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A decomposition of open sets through operator interior

Research Article

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Abstract: The main idea of this article is to define an operator interior and then to establish a new classes of open sets which will be denoted by ψ. Besides, it shows some of their properties and proves some relations between other types of sets.

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1. Introduction

Elez and Papaz in 2013 [4] defined the concept of operator A^{φ} and showed some of its properties, indeed many authors have been interested in study and define open sets through either operator closure or operator interior. Recently in 2013, Al-Omari and Noiri [1] defined the local closure functions and found out some relations between some classes of sets. In this paper, motivated by the authors mentioned above, we defined an operator interior denoted by ψ, furthermore we showed that ψ induces a topology finer than τ, besides we defined some open sets with the notion of operator interior ψ and we proved some directly relations with semi-open [8], pre-open [5], b-open [2], $^{\star}b$-open [6], α-open[9] and β-open [3] sets.

Throughout this paper the terms (X,τ) and (X,τ,ψ) are a topological spaces on which no separation axioms are assumed, unless otherwise be mentioned. Besides, we sometimes write X instead of (X,τ) or (X,τ,ψ). On the other hand, for any subset A of X, $Int(A)$ and $Cl(A)$ represent the interior and closure of A respectively.

2. Operator interior ψ

In this section, we introduce the operator interior ψ and show some of its properties.

Definition 2.1.
[4]Let (X,τ) be a topological space, the operator φ on X is defined by $A^{\varphi} = A - Int(A)$.

Theorem 2.1.
Let (X,τ) be a topological space and $A, B \subseteq X$. If the following conditions hold for an operator φ on X:

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1. $\emptyset^{\varphi} = \emptyset$.
2. $X^{\varphi} = \emptyset$.
3. $A^{\varphi} \subseteq A$.
4. $(A \cap B)^{\varphi} = (A^{\varphi} \cap B) \cup (A \cap B^{\varphi})$.
5. $(A^{\varphi})^{\varphi} = A^{\varphi}$.

Then, we define the operator interior ψ as $Int(A) \subseteq A - A^{\varphi}$ for every subset A in X.

Proof. The conditions (1), (2) and (3) are trivially.

$$(4)\ (A\cap B)^{\varphi} = (A\cap B) - Int(A\cap B) = (A\cap B)\cap(X-(Int(A)\cap Int(B))) =$$
$$(A\cap(X-Int(A))\cap B)\cup(A\cap B\cap(X-Int(B)))) = (A^{\varphi}\cap B)\cup(A\cap B^{\varphi}).$$

$$(5)\ (A^{\varphi})^{\varphi} = (A-Int(A)) - Int((A-Int(A))) = A - Int(A) = A^{\varphi}$$

Now. let the operator ψ on X satisfies the above conditions. For the operator interior defined by $Int(A) \subseteq A - A^{\varphi} = A - (A - Int(A))$, we have:

1. $Int(X) = X - (X - Int(X)) = X - \emptyset = X \subseteq X$.
2. $Int(\emptyset) = \emptyset - (\emptyset - Int(\emptyset)) = \emptyset$.
3. $Int(A\cap B) = (A\cap B) - (A\cap B)^{\varphi} = (A\cap B) - ((A^{\varphi}\cap B)\cup(A\cap B^{\varphi})) = (A - A^{\varphi})\cap(B\cap B^{\varphi}) = Int(A)\cap Int(B)$.
4. $Int(Int(A)) = (A - A^{\varphi}) - (A - A^{\varphi})^{\varphi} \supseteq$
   $(A - A^{\varphi}) - (A^{\varphi} \cap (X - A^{\varphi})^{\varphi} \supseteq$
   $(A - A^{\varphi}) - (A^{\varphi} \cap (X - A^{\varphi})) = Int(A)$

□

Theorem 2.2.
Let X be any set and ψ be the operator interior, then ψ induces a topology denoted by $\tau^{\psi} = \{Int(A) \subseteq A - A^{\varphi} : A \in X\}$.

Proof. By the Theorem 2.1, X and $\emptyset$ are in τ^{ψ}.
Now, let $\{A_\delta : \delta \in \Delta\}$ be a collection of elements of X, then

$$\bigcup_{\delta\in\Delta} A_\delta \subseteq \bigcup_{\delta\in\Delta} Int(A_\delta) \subseteq \bigcup_{\delta\in\Delta}(A_\delta - A_\delta^{\varphi})$$

$$\bigcup_{\delta\in\Delta} A_\delta \subseteq \bigcup_{\delta\in\Delta} Int(A_\delta) \subseteq \bigcup_{\delta\in\Delta}(A_\delta - (A_\delta - Int(A_\delta))$$

$$\bigcup_{\delta\in\Delta} A_\delta \subseteq \bigcup_{\delta\in\Delta} Int(A_\delta) \subseteq \bigcup_{\delta\in\Delta}(A_\delta \cap Cl(A_\delta \cap Cl(Int(A_\delta)))$$

$$\bigcup_{\delta\in\Delta} A_\delta \subseteq Int(\bigcup_{\delta\in\Delta}(A_\delta) \subset (\bigcup_{\delta\in\Delta}(A_\delta)\cap Cl(\bigcup_{\delta\in\Delta}(A_\delta)\cap Cl(Int(\bigcup_{\delta\in\Delta}(A_\delta))))$$

Thus, $\bigcup_{\delta\in\Delta} A_\delta \in \tau^{\psi}$.
Now, let $\{A_i : i = 1, 2, ..., n; n \in \mathbb{N}\}$ be a finite collection of elements of X, then

$$\bigcap_{i=1}^{n} A_i \subseteq \bigcap_{i=1}^{n} Int(A_i) \subseteq \bigcap_{i=1}^{n}(A_i - A_i^{\varphi})$$

$$\bigcap_{i=1}^{n} A_i \subseteq \bigcap_{i=1}^{n} Int(A_i) \subseteq \bigcap_{i=1}^{n}(A_i - (A_i - Int(A_i))$$

$$\bigcap_{i=1}^{n} A_i \subseteq \bigcap_{i=1}^{n} Int(A_i) \subseteq \bigcap_{i=1}^{n}(A_i \cap Cl(A_i \cap Cl(Int(A_i)))$$

$$\bigcap_{i=1}^{n} A_i \subseteq Int(\bigcap_{i=1}^{n}(A_i) \subseteq (\bigcap_{i=1}^{n}(A_i)\cap Cl(\bigcap_{i=1}^{n}(A_i)\cap Cl(Int(\bigcap_{i=1}^{n}(A_i))))$$

Hence, $\bigcap_{i=1}^{n} A_i \in \tau^{\psi}$.
Therefore, τ^{ψ} is a topology on X and the term (X,τ^{ψ}) is a topological space. □

Definition 2.2.
If A is an subset of τ^{ψ}. Then A is said to be ψ-open and its complement is called ψ-closed. The collection of all ψ-open sets is denoted by $\psi O(X)$ and the collection of all ψ-closed sets is denoted by $\psi C(X)$.

Definition 2.3.
Let (X,τ^{ψ}) be a ψ-topological space and $A \subset X$. An element $x \in A$ is said to be ψ-interior point of A if there exits a ψ-open set U such that $x \in U \subseteq A$. The set of all ψ-interior points of A is said to be ψ-interior of A an it is denoted by $\psi\text{-}Int(A)$.

Lemma 2.1.
If A is a open set of τ^{ψ}, then $A = \psi\text{-}Int(A)$.

Proof. The proof is followed by the Definition 2.3. □

Definition 2.4.
Let (X,τ^{ψ}) be a ψ-topological space and $A \subseteq X$. The ψ-closure of A is defined as the intersection of all ψ-closed sets containing A and it is denoted by $\psi\text{-}Cl(A)$.

Lemma 2.2.
If A is a closed set of τ^{ψ}, then $A = \psi\text{-}Cl(A)$.

Proof. The proof is followed by the Definition 2.4. □

Theorem 2.3.
Let (X,τ) be a topological space. Then, every open is ψ-open.

Proof. Let A be an open set of τ. Then, $A = Int(A) \subseteq A-(A-Int(A)) = A$. Therefore, A is a ψ-open set. □

The following example shows that the converse of the above Theorem, it is not always true.

Example 2.1.
Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Then $\{b,c\}$ is a ψ-open set, but it is not a open set.

3. Open sets through operator interior ψ

We first introduce the following definitions.

Definition 3.1.
Let (X,τ) be a topological space and $A \subseteq X$, then:

1. A is said to be semi-open [8] if $A \subseteq Cl(Int(A))$.
2. A is said to be pre-open [5] if $A \subseteq Int(Cl(A))$.
3. A is said to be b-open [2] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$.
4. A is said to be $^{\star}b$-open [6] if $A \subseteq Cl(Int(A)) \cap Int(Cl(A))$.
5. A is said to be α-open [9] if $A \subseteq Int(Cl(Int(A)))$
6. A is said to be β-open [3] if $A \subseteq Cl(Int(Cl(A)))$

Now, we define the sets mentioned in the above Definition using the notion of operator interior ψ.

Definition 3.2.
The term (X,τ,ψ) will be called ψ-topological space, where τ and ψ are the topology and the interior operator on X respectively. We sometimes write X instead of (X,τ,ψ).

Definition 3.3.
Let (X,τ,ψ) be a ψ-topological space and $A \subseteq X$, then we define $Int^{\psi}(A) = A-(A-Int(A))$, where $Int^{\psi}(A)$ satisfies the conditions of operator interior ψ.

Definition 3.4.
Let (X,τ,ψ) be a ψ-topological space and $A \subseteq X$. Then, A is said to be semi-ψ-open if $A \subseteq Cl(Int^{\psi}(A))$. The complement of a semi-ψ-open is called semi-ψ-closed.

Remark 3.1.
The collection of all semi-ψ-open sets is denoted by $S\psi O(X)$ and the collection of all semi-ψ-closed is denoted by $S\psi C(X)$.

Theorem 3.1.
The arbitrary union of semi-ψ-open sets is a semi-ψ-open set.

Proof. Let $\{A_\delta : \delta \in \Delta\}$ be a collection of semi-ψ-open sets of X, then
$A_\delta \subseteq Cl(Int^{\psi}(A_\delta))$
$\bigcup_{\delta\in\Delta} A_\delta \subseteq \bigcup_{\delta\in\Delta}(Cl(Int^{\psi}(A_\delta)))$
$\bigcup_{\delta\in\Delta} A_\delta \subset Cl(\bigcup_{\delta\in\Delta}(Int^{\psi}(A_\delta)))$
By the Theorem 2.1, we have that
$\bigcup_{\delta\in\Delta} A_\delta \subset Cl(Int^{\psi}(\bigcup_{\delta\in\Delta}(A_\delta)))$.
Therefore, $\bigcup_{\delta\in\Delta} A_\delta$ is a semi-ψ-open set. □

Lemma 3.1.
Arbitrary intersection of semi-ψ-closed sets is semi-ψ-closed set.

Proof. The proof is followed by the Theorem 3.1. □

Theorem 3.2.
Every open set is semi-ψ-open.

Proof. Let A be an open set, then $A \subseteq Cl(Int^{\psi}(A)) = Cl(A-(A-Int(A))) = Cl(A)$. Therefore, A is semi-ψ-open. □

The following example shows that the converse of the above Theorem not need be true.

Example 3.1.
Let $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,b,d\}\}$. $\{b,c,d\}$ is a semi-ψ-open set, but it is not an open set.

Theorem 3.3.
Let A be a subset of a ψ-topological space X, then the following statements are equivalent:

1. *If A semi-ψ-open, then A semi-open.*
2. *If A is semi-open, then A is semi-ψ-open.*

Proof. 1. Let A be a semi-ψ-open set, then

$A \subseteq Cl(Int^{\psi}(A))$

$A \subseteq Cl(A-(A-Int(A)))$

$A \subseteq Cl(A \cap Cl(A \cap Cl(Int(A))))$

$A \subset Cl(A) \cap Cl(Cl(A \cap Cl(Int(A))))$

$A \subset Cl(Cl(A \cap Cl(Int(A))))$

$A \subseteq Cl(A \cap Cl(Int(A)))$

$A \subset Cl(A) \cap Cl(Cl(Int(A)))$

$A \subset Cl(Cl(Int(A)))$

$A \subseteq Cl(Int(A))$

Therefore, A is semi-open.

2. Let A be a semi-open set, then

$A \subseteq Cl(Int(A))$

$A \subseteq Cl(Int(A) \cap A)$

$A \subseteq Cl(A - (A - Int(A)))$

$A \subseteq Cl(Int^{\psi}(A))$

Therefore, A is semi-ψ-open.

□

Definition 3.5.
Let (X, τ, ψ) be a ψ-topological space and $A \subseteq X$. Then, A is said to be pre-ψ-open if $A \subseteq Int^{\psi}(Cl(A))$. The complement of a pre-ψ-open is called pre-ψ-closed.

Remark 3.2.
The collection of all pre-ψ-open sets is denoted by $P\psi O(X)$ and the collection of all pre-ψ-closed is denoted by $P\psi C(X)$.

Theorem 3.4.
The arbitrary union of pre-ψ-open sets is a pre-ψ-open set.

Proof. Let $\{A_\delta : \delta \in \Delta\}$ be a collection of pre-ψ-open sets of X, then

$A_\delta \subseteq Int^{\psi}(Cl(A_\delta))$

$\bigcup_{\delta\in\Delta} A_\delta \subseteq \bigcup_{\delta\in\Delta}(Int^{\psi}(Cl(A_\delta)))$

By the Theorem 2.1, we have

$\bigcup_{\delta\in\Delta} A_\delta \subset Int^{\psi}(\bigcup_{\delta\in\Delta}(Cl(A_\delta)))$

$\bigcup_{\delta\in\Delta} A_\delta \subset Int^{\psi}(Cl(\bigcup_{\delta\in\Delta}(A_\delta)))$

Therefore, $\bigcup_{\delta\in\Delta} A_\delta$ is a pre-ψ-open set. □

Lemma 3.2.
Arbitrary intersection of pre-ψ-closed sets is a pre-ψ-closed set.

Proof. The proof is followed by the Theorem 3.4. □

Theorem 3.5.
Every open set is pre-ψ-open.

Proof. Let A be a open set, then $A \subseteq Int^{\psi}(Cl(A))$. Now, $Cl(A) = B$, where B is a closed set or $Cl(A) = X$, thus $A \subseteq Int^{\psi}(B) = B - (B - Int(B)) = B \cap Int(B)$, but $A \in Int(B)$. Therefore, A is a pre-ψ-open set. □

The following example shows that the converse of the above Theorem not need be true.

Example 3.2.
Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$. $\{a, b, c\}$ is a pre-ψ-open set, but it is not an open set.

Theorem 3.6.
Let A be subset of a ψ-topological space X, then the following statements are equivalent:

1. *If A is pre-ψ-open, then A is pre-open.*
2. *If A is pre-open, then A is pre-ψ-open.*

Proof. 1. Let A be a pre-ψ-open set, then

$A \subseteq Int^{\psi}(Cl(A))$

$A \subseteq Cl(A) - (Cl(A) - Int(Cl(A)))$

$A \subseteq Cl(A) \cap Int(Cl(A) - Int(Cl(A)))$

$A \subset Int(Cl(A) \cap Cl(Int(Cl(A))))$

$A \subset Int(Cl(A)) \cap Int(Cl(Int(Cl(A))))$

$A \subset Int(Cl(A))$

Therefore, A is a pre-open set.

2. Let A be a pre-open set, then

$A \subseteq Int(Cl(A))$

$A \subseteq Int(Cl(A) \cap Int(Cl(A)))$

$A \subseteq Int(Cl(A) - (Cl(A) - Int(Cl(A))))$

$A \subseteq Int(Int^{\psi}(Cl(A)))$

$A \subset Int^{\psi}(Cl(A))$

Therefore, A is pre-ψ-open.

□

Definition 3.6.
Let (X, τ, ψ) be a ψ-topological space and $A \subseteq X$. Then, A is said to be b-ψ-open if $A \subseteq Cl(In^{\psi}(A)) \cup Int^{\psi}(Cl(A))$. The complement of a b-ψ-open is called b-ψ-closed.

Remark 3.3.
The collection of all b-ψ-open sets is denoted by $b\psi O(X)$ and the collection of all b-ψ-closed is denoted by $b\psi C(X)$.

Theorem 3.7.
The arbitrary union of b-ψ-open sets is a b-ψ-open set.

Proof. Let $\{A_\delta : \delta \in \Delta\}$ be a collection of b-ψ-open sets of X, then

$A_\delta \subseteq Cl(Int^{\psi}(A_\delta)) \cup Int^{\psi}(Cl(A_\delta))$

$\bigcup_{\delta\in\Delta} A_\delta \subseteq \bigcup_{\delta\in\Delta}(Cl(Int^{\psi}(A_\delta)) \cup Int^{\psi}(Cl(A_\delta)))$

$\bigcup_{\delta\in\Delta} A_\delta \subseteq \bigcup_{\delta\in\Delta}(Cl(Int^{\psi}(A_\delta))) \cup \bigcup_{\delta\in\Delta}(Int^{\psi}(Cl(A_\delta)))$

$\bigcup_{\delta\in\Delta} A_\delta \subset Cl \bigcup_{\delta\in\Delta}(Int^{\psi}(A_\delta)) \cup \bigcup_{\delta\in\Delta}(Int^{\psi}(Cl(A_\delta)))$

By the Theorem 2.1, we have

$\bigcup_{\delta\in\Delta} A_\delta \subset Cl(Int^{\psi}(\bigcup_{\delta\in\Delta} A_\delta)) \cup Int^{\psi}(Cl(\bigcup_{\delta\in\Delta} A_\delta)))$

Therefore, $\bigcup_{\delta\in\Delta} A_\delta$ is a b-ψ-open set. □

Lemma 3.3.
Arbitrary intersection of b-ψ-closed sets is a b-ψ-closed set.

Proof. The proof is followed by the Theorem 3.7. □

Theorem 3.8.
Every semi-ψ-open set is b-ψ-open set.

Proof. Let A be a semi-ψ-open set, then $A \subseteq Cl(Int^{\psi}(A)) \subset Cl(Int^{\psi}(A)) \cup Int^{\psi}(Cl(A))$, therefore A is b-ψ-open. □

Remark 3.4.
If a subset A of X is not semi-ψ-open when $Cl(In^{\psi}(A)) = \emptyset$ and $A \subseteq Int^{\psi}(Cl(A)) \neq \emptyset$.

Theorem 3.9.
Every pre-ψ-open set is b-ψ-open set.

Proof. Let A be a semi-ψ-open set, then $A \subseteq Int^{\psi}(Cl(A)) \subset Cl(Int^{\psi}(A)) \cup Int^{\psi}(Cl(A))$, therefore A is b-ψ-open. □

The following example shows that the converse of the above Theorem not need be true.

Example 3.3.
Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$. $\{a, d\}$ is a b-ψ-open set, but it is not a pre-ψ-open set.

Lemma 3.4.
Every open set is b-ψ-open set.

Proof. Let A be a open set, then A is semi-ψ-open and by the Theorem 3.8, A is b-ψ-open. □

The following example shows that the converse of the above Theorem not need be true.

Example 3.4.
Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$. $\{a, b, c\}$ is a b-ψ-open set, but it is not an open set.

Theorem 3.10.
Let A be a subset of a ψ-topological space X, then the following statements are equivalent:

1. *If A is b-ψ-open, then A is b-open.*
2. *If A is b-open, then A is b-ψ-open.*

Proof. 1. Let A be a b-ψ-open set, then

$A \subseteq Cl(Int^{\psi}(A)) \cup Int^{\psi}(Cl(A))$

$A \subseteq Cl((A - (A - Int(A)))) \cup (Cl(A) - (Cl(A) - Int(Cl(A))))$

$A \subseteq Cl(A \cap Int(A)) \cup (Cl(A) \cap Int(Cl(A)))$

$A \subseteq (Cl(A \cap Int(A)) \cup Cl(A)) \cap (Cl(A \cap Int(A)) \cup Int(Cl(A))$

$A \subset Cl(A \cap Int(A)) \cup Int(Cl(A))$

$A \subset (Cl(A) \cap Cl(Int(A))) \cup Int(Cl(A))$

$A \subset (Cl(A) \cup Int(Cl(A))) \cap (Cl(Int(A)) \cup Cl(Int(A)))$

$A \subset Cl(Int(A)) \cup Cl(Int(A))$

Therefore, A is a b-open set.

2. The proof of this part is followed by the Theorems 3.3 and 3.6.

□

Lemma 3.5.
For a subset A of X the following statements hold:

1. *If A is semi-ψ-open, then A is b-open.*
2. *If A is pre-ψ-open, then A is b-open*

Proof. The proof is followed by the Theorems 3.8, 3.9 and 3. □

Definition 3.7.
Let (X, τ, ψ) be a ψ-topological space and $A \subseteq X$. Then, A is said to be $^{\star}b$-ψ-open if $A \subseteq Cl(Int^{\psi}(A)) \cap Int^{\psi}(Cl(A))$. The complement of a $^{\star}b$-ψ-open is called $^{\star}b$-ψ-closed.

Remark 3.5.
The collection of all $^\star b$-ψ-open sets is denoted by $^\star b\psi O(X)$ and the collection of all $^\star b$-ψ-closed is denoted by $^\star b\psi C(X)$.

Theorem 3.11.
The arbitrary union of $^\star b$-ψ-open sets is a $^\star b$-ψ-open set.

Proof. Let $\{A_\delta : \delta \in \Delta\}$ be a collection of $^\star b$-ψ-open sets of X, then
$A_\delta \subseteq Cl(Int^\Psi(A_\delta)) \cap Int^\Psi(Cl(A_\delta))$
$\bigcup\limits_{\delta\in\Delta} A_\delta \subseteq \bigcup\limits_{\delta\in\Delta} (Cl(Int^\Psi(A_\delta)) \cap Int^\Psi(Cl(A_\delta)))$
$\bigcup\limits_{\delta\in\Delta} A_\delta \subset \bigcup\limits_{\delta\in\Delta} (Cl(Int^\Psi(A_\delta))) \cap \bigcup\limits_{\delta\in\Delta} (Int^\Psi(Cl(A_\delta)))$
$\bigcup\limits_{\delta\in\Delta} A_\delta \subset Cl \bigcup\limits_{\delta\in\Delta} (Int^\Psi(A_\delta)) \cap \bigcup\limits_{\delta\in\Delta} (Int^\Psi(Cl(A_\delta)))$
By the Theorem 2.1, we have
$\bigcup\limits_{\delta\in\Delta} A_\delta \subset Cl(Int^\Psi(\bigcup\limits_{\delta\in\Delta} A_\delta)) \cap Int^\Psi(Cl(\bigcup\limits_{\delta\in\Delta} A_\delta)))$
Therefore, $\bigcup\limits_{\delta\in\Delta} A_\delta$ is a $^\star b$-ψ-open set. □

Lemma 3.6.
Arbitrary intersection of $^\star b$-ψ-closed sets is a $^\star b$-ψ-closed set.

Proof. The proof is followed by the Theorem 3.11. □

Theorem 3.12.
Every open set is a $^\star b$-ψ-open.

Proof. Let A be a open set, then $A \subseteq Cl(Int^\Psi(A)) \cap Int^\Psi(Cl(A)) = A \cap A = A$. Therefore, A is $^\star b$-ψ-open. □

The following example shows that the converse of the above Theorem not need be true.

Example 3.5.
Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$. $\{a, b, c\}$ is a $^\star b$-ψ-open set, but it is not an open set.

Theorem 3.13.
Every $^\star b$-ψ-open set is a semi-ψ-open set.

Proof. Let A be a $^\star b$-ψ-open set, then $A \subseteq Cl(Int^\Psi(A)) \cap Int^\Psi(Cl(A)) \subset Cl(Int^\Psi(A))$, therefore A is semi-ψ-open. □

The following example shows that the converse of the above Theorem not need be true.

Example 3.6.
Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$. $\{b, c\}$ is a semi-ψ-open set, but it is not a $^\star b$-ψ-open set.

Theorem 3.14.
Every $^\star b$-ψ-open set is a pre-ψ-open set.

Proof. Let A be a $^\star b$-ψ-open set, then $A \subseteq Cl(Int^\Psi(A)) \cap Int^\Psi(Cl(A)) \subset Int^\Psi(Cl(A))$, therefore A is pre-ψ-open. □

Remark 3.6.
For a subset A of X, A is pre-ψ-open when $Cl(In^\Psi(A)) = \emptyset$ and $A \subseteq Int^\Psi(Cl(A)) \neq \emptyset$.

Lemma 3.7.
Let A be a subset of X, the the following statements hold:

1. *Every $^\star b$-ψ-open set is a b-ψ-open set.*

2. *Every* $^{\star}b$-ψ*-open set is a* b*-open set.*

3. *Every* $^{\star}b$-ψ*-open set is a semi-open set.*

4. *Every* $^{\star}b$-ψ*-open set is a pre-open set.*

Proof. The proof is followed by the Theorems 3.3, 3.6, 3.8, 3.13 and 3. □

Theorem 3.15.
Let A *be a subset of a* ψ*-topological space* X*, then the following statements are equivalent:*

1. *If* A *is* $^{\star}b$-ψ*-open, then* A *is* $^{\star}b$*-open.*

2. *If* A *is* $^{\star}b$*-open, then* A *is* $^{\star}b$-ψ*-open.*

Proof. 1. Let A be a $^{\star}b$-ψ-open set, then

$A \subseteq Cl(Int^{\psi}(A)) \cap Int^{\psi}(Cl(A))$

$A \subseteq Cl((A-(A-Int(A)))) \cap (Cl(A)-(Cl(A)-Int(Cl(A))))$

$A \subseteq Cl(A \cap Int(A)) \cap (Cl(A) \cap Int(Cl(A)))$

$A \subseteq (Cl(A \cap Int(A)) \cap Cl(A)) \cap (Cl(A \cap Int(A)) \cup Int(Cl(A))$

$A \subset Cl(A \cap Int(A)) \cap Int(Cl(A))$

$A \subset (Cl(A) \cap Cl(Int(A))) \cap Int(Cl(A))$

$A \subset Cl(Int(A)) \cap Cl(Int(A))$

Therefore, A is $^{\star}b$-open.

2. The proof of this part is followed by the Theorems 3.3 and 3.6.

□

Definition 3.8.
Let (X,τ,ψ) be a ψ-topological space and $A \subseteq X$. Then, A is said to be α-ψ-open if $A \subseteq Int^{\psi}(Cl(Int^{\psi}(A)))$. The complement of a α-ψ-open is called α-ψ-closed.

Remark 3.7.
The collection of all α-ψ-open sets is denoted by $\alpha\psi O(X)$ and the collection of all α-ψ-closed is denoted by $\alpha\psi C(X)$.

Theorem 3.16.
The arbitrary union of α-ψ*-open sets is a* α-ψ*-open set.*

Proof. Let $\{A_\delta : \delta \in \Delta\}$ be a collection of α-ψ-open sets of X, then

$A_\delta \subseteq Int^{\psi}(Cl(Int^{\psi}(A_\delta)))$

$\bigcup_{\delta\in\Delta} A_\delta \subseteq \bigcup_{\delta\in\Delta}(Int^{\psi}(Cl(Int^{\psi}(A_\delta)))$

By the Theorem 2.1, we have

$\bigcup_{\delta\in\Delta} A_\delta \subset Int^{\psi} \bigcup_{\delta\in\Delta}(Cl(Int^{\psi}(A_\delta)))$

$\bigcup_{\delta\in\Delta} A_\delta \subset Int^{\psi}(Cl \bigcup_{\delta\in\Delta}(Int^{\psi}(A_\delta)))$

Now, by the Theorem 2.1, we have

$\bigcup_{\delta\in\Delta} A_\delta \subset Int^{\psi}(Cl(Int^{\psi}(\bigcup_{\delta\in\Delta} A_\delta)))$

Therefore, $\bigcup_{\delta\in\Delta} A_\delta$ is α-ψ-open. □

Lemma 3.8.
Arbitrary intersection of α-ψ*-closed sets is* α-ψ*-closed set.*

Proof. The proof is followed by the Theorem 3.16. □

Theorem 3.17.
Every open set is α-ψ-open.

Proof. Let A be a open set, then $A \subseteq Int^{\psi}(Cl(Int^{\psi}(A))) = Int^{\psi}(Cl(A))$. Now, $Cl(A) = X$ or $Cl(A) = B$, where B is a closed set. Then, $A \subseteq Int^{\psi}(B) = B \cap Int(B)$, since $A \in Int(B) \in B$, therefore A is a α-ψ-open set. □

The following example shows that the converse of the above Theorem not need be true.

Example 3.7.
Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$. $\{a, b, c\}$ is a α-ψ-open set, but it is not an open set.

Theorem 3.18.
Let A be a subset of a ψ-topological space X, then the following statements are equivalent:

1. *If A is α-ψ-open, then A is α-open.*
2. *If A is α-open, then A is α-ψ-open.*

Proof. 1. Let A be a α-ψ-open set, then

$A \subseteq Int^{\psi}(Cl(Int^{\psi}(A)))$

$A \subseteq Cl(Int^{\psi}(A)) - (Cl(Int^{\psi}(A)) - Int(Cl(Int^{\psi}(A))))$

$A \subseteq Cl(Int^{\psi}(A)) \cap Int(Cl(Int^{\psi}(A)))$

$A \subset Int(Cl(Int^{\psi}(A)))$

$A \subseteq Int(Cl(A - (A - Int(A))))$

$A \subset Int(Cl(A \cap Int(A)))$

$A \subset Int(Cl(A) \cap Cl(Int(A)))$

$A \subset Int(Cl(A)) \cap Int(Cl(Int(A)))$

$A \subset Int(Cl(Int(A)))$

Therefore, A is α-open.

2. Let A be a α-open set, then

$A \subseteq Int(Cl(Int(A)))$

$A \subseteq Int(Cl(A \cap Int(A)))$

$A \subset Int(Cl(A) \cap Cl(Int(A)))$

By the Theorem 3.3, we have

$A \subseteq Int(Cl(A) \cap Cl(Int^{\psi}(A)))$

$A \subseteq Int(Cl(A)) \cap Int(Cl(Int^{\psi}(A)))$

$A \subset Int(Cl(Int^{\psi}(A)))$

Since $Int(A) \subseteq Int^{psi}(A)$, this implies that

$A \subset Int^{\psi}(Cl(Int^{\psi}(A)))$

Therefore, A is α-ψ-open.

□

Theorem 3.19.
Every α-ψ-open set is semi-ψ-open.

Proof. Let A be a α-ψ-open set, then
$A \subseteq Int^{\psi}(Cl(Int^{\psi}(A)))$
$A \subseteq Cl(Int^{\psi}(A)) - (Cl(Int^{\psi}(A)) - Int(Cl(Int^{\psi}(A))))$
$A \subseteq Cl(Int^{\psi}(A)) \cap Int(Cl(Int^{\psi}(A)))$
$A \subset Cl(Int^{\psi}(A))$
Therefore, A is semi-ψ-open. □

The following example shows that the converse of the above Theorem not need be true.

Example 3.8.
Let $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,b,d\}\}$. $\{a,c\}$ is a semi-ψ-open set, but it is not a α-ψ-open set.

Theorem 3.20.
Every α-ψ-open set is pre-open.

Proof. Let A be a α-ψ-open set, then
$A \subseteq Int^{\psi}(Cl(Int^{\psi}(A)))$
$A \subseteq Cl(Int^{\psi}(A)) - (Cl(Int^{\psi}(A)) - Int(Cl(Int^{\psi}(A))))$
$A \subseteq Cl(Int^{\psi}(A)) \cap Int(Cl(Int^{\psi}(A)))$
$A \subset Int(Cl(Int^{\psi}(A)))$
$A \subseteq Int(Cl(A - (A - Int(A))))$
$A \subset Int(Cl(A \cap Int(A)))$
$A \subset Int(Cl(A) \cap Cl(Int(A)))$
$A \subset Int(Cl(A)) \cap Int(Cl(Int(A)))$
$A \subset Int(Cl(A))$
Therefore, A is pre-open. □

Lemma 3.9.
For any subset A of X, the following statements hold:

1. *If A is α-ψ-open, then A is b-ψ-open.*
2. *If A is α-ψ-open, then A is b-open.*

Proof. The proof is followed by the Theorems 3.8, 3 and 3.19. □

Definition 3.9.
Let (X,τ,ψ) be a ψ-topological space and $A \subseteq X$. Then, A is said to be β-ψ-open if $A \subseteq Cl(Int^{\psi}(Cl(A)))$. The complement of a β-ψ-open is called β-ψ-closed.

Remark 3.8.
The collection of all β-ψ-open sets is denoted by $\beta\psi O(X)$ and the collection of all β-ψ-closed is denoted by $\beta\psi C(X)$.

Theorem 3.21.
The arbitrary union of β-ψ-open sets is β-ψ-open set.

Proof. Let $\{A_\delta : \delta \in \Delta\}$ be a collection of β-ψ-open sets of X, then
$A_\delta \subseteq Cl(Int^{\psi}(Cl(A_\delta)))$
$\bigcup_{\delta\in\Delta} A_\delta \subseteq \bigcup_{\delta\in\Delta}(Cl(Int^{\psi}(Cl(A_\delta))))$
$\bigcup_{\delta\in\Delta} A_\delta \subset Cl(\bigcup_{\delta\in\Delta}(Int^{\psi}(Cl(A_\delta))))$
By the Theorem 2.1, we have
$\bigcup_{\delta\in\Delta} A_\delta \subset Cl(Int^{\psi}(\bigcup_{\delta\in\Delta}(Cl(A_\delta))))$
$\bigcup_{\delta\in\Delta} A_\delta \subset Cl(Int^{\psi}(Cl(\bigcup_{\delta\in\Delta} A_\delta)))$
Therefore, $\bigcup_{\delta\in\Delta} A_\delta$ is a α-ψ-open set. □

Lemma 3.10.
Arbitrary intersection of α-ψ-closed sets is α-ψ-closed set.

Proof. The proof is followed by the Theorem 3.21. □

Theorem 3.22.
Every open set is β-ψ-open.

Proof. Let A be a open set, then $A \subseteq Cl(Int^{\psi}(Cl(A)))$. Now, $Cl(A) = X$ or $Cl(B)$ where B is a closed set, thus $A \subseteq Cl(Int^{\psi}(B)) = Cl(B \cap Int(B))$, but $A \subseteq Int(B)$, therefore $B \cap A \subseteq B \cap Int(B)$. Hence, A is a β-ψ-open set. □

The following example shows that the converse of the above Theorem not need be true.

Example 3.9.
Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$. $\{b, c, d\}$ is a β-ψ-open set, but it is not an open set.

Theorem 3.23.
Let A be a subset of a ψ-topological space X, then the following statements are equivalent:

1. *If A is β-ψ-open, then A is β-open.*
2. *If A is β-open, then A is β-ψ-open.*

Proof. 1. Let A be a β-ψ-open set, then

$A \subseteq Cl(Int^{\psi}(Cl(A)))$

$A \subseteq Cl(Cl(A) - (Cl(A) - Int(Cl(A))))$

$A \subseteq Cl(Cl(A) \cap Int(Cl(A)))$

$A \subset Cl(A) \cap Cl(Int(Cl(A)))$

$A \subset Cl(Int(Cl(A)))$

Therefore, A is β-open.

2. Let A be a β-open set, then

$A \subseteq Cl(Int(Cl(A)))$

$A \subset Cl(Int(Cl(A) \cap Int(Cl(A)))$

$A \subseteq Cl(Int^{\psi}(Cl(A)))$

Therefore, A is a β-open set.

□

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