\epsilon)) \tag{5.12}$$

and

$$K_{\alpha\beta} = \tilde{e}_{\alpha\beta}(u^l),\ \ K_{\alpha 3} = 0,\ \ K_{33} = -e^2 \frac{\lambda}{\lambda + 2\mu} \tilde{e}_{\alpha\alpha}(u^l). \tag{5.13}$$

Claim: $K(\epsilon) = (K_{ij}(\epsilon)) \rightharpoonup K = (K_{ij})$ weakly in $L^2(\Omega)$.

From the definition (5.12) and relation (3.8), we have

$$||K(\epsilon)||^2_{0,\Omega} \leq 2\Sigma_{i,j}||e_{i||j}(\epsilon; u(\epsilon))||^2_{0,\Omega} + 2\epsilon^4 \Sigma_{\alpha\beta}||\tilde{e}^\sharp(\epsilon; u(\epsilon))||^2_{0,\Omega} + 4\epsilon^2 \Sigma_\alpha ||\tilde{e}^\sharp(\epsilon; u(\epsilon))||^2_{0,\Omega}. \tag{5.14}$$

From the boundedness of $(e_{i||j}(\epsilon, u^l(\epsilon)))$ and the relation (3.10) it follows that $(K(\epsilon))$ is bounded and hence $K(\epsilon)$ converges weakly to some $K \in (L^2(\Omega))^9$.

From the weak convergence of $u^l(\epsilon)$ to $u^l$ and the definition of $K_{\alpha\beta}(\epsilon)$, it follows that $K_{\alpha\beta} = \tilde{e}_{\alpha\beta}(u^l)$.

We next note the following result

$$\int_\Omega u\partial_3 v dx = 0\ \ for\ \ all\ \ v \in H^1(\Omega)\ \ with\ \ v = 0\ \ on\ \ \Gamma \Rightarrow u = 0.$$

Multiplying (3.6) by $\epsilon$, taking $v_3 = 0$ and letting $\epsilon \to 0$ and using the relations (3.8)-(3.11) we get

$$\int_\Omega K_{\alpha 3}\partial_3 v_\alpha e dx = 0\ \ for\ \ all\ \ v_\alpha. \tag{5.15}$$

Hence by the above result $K_{\alpha 3} = 0$.

Multiplying (3.6) by $\epsilon^2$, letting $v_\alpha = 0$ and passing to the limit as $\epsilon \to 0$ we get

$$\int_\Omega \{\frac{\lambda}{e^2} K_{\sigma\sigma} + \frac{(\lambda + 2\mu)}{e^4} K_{33}\} e dx = 0\ \ \ \forall v_3. \tag{5.16}$$

Hence $K_{33} = -e^2 \frac{\lambda}{\lambda+2\mu} \tilde{e}_{\alpha\alpha}(u)$.

Using the relation (5.13) and (3.8)-(3.11) it follows that for $v = (\eta_\alpha - x_3\partial_\alpha\eta_3, \eta_3) \in V_{KL}(\Omega)$

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\epsilon; u^l(\epsilon)) e_{i||j}(\epsilon)(v)\sqrt{g(\epsilon)} dx \\
&\to -\int_\omega m_{\alpha\beta}(\zeta^l_3)\partial_{\alpha\beta}\eta_3 e d\omega - \int_\omega [n_{\alpha\beta}(\zeta^l)\partial_{\alpha\beta}\phi\eta_3 + m_{\alpha\beta}(\zeta^l_3)\partial_{\alpha\beta}e]\eta_3 e d\omega + \int_\omega n_{\alpha\beta}(\zeta^l)\partial_\alpha\eta_\beta e d\omega.
\end{aligned} \tag{5.17}$$

as $\epsilon \to 0$.

Hence passing to the limit in (3.6) by taking $v = (\eta_\alpha - x_3\partial_\alpha\eta_3, \eta_3) \in V_{KL}(\Omega)$, we get

$$\begin{aligned}
-\int_\omega m_{\alpha\beta}(\zeta^l_3)\partial_{\alpha\beta}\eta_3 e d\omega\ \ &-\ \ \int_\omega [n_{\alpha\beta}(\zeta^l)\partial_{\alpha\beta}\phi + m_{\alpha\beta}(\zeta^l_3)\partial_{\alpha\beta}e]\eta_3 e d\omega + \int_\omega n_{\alpha\beta}(\zeta^l)\partial_\alpha\eta_\beta e d\omega \\
&=\ \ \xi^l \int_\omega \zeta^l_3 \eta_3 e d\omega\ \ \ \forall\ (\eta_i) \in (H^1_0(\omega))^2 \times H^2_0(\omega).
\end{aligned} \tag{5.18}$$

.

This completes the proof. □

Taking $\eta_3 = 0$ in (5.2), we have

$$\int_\omega n_{\alpha\beta}(\zeta^l)\partial_\beta\eta_\alpha\ e d\omega = 0\ \ \ \forall\ (\eta_\alpha) \in (H^1_0(\omega))^2. \tag{5.19}$$

Denoting $\tilde{\zeta} = (\zeta_\alpha)$ and $\tilde{\eta} = (\eta_\alpha)$ the above equation can be written as

$$\int_\omega \left[\frac{2\lambda\mu}{\lambda + 2\mu} e_{\alpha\alpha}(\tilde{\zeta}) e_{\sigma\sigma}(\tilde{\eta}) + 2\mu e_{\alpha\beta}(\tilde{\zeta}) e_{\alpha\beta}(\tilde{\eta})\right]\ e d\omega = < g_\alpha, \eta_\alpha > \tag{5.20}$$

where

$$< g_\alpha, \eta_\alpha > = \int_\omega p_{\alpha\beta}(\zeta_3)\partial_\beta\eta_\alpha e d\omega,$$

$$p_{\alpha\beta}(\zeta_3) = \frac{2\lambda\mu}{\lambda+2\mu}(\frac{\zeta_3}{e}\partial_{\sigma\sigma}\phi)\delta_{\alpha\beta} + \mu(\frac{\zeta_3}{e}\partial_{\alpha\beta}\phi).$$

Clearly the bilinear form associated with left hand side of (5.20) is symmetric and $(H_0^1(\omega))^2-$ elliptic and hence, by the Lax-Milgram lemma, given $(g_\alpha) \in (H^{-1})^2$, there exists a unique $\tilde{\zeta} \in (H_0^1(\omega))^2$ satisfying (5.20).

Thus, given $\zeta_3 \in H_0^2(\omega)$, (implies $p_{\alpha\beta}(\zeta_3) \in L^2(\omega)$) we denote by $T\zeta_3 \in (H_0^1(\omega))^2 \times H_0^2(\omega)$ the vector $(\zeta_\alpha, \zeta_3)$ where $(\zeta_\alpha) \in (H_0^1(\omega))^2$ is the solution of (5.20).

Then the limit problem (5.2) becomes

$$b(\zeta_3^l, \eta_3) = \xi^l \int_\omega \zeta_3^l \eta_3 \; ed\omega \;\; \forall\, \eta_3 \in H_0^2(\omega), \tag{5.21}$$

where

$$b(\zeta_3, \eta_3) = -\int_\omega m_{\alpha\beta}(\zeta_3)\partial_{\alpha\beta}\eta_3 \; ed\omega - \int_\omega n_{\alpha\beta}(T\zeta_3)\partial_{\alpha\beta}\phi\eta_3 \; ed\omega - \int_\omega m_{\alpha\beta}(\zeta_3)(\partial_{\alpha\beta}e)\eta_3 \; ed\omega \tag{5.22}$$

which depends only on the third unknown $\zeta_3^l$.

The ellipticity of the bilinear operatoe $b(.,.)$ follows as in [21]. Thus by Lax-Milgram theorem there exists a sequence of eigensolutions for the limit problem.

**Theorem 5.2.**
Let $\xi^l(\epsilon) \to \xi^l$ and $u^l(\epsilon) \to u^l$ in $V(\Omega)$. Then $\xi^l$ is the $l-th$ eigenvalue of the limit problem (5.21) and $\{u_3^l\}$ forms an orthonormal basis for $L^2(\omega)$. Thus all the eigensolutions of the limit problem are obtained as limits of $\{\xi^l(\epsilon), u^l(\epsilon)\}$.

*Proof.* . Proof is similar to the proof of theorem 6.1 in [17]. □

## 6. Flexural Shells

For all $\eta = (\eta_i) \in (H^1(\omega))^2 \times H^2(\omega)$, let

$$\gamma_{\alpha\beta}(\eta) = \frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - \frac{1}{e}b_{\alpha\beta}\eta_3. \tag{6.1}$$

Define the space of inextensional displacements by

$$V_F(\omega) = \{\eta = (\eta_i) \in (H^1(\omega))^2 \times H^2(\omega) | \eta_i = \partial_\nu\eta_3 = 0 \text{ on } \gamma, \gamma_{\alpha\beta}(\eta) = 0 \text{ in } \omega\}. \tag{6.2}$$

**Assumption**: We assume henceforth that $V_F(\omega) \neq 0$.

In this case we make the following assumptions on the unknowns.

$$u^\epsilon = u(\epsilon), \;\; \xi^\epsilon = \epsilon^2\xi(\epsilon). \tag{6.3}$$

Then the eigenvalue problem (3.6) becomes: find $(u(\epsilon), \xi(\epsilon)) \in V(\Omega) \times R$ such that

$$\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon, u(\epsilon))e_{i||j}(\epsilon, v)\sqrt{g(\epsilon)}dx = \epsilon^2\xi(\epsilon)\int_\Omega u_i(\epsilon).v_i\sqrt{g(\epsilon)}dx \;\;\; \forall v \in V. \tag{6.4}$$

For $v \in V(\Omega)$, define

$$\begin{aligned}\rho_{\alpha\beta}(v) \;&=\; v_{3|\alpha\beta} + eb^\sigma_\alpha v_{\sigma|\beta} + eb^\sigma_\beta v_{\sigma|\alpha} + eb^\sigma_{\alpha|\beta}v_\sigma - (c_{\alpha\beta} + \frac{1}{e}e_{|\alpha\beta})v_3 \\ &+\; \frac{2}{e^2}\partial_\alpha e\partial_\beta e v_3 - \frac{1}{e}\partial_\sigma e\partial_\beta v_3 - \frac{1}{e}\partial_\beta e\partial_\alpha v_3\end{aligned} \tag{6.5}$$

where

$$v_{\alpha|\beta} = \partial_\alpha v_\beta - \Gamma^\rho_{\sigma\alpha}v_\rho, \;\;\; v_{|\alpha\beta} = \partial_{\alpha\beta}v - \Gamma^\rho_{\alpha\beta}\partial_\rho v. \tag{6.6}$$

$$\begin{aligned} e^1_{\alpha||\beta}(\epsilon)(v) &= \frac{1}{2\epsilon}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \frac{1}{\epsilon}\Gamma^\sigma_{\alpha\beta} v_\sigma - \frac{1}{\epsilon e} b_{\alpha\beta} v_3 \\ &+ x_3(eb^\sigma_{\beta|\alpha} + \partial_\beta e b^\sigma_\alpha + \partial_\alpha e b^\sigma_3) v_\sigma + x_3(c_{\alpha\beta} - \frac{1}{e} e_{|\alpha\beta}) v_3. \end{aligned} \tag{6.7}$$

Then we have the following inequalities(cf. lemma 5.1 in [2])

$$||\Gamma^\sigma_{\alpha\beta}(\epsilon) - \Gamma^\sigma_{\alpha\beta}||_{0,\infty} + ||\Gamma^3_{\alpha\beta}(\epsilon) - \frac{1}{e} b_{\alpha\beta}||_{0,\infty} + ||\Gamma^\sigma_{\alpha 3}(\epsilon) + eb^\sigma_\alpha||_{0,\infty} + ||\Gamma^3_{\alpha 3}(\epsilon) - \frac{1}{e}\partial_\alpha e||_{0,\infty} \leq C\epsilon, \tag{6.8}$$

$$||g(\epsilon) - e^2 a||_{0,\infty} \leq C\epsilon, \tag{6.9}$$

$$||A^{ijkl}(\epsilon) - A^{ijkl}(0)||_{0,\infty} \leq C\epsilon, \tag{6.10}$$

with

$$A^{\alpha\beta\sigma\tau}(0) = \lambda a^{\alpha\beta} a^{\sigma\tau} + \mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma},\ A^{\alpha\beta\sigma 3}(0) = 0,$$

$$A^{\alpha\beta 33}(0) = \tfrac{1}{e^2}\lambda a^{\alpha\beta},\ \ A^{\alpha 3\sigma 3}(0) = \tfrac{1}{e^2}\mu a^{\alpha\sigma},$$

$$\left.\begin{aligned} ||\Gamma^\sigma_{\alpha\beta}(\epsilon) - \{\Gamma^\sigma_{\alpha\beta} + \epsilon x_3[-eb^\sigma_\beta|\alpha) - \partial_\beta e b^\sigma_\alpha - \partial_\alpha e b^\sigma_\beta]\}||_{0,\infty,\Omega} \leq C\epsilon^2, \\ ||\Gamma^3_{\alpha\beta}(\epsilon) - \{\tfrac{1}{e} b_{\alpha\beta} + \epsilon x_3[\tfrac{1}{e} e_{|\alpha\beta} - c_{\alpha\beta}]\}||_{0,\infty,\Omega} \leq C\epsilon^2, \\ ||\Gamma^\sigma_{\alpha 3}(\epsilon) - \{eb^\sigma_\alpha - \epsilon x_3 e^2 b^\sigma_\tau b^\tau_\alpha\}||_{0,\infty,\Omega} \leq C\epsilon^2, \\ ||\Gamma^3_{\alpha 3}(\epsilon) - \{\tfrac{1}{e}\partial_\alpha e + \epsilon x_3 \partial_\beta e b^\beta_\alpha\}||_{0,\infty,\Omega} \leq C\epsilon^2 \end{aligned}\right\} \tag{6.11}$$

$$||\frac{1}{\epsilon} e_{\alpha||\beta}(\epsilon)(v) - e^1_{\alpha||\beta}(\epsilon)(v)||_{0,\Omega} \leq C\epsilon \sum_i ||v_i||_{0,\Omega}, \tag{6.12}$$

$$||\frac{1}{\epsilon}\partial_3 e_{\alpha||\beta}(\epsilon)(v) + \rho_{\alpha\beta}(v)||_{-1,\Omega} \leq C\{\Sigma_i ||e_{i||3}(\epsilon)(v)||_{0,\Omega} + \epsilon\Sigma_\alpha ||v_\alpha||_{0,\Omega} + \epsilon||v_3||_{1,\Omega}\}, \tag{6.13}$$

$$||\eta||^2_{V_F} \leq C||\rho_{\alpha\beta}||^2_0 \ \ \forall \eta \in V_F(\omega). \tag{6.14}$$

## 7. Apriori estimates

Note that in the case of flexural shells the $l$-th eigenvalue is characterized by

$$\xi^l(\epsilon) = \min_{W\in V_l} \max_{v \in W - \{0\}} \tilde{R}(\epsilon)(v) \tag{7.1}$$

where $V_l$ is the collection of all $l$-dimensional subspaces of $V(\Omega)$ and

$$\tilde{R}(\epsilon)(v) = \frac{\int_\Omega A^{ijkl}(\epsilon) e_{i||j}(\epsilon, v) e_{k||l}(\epsilon, v)\sqrt{g(\epsilon)}dx}{\epsilon^2 \int_\Omega v_i . v_i \sqrt{g(\epsilon)} dx} \tag{7.2}$$

**Theorem 7.1.**
Assume that $V_F(\omega)$ is an infinite dimensional subspace of $V(\Omega)$. Then for each $l \geq 1$, the sequence $\xi^l(\epsilon)$ is bounded uniformly with respect to $\epsilon$.

*Proof.* Let $\eta = (\eta_i) \in V_F(\omega)$. Define $v_\epsilon(\eta) \in V(\Omega)$ by

$$(v_\epsilon(\eta))_\alpha = \eta_\alpha - \epsilon x_3(\partial_\alpha \eta_3 + 2eb^\sigma_\alpha \eta_\sigma - \frac{2}{e}\partial_\alpha e\eta_3), \quad (7.3)$$

$$(v_\epsilon(\eta))_3 = \eta_3. \quad (7.4)$$

Let

$$\theta_\alpha = \partial_\alpha \eta_3 + 2b^\sigma_\alpha \eta_\sigma. \quad (7.5)$$

Since $\gamma_{\alpha\beta}(\eta) = 0$ we have

$$\begin{aligned} e^1_{\alpha||\beta}(\epsilon)(v(\epsilon)) &= -x_3\{\frac{1}{2}(\partial_\alpha\theta_\beta) - \Gamma^\sigma_{\alpha\beta}\theta_\sigma - (eb^\sigma_{\beta|\alpha} + \partial_\beta eb^{\sigma\alpha} + \partial_\alpha eb^\sigma_\beta)\eta_\sigma - (c_{\alpha\beta} - \frac{1}{e}e_{|\alpha\beta})\eta_3 \\ &\quad -\epsilon x_3^2(eb^\sigma_{\beta|\alpha} + \partial_\beta eb^\sigma_\alpha + \partial_\alpha eb^\sigma_\beta)\theta_\sigma \end{aligned} \quad (7.6)$$

and

$$\begin{aligned} \frac{1}{2}(\partial_\alpha\theta_\beta) &- \Gamma^\sigma_{\alpha\beta}\theta_\sigma - (eb^\sigma_{\beta|\alpha} + \partial_\beta eb^{\sigma\alpha} + \partial_\alpha eb^\sigma_\beta)\eta_\sigma - (c_{\alpha\beta} - \frac{1}{e}e_{|\alpha\beta})\eta_3 \\ &= \partial_{\alpha\beta}\eta_3 + eb^\sigma_\alpha\partial_\beta\eta_\alpha + eb^\sigma_\beta\partial_\beta\eta_\alpha + e(\partial_\beta b^\sigma_\alpha + \partial_\alpha b^\sigma_\beta - 2\Gamma^\tau_{\alpha\beta}b^\sigma_\tau)\eta_\sigma \\ &- \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 + \frac{2}{e}\Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 - eb^\sigma_{\beta|\alpha}\eta_\sigma - c_{\alpha\beta}\eta_3 + \frac{1}{e}e_{|\alpha\beta}\eta_3 \\ &+ \frac{2}{e^2}\partial_\alpha e\partial_\beta\eta_3 - \frac{2}{e}\partial_{\alpha\beta}e\eta_3\frac{1}{e}\partial_\alpha e\partial_\beta e\eta_3 - \frac{1}{e}\partial_b etae\partial_\alpha\eta_3 \\ &= \eta_{3|\alpha\beta} + eb^\sigma_\alpha\eta_{\sigma|\beta} + eb^\sigma_\beta\eta_{\sigma|\alpha} + eb^\sigma_{\alpha|\beta}\eta_\sigma - (c_{\alpha\beta} + \frac{1}{e}e_{|\alpha\beta})\eta_3 \\ &+ \frac{2}{e^2}\partial_\alpha e\partial_\beta e\eta_3 - \frac{1}{e}\partial_\sigma e\partial_\beta\eta_3 - \frac{1}{e}\partial_\beta e\partial_\alpha\eta_3 \\ &= \rho_{\alpha\beta}(\eta) \end{aligned} \quad (7.7)$$

Thus

$$e^1_{\alpha||\beta}(\epsilon)(v(\epsilon)) = -x_3\rho_{\alpha\beta}(\eta) - \epsilon x_3^2(eb^\sigma_{\beta|\alpha} + \partial_\beta eb^\sigma_\alpha + \partial_\alpha eb^\sigma_\beta)\theta_\sigma \quad (7.8)$$

Hence from (6.12) it follows that

$$\epsilon^{-1}e_{\alpha||\beta}(\epsilon)(v_\epsilon(\eta)) \to -x_3\rho_{\alpha\beta}(\eta) \text{ in } L^2(\Omega) \text{ as } \epsilon \to 0, \quad (7.9)$$

Also

$$\epsilon^{-1}e_{\alpha||3}(\epsilon)(v(\epsilon)) = \frac{1}{\epsilon}[(\Gamma^\sigma_{\alpha3}(\epsilon) + \epsilon b^\sigma_\alpha)\eta_\alpha + (\Gamma^3_{\alpha3}(\epsilon) - \frac{1}{e}\partial_\alpha e)\eta_3 - \epsilon x_3\Gamma^\sigma_{\alpha3}(\epsilon)\theta_\sigma]. \quad (7.10)$$

Using the relations (6.8)-(6.11) we have

$$||\frac{1}{\epsilon}e_{\alpha||3}(\epsilon)(v(\epsilon))||^2_{0,\Omega} \le C(||\eta_\alpha||^2_{0,\omega} + ||\eta_3||^2_{1,\omega}) \quad (7.11)$$

Clearly

$$u_{3||3}(\epsilon)(v(\epsilon)) = 0 \quad (7.12)$$

Let $\mathscr{W}_l$ denote the collection of all $l$-dimensional subspaces of $V_F(\omega)$.

The map $A_\epsilon : V_F(\omega) \to V(\Omega)$ defined by

$$A_\epsilon(\eta) = v_\epsilon(\eta). \quad (7.13)$$

is one-one for sufficiently small $\epsilon$. Thus if $W \in \mathscr{W}_t$, then $A_\epsilon(W) \in \mathscr{V}_l$. Consequently, we have

$$\xi^l(\epsilon) \le \min_{W\in\mathscr{W}_l}\max_{\eta\in W\setminus\{0\}} R_\epsilon(v_\epsilon(\eta)). \quad (7.14)$$

We have,

$$\begin{aligned} \int_\Omega (v_\epsilon(\eta))_i\sqrt{g(\epsilon)}dx &\ge g_0\int_\Omega (v_\epsilon(\eta))_i(v_\epsilon(\eta))_i dx \quad (7.15) \\ &= 2g_0\int_\omega \eta_3^2 d\omega + g_0\sum_\alpha\int_\Omega[\eta_\alpha - \epsilon x_3(\partial_\alpha\eta_3 + 2eb^\sigma_\alpha\eta_\sigma - \frac{2}{e}\partial_\alpha e\eta_3)]^2 dx \\ &\ge 2g_0\int_\omega \eta_i\eta_i d\omega. \quad (7.16) \end{aligned}$$

for sufficienlty smal $\epsilon$. Using the symmetries of $A^{ijkl}(\epsilon)$, the fact that $A^{\alpha\beta\sigma 3}(\epsilon) = A^{\alpha 333}(\epsilon) = 0$, we have

$$\begin{aligned}\frac{1}{\epsilon^2}\int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\epsilon)(v_\epsilon(\eta)) e_{i||j}(\epsilon)(v_\epsilon(\eta))\sqrt{g(\epsilon)}dx \\ = \left\{\int_\Omega A^{\alpha\beta\sigma\tau}(\epsilon)\left[\frac{1}{\epsilon}e_{\sigma\tau}(\epsilon)(v_\epsilon(\eta))\right]\left[\frac{1}{\epsilon}e_{\alpha||\beta}(\epsilon)(v_\epsilon(\eta))\right]\sqrt{g(\epsilon)}dx\right. \\ + \left. 4\int_\Omega A^{\alpha 3\sigma 3}(\epsilon)\left[\frac{1}{\epsilon}e_{\sigma||3}(\epsilon)(v_\epsilon(\eta))\right]\left[\frac{1}{\epsilon}e_{\alpha||\beta}(\epsilon)(v_\epsilon(\eta))\right]\sqrt{g(\epsilon)}dx\right\}. \end{aligned} \tag{7.17}$$

By virtue of relations (7.9)-(7.12) above and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\frac{1}{\epsilon^2}\int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\epsilon)(v_\epsilon(\eta)) e_{i||j}(\epsilon)(v_\epsilon(\eta))\sqrt{g(\epsilon)}dx \\ &\leq C\left[\sum_{\alpha,\beta}||\rho\alpha\beta(\eta)||_{0,\omega} + \epsilon\left(\sum_\alpha||\eta_\alpha||_{0,\omega} + ||\eta_3||_{1,\omega}\right)\right]^2 + \left(\sum_\alpha||\eta_\alpha||_{0,\omega} + ||\eta_3||_{1,\omega}\right)^2 \\ &\leq C\left[\sum_{\alpha,\beta}||\rho\alpha\beta(\eta)||^2_{0,\omega} + \sum_\alpha||\eta_\alpha||^2_{0,\omega} + ||\eta_3||^2_{1,\omega}\right] \\ &\leq C\left[\sum_{\alpha,\beta}||\rho_{\alpha\beta}(\eta)||^2_{0,\omega}\right]\end{aligned} \tag{7.18}$$

Hence

$$R_\epsilon(v_\epsilon(\eta)) \leq C\frac{\sum_{\alpha,\beta}||\rho_{\alpha\beta}(\eta)||^2_{0,\omega}}{\sum_i||\eta_i||^2_{0,\omega}} \leq C\frac{\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\alpha\beta}(\eta)\rho_{\sigma\tau}(\eta)d\omega}{\int_\omega \eta_i^2 d\omega} \tag{7.19}$$

where

$$a^{\alpha\beta\sigma\tau} = \frac{4\tau\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}\alpha^{\alpha\sigma}). \tag{7.20}$$

Thus $\xi^l(\epsilon) \leq C\kappa^l$, where $\kappa^l$ is the $l-th$ eigenvalue of the two dimensional problem: find $(\zeta,\kappa)$ such that

$$\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\alpha\beta}(\zeta)\rho_{\sigma\tau}(\eta)d\omega = \kappa\int_\omega \zeta\eta d\omega, \ \ \forall\ \eta \in V_F(\omega) \tag{7.21}$$

□

## 8. Limit Problem

**Theorem 8.1.**
a) For each positive integers there exists a subsequence (still indexed by $\epsilon$) such that

$$u^l(\epsilon) \to u^l \text{ in } H^1(\Omega), \quad \xi^l(\epsilon) \to \xi^l \tag{8.1}$$

b) $u^l$ is independent of $x_3$, $\overline{u}^l = \frac{1}{2}\int_{-1}^1 u^l dx_3 \in V_F(\omega)$ and $(\overline{u})$ solves the two-dimensional eigenvalue problem for flexural shell with variable thickness, viz; find $(\zeta^l,\xi^l) \in V_F(\omega)\backslash\{0\} \times R$ such that

$$\frac{1}{6}\int_\omega A^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\zeta^l)\rho_{\alpha\beta}(\eta)e\sqrt{a}dy = \xi^l\int_\omega \zeta_i\eta_i e\sqrt{a}dy \text{ for all } \eta = \eta_i \in V_F(\omega). \tag{8.2}$$

*Proof.* Letting $v = u^l(\epsilon)$ in (6.4), we have

$$\begin{aligned}\frac{1}{\epsilon^2}\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon,u^l(\epsilon))e_{i||j}(\epsilon,u^l(\epsilon)\sqrt{g(\epsilon)}dx &= \xi^l(\epsilon)\int_\Omega u_i^l(\epsilon).u_i^l(\epsilon)\sqrt{g(\epsilon)}dx \\ &= \xi^l(\epsilon).\end{aligned} \tag{8.3}$$

Using the generalized Korn's inequality [2]

$$\sum_i||u_i||^2 \leq C\sum_{i,j}||\frac{1}{\epsilon}e_{i||j}(\epsilon,u)||^2 \ \ \forall u \in V(\Omega) \tag{8.4}$$

the coersivity relation (2.6) and the boundedness of $\xi^l(\epsilon)$ it follows that there exists a constant $C$ such that

$$||u^l(\epsilon)||^2_{1,\Omega} \le C\sum_{i,j}||\frac{1}{\epsilon}e_{i||j}(\epsilon,u^l(\epsilon))||^2 \le C \;\; \forall u \in V(\Omega). \tag{8.5}$$

Consequently there exists a subsequence such that $u^l(\epsilon) \rightharpoonup u^l$ weakly in $H^1(\Omega)$ (hence strongly in $L^2(\Omega)$) and $\frac{1}{\epsilon}e_{i||j}(\epsilon)(u^l(\epsilon)) \rightharpoonup e^{1,l}_{i||j}$ weakly in $L^2(\Omega)$. Hence it follows from Lemma (5.2) in [2] that $u^l$ is independent of $x_3$, $\gamma_{\alpha\beta}(u^l) = 0, i.e. \overline{u^l} \in V_F(\omega)$ and

$$-\partial_3 e^{1,l}_{\alpha||\beta} = \rho_{\alpha\beta}(u^l). \tag{8.6}$$

Claim: The limit functions $e^{1,l}_{\alpha||3}$ and $e^{1,l}_{3||3}$ are related to the limit function $u^l$ by

$$e^{1,l}_{\alpha||3} = 0, \tag{8.7}$$

$$e^1_{3||3} = \frac{\lambda}{\lambda+\mu}e^2 a^{\alpha\beta}e^1_{\alpha||\beta}. \tag{8.8}$$

Let $v = (v_i)$ be an arbitrary function in the space $V(\Omega)$. Then

$$\epsilon e_{\alpha||\beta}(\epsilon)(v) \to 0 \text{ in } L^2(\Omega), \tag{8.9}$$

$$\epsilon e_{\alpha||3}(\epsilon)(v) \to \frac{1}{2}\partial_3 v_\alpha \text{ in } L^2(\Omega), \tag{8.10}$$

$$\epsilon e_{3||3}(\epsilon)(v) = \partial_3 v_3 \text{ for all } \epsilon > 0. \tag{8.11}$$

Keep $v \in V(\Omega)$ fixed in (6.4) and letting $\epsilon \to 0$ and using the above relations we obtain

$$\int_\Omega \{\frac{2}{e^2}\mu a^{\alpha\sigma}e^1_{\sigma||3}\partial_3 v_\alpha + [\frac{\lambda}{e^2}a^{\sigma\tau}e^1_{\sigma||\tau} + \frac{(\lambda+2\mu)}{e^4}e^1_{3||3}]\partial_3 v_3\}e\sqrt{a}dx = 0 \;\; \forall \;\; v \in V(\Omega). \tag{8.12}$$

Letting $v$ vary in $V(\Omega)$ gives relations (8.7)-(8.8).

Taking $v$ in equation (6.4) of the form $v = v_\epsilon(\eta)$, $\eta \in V_F(\omega)$ where

$$(v_\epsilon(\eta))_\alpha = \eta_\alpha - \epsilon x_3(\partial_\alpha\eta_3 + 2eb^\sigma_\alpha\eta_\sigma - \frac{2}{e}\partial_\alpha e\eta_3), \tag{8.13}$$

$$(v_\epsilon(\eta))_3 = \eta_3. \tag{8.14}$$

and passing to the limit as $\epsilon \to 0$, taking into account of the relations (8.6) to (8.8) it follows that $\overline{u}^l$ satisfies

$$\frac{1}{6}\int_\omega A^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\overline{u^l})\rho_{\alpha\beta}(\eta)e\sqrt{a}dy = \xi^l\int_\omega u^l_i.\eta_i e\sqrt{a}dy \text{ for all } \eta \in V_F(\omega). \tag{8.15}$$

The strong convergence of $(u^l(\epsilon))$ to $u^l$ in $(H^1(\Omega))^3$ and that these limits are the only eigenvalues of the two dimensional problem can be proved as in theorem (6.1) in [15].

□

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