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Analysis of Bifurcations for Quadratic Families

Research Article

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Abstract: Qualitative deflect in iteration dynamics when parameter c is deflected is called bifurcation. Let $Q_c(x)$ prevail the family of quadratic functions $Q_c(x) = x^2+c$, where c is a parameter for each detached c we obtain a isolated dynamical system Q_c . We have noticed that how the dynamics of Q_c change as c varies. When c decreases below 0.25, this situation changes first bifurcation. For $c > 0.25$, Q_c has no fixed points; for $c = 0.25$, Q_c has exactly one fixed point. But for $c < 0.25$, this fixed-point split into two, once at p_+ and one at p_- .Animation of a point "rolling" along the quadratic curve, bifurcation diagram on quadratic map for several iterations are analyzed. Some propositions for different parameter values of c and their solutions are given. Finally, we have manifested that the bifurcation sketches which are chaotic and non-chaotic for isolated parameter values.

Keywords: Bifurcation • Attracting fixed point • Neutral fixed point • Repelling fixed point • Chaos

1. Introduction

Bifurcation means a division in two, a splitting apart, and that is exactly what has happened to the fixed points of Q_c. Bifurcations occur in a one –parameter family of functions when there is a switch in the fixed or periodic point composition as c passes through some certain value [1]. The combing and extermination of periodic points as we vary a parameter is a common theme in dynamics [2]. In dynamical systems, the object of bifurcation theory is to study the barters that maps sustain as parameter change. These switches often embroil the periodic point structure, but may also embroil other switches as well [3]. A dynamical system is a process of narrating the passage in time of all points of yield states of some physical system. The higher dimensional systems may exhibit chaotic behaviour, a property that makes knowing a particular explicit solution essentially worthless in the larger scheme of understanding the behaviour of the system [4]. Fruitful applications of the conception of chaos as a natural affair followed. Such as, at times a fluid flows smoothly, but at times it becomes turbulent and irregular for no patent reason. There are many dynamical systems that can yield chaos. The quadratic map, the properties of chaos can be observed and completely analyzed [5]. Our motive is in exposing what kinds of parameter changes quadratic map can operate to chaos and non chaos.

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2. Families of the Quadratic map: $Q_c(x) = x^2 + c$

Here we consider a family of quadratic function where we get different dynamical system for each different parameter value. We are interested in understanding how the dynamics of Q_c changes as c varies. Let $Q_c(x) = x^2 + c\,(c \in R)$ denote the family of quadratic function, where c is a parameter. We need to endure the fixed points of Q_c by solving the quadratic equation

$$x^2 + c = x \Rightarrow x = p_+(\text{say}) = \frac{1+\sqrt{1-4c}}{2},\ p_-(\text{say}) = \frac{1-\sqrt{1-4c}}{2},$$

which are the two fixed points for Q_c. The fixed points p_+ and p_- are if and only if $1-4c \geq 0 \quad \Rightarrow c \leq \frac{1}{4}$. Notwithstanding $c = \frac{1}{4} \Rightarrow 1-4c = 0$ and hence $p_+ = p_- = \frac{1}{2}$ and when $c < \frac{1}{4} \Rightarrow 1-4c > 0$ and hence p_+ and p_- are authentic & variant and in this case $p_+ > p_-$. When $c > \frac{1}{4}$, Q_c has no fixed points.

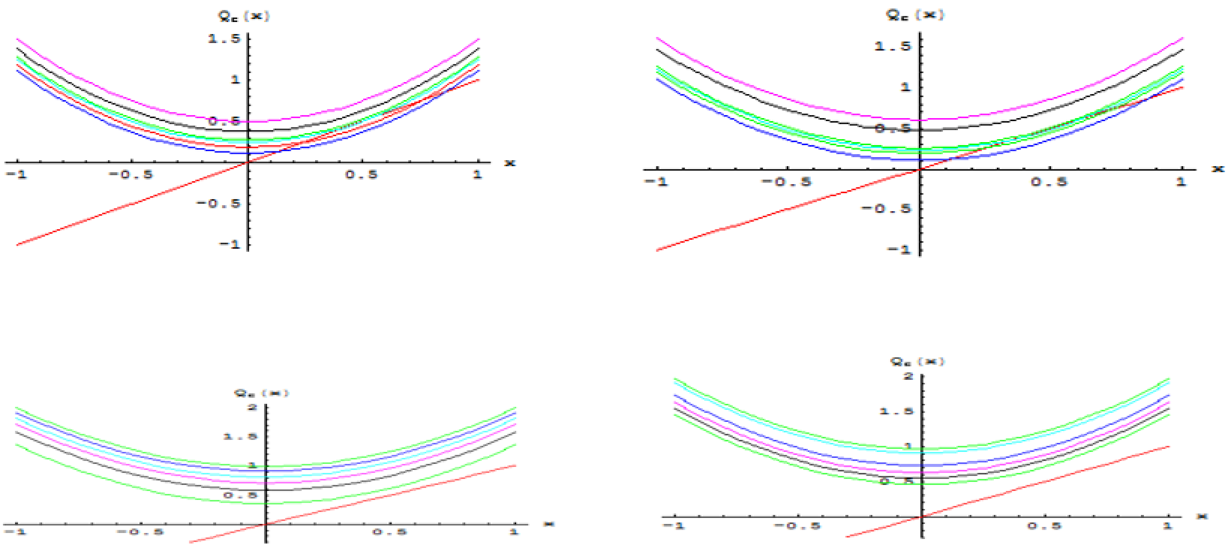


Fig. 1. Quadratic maps $c = 0.5, 0, -0.5, -1, -2$

2.1. The dynamical behaviour of quadratic map, when $c > \frac{1}{4}$

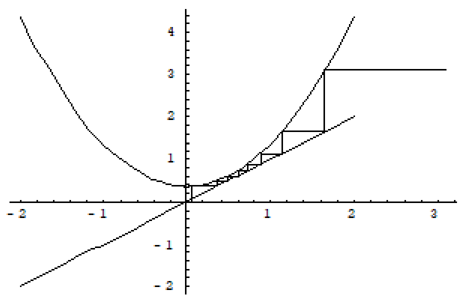


Fig. 2. All orbits of $Q_c(x) = x^2 + c$ for $c > 1/4$ tend to infinity.

We view that the graph of Q_c is a parabola inaugural uphill. At what time $c > \frac{1}{4}$, this graph does not meet the diagonal line $y = x$. Hence the graphical analysis views that all orbits of Q_c (when $c > \frac{1}{4}$) tends to infinity as shown in Fig. 2.2. Thus this is our dynamics in this case, when c decreases below $\frac{1}{4}$, this situation changes. That is, we encounter our first bifurcation.

(i) For $c > \frac{1}{4}$, Q_c has no fixed point.

(ii) For $c = \frac{1}{4}$, Q_c has appropriately one fixed point.

(iii) For $c < \frac{1}{4}$, this fixed-point transgress into two, one at p_+ and one at p_- .

Enunciation in case the fixed points $p_=$ and p_- are attracting, repelling, or neutral.
Since $Q_c'(x) = 2x$, then we have

$$Q_c'(p_+) = 2.\frac{1+\sqrt{1-4c}}{2} = 1+\sqrt{1-4c},\quad Q_c'(p_-) = 2.\frac{1-\sqrt{1-4c}}{2} = 1-\sqrt{1-4c}.$$

In case $c = \frac{1}{4}$, so $Q_c'(p_+) = 1$, consequently, p_+ is a neutral fixed point. If $c < \frac{1}{4}$ pursuant to $\left|Q'_c(p_+)\right| > 1$ $\sqrt{1-4c} > 0$ for these $c-$ values and hence p_+ is a repelling fixed point.
The circumstances for p_- is slightly more complicated.

If $c = \frac{1}{4}$, then $Q_c'(p_+) = 1$ ($p_+ = p_- = \frac{1}{2}$), If c is slightly below, then $Q_c'(p_-) < 1$ and so p_- becomes attracting. To find all of the c-values for which $|Q_c'(p_-)| < 1$, we solve $-1 < Q_c'(p_-) < 1 \Rightarrow -1 < 1-\sqrt{1-4c} < 1 \Rightarrow 2 > \sqrt{1-4c} > 0 \Rightarrow 4 > 1-4c > 0 \Rightarrow -3/4 < c < 1/4$. That is, when $-3/4 < c < 1/4$, $|Q_c'(p_-)| < 1$ and consequently p_- is an attracting fixed point for Q_c .As $c = -3/4$, $Q_c'(p_-) = 1-\sqrt{1+3} = -1$, and hereafter p_- is a neutral for Q_c . So $Q_c'(p_-) < -1$ and hereafter p_- is a repelling fixed point for Q_c.

3. Iterations of the quadratic map $Q_c(x) = x^2 + c$

Table 1. Iterations of the quadratic map, $Q_c(x)$

Iteration Number	$c = 0.16$	$c = 0.19$	$c = 0.20$	$c = 0.25$	$c = -0.23$	$c = -0.25$
	Initial Seed	Initial Seed	Initial Seed	Initial Seed	Initial Seed	Initial Seed
	$x_0 = 0.001$	$x_0 = 0$	$x_0 = 0.0001$	$x_0 = 0.1$	$x_0 = 0.0002$	$x_0 = 0.01$
0	0.001	0	0.0001	0.1	0.0002	0.1
1	0.160001	0.19	0.2	0.26	-0.23	-0.2499
2	0.1856	0.2261	0.24	0.3176	-0.1771	-0.18755
3	0.194447	0.241121	0.2576	0.35087	-0.198636	-0.214825
5	0.199129	0.251573	0.270946	0.389211	-0.193693	-0.208445
10	0.199991	0.254933	0.276118	0.431157	-0.192813	-0.20709
15	0.2	0.255047	0.276379	0.449617	-0.19282	-0.207107
30	0.2	0.255051	0.276393	0.471783	-0.19282	-0.207107
50	0.2	0.255051	0.276393	0.481434	-0.19282	-0.20717
100	0.2	0.255051	0.276393	0.490615	-0.19282	-0.207107
160	0.2	0.255051	0.276393	0.494012	-0.19282	-0.207107

Remark 3.1.
We have regarded that when $c = 0.16\,to\,-0.25$ and the number of iterations is captive 5 to 160 times, the illustrated figures becoming narrow to narrower.

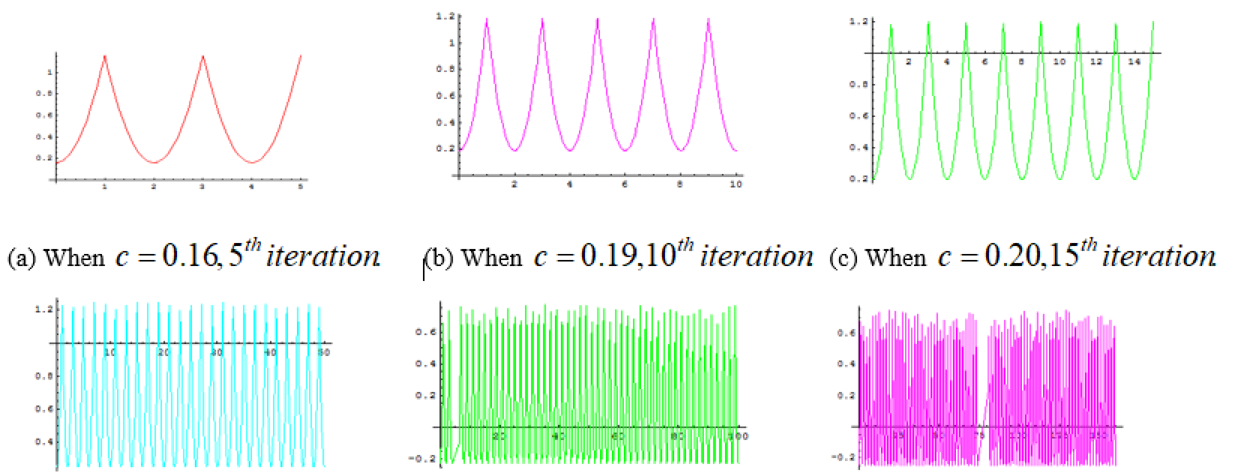
4. Results and Discussions

4.1. The Transmigration to Chaos

The objective in this assay will be to remark the dynamical behaviour of $Q_c(x) = x^2 + c$ for large number of c-values in $-2 \leq c \leq 0.25$. We are to compute the orbit of 0 under $Q_c(x) = x^2 + c$ for at least 50 different c -values as specified below. For each such c, the goal is to record the ultimate or "asymptotic" behaviour of the orbit of 0. This means that we are only interested in what eventually happens to the orbit of 0. In the Figure 4.1, we pretend the graphs of Q_c^2 for c-values before, at, and after the period-doubling bifurcation.

5. Some Propositions Apply to the Family $Q_c(x)$

Proposition 5.1.
The formulas for the fixed points $p_\pm$ and the 2-cycle $q_\pm$.



(a) When $c = 0.16, 5^{th}$ *iteration* (b) When $c = 0.19, 10^{th}$ *iteration* (c) When $c = 0.20, 15^{th}$ *iteration*

(d) When $c = 0.25, 50^{th}$ *iteration* (e) When $c = -0.23, 100^{th}$ *iteration* (f) When $c = -0.25, 160^{th}$ *iteration*

Fig. 3. A bifurcation in the Quadratic families, $Q_c(x) = x^2 + c$ for some parameter values of c.

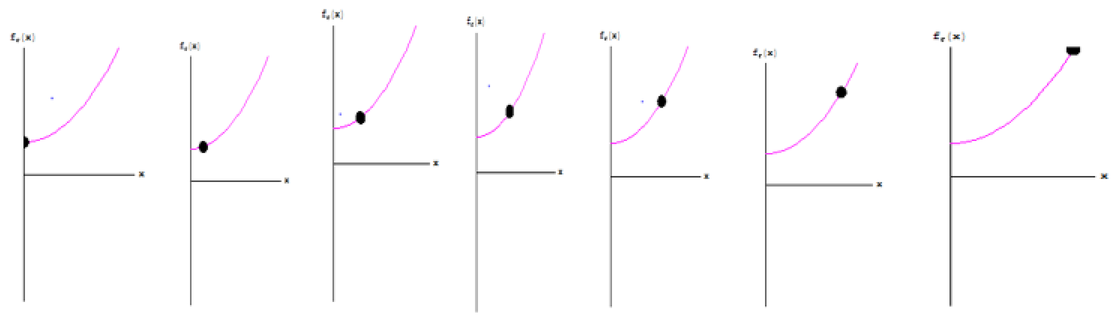


Fig. 4. Animation of a point "rolling" along the quadratic curve, $Q_c(x) = x^2 + c$ curve from 0 to 1, when $c = 0.25$

Recall that

$$p\pm = \frac{1 \pm \sqrt{1-4c}}{2}$$

And note that these points are real when $1 - 4c > 0$ that is, when $c < \frac{1}{4}$.
Now

$$Q_c(p_+) = Q_c\left(\left(1+\sqrt{1-4c}\right)/2\right) = \left(1+\sqrt{1-4c}\right)^2/4 + c$$
$$= \frac{\left(1+2\sqrt{1-4c}-4c+1\right)}{4} + c = \left(2+2\sqrt{1-4c}-4c+4c\right)/4 = \frac{1+\sqrt{1-4c}}{2} = p_+$$

We leave it to the reader to show that $p_-$is fixed. Verify that

$$q\pm = \frac{-1 \pm \sqrt{-4c-3}}{2}$$

Constitute a 2-cycle for Q_c

$$Q_c(q_+) = Q_c\left(\left(-1+\sqrt{-4c}-3\right)/2\right)$$

$$= \left(-1+\sqrt{-4c-3}\right)^2/4 + c$$

$$= \frac{1-2\sqrt{-4c-3}-4c-3}{4} + c$$

$$= \left(-2-2\sqrt{-4c-3}-4c+4c\right)/4 = \frac{-1-\sqrt{-4c-3}}{2} = q_-.$$

The reader may show that $Q_c(q_-) = q_+$ as well. Thus $\{q_+, q_-\} \subset per_2 Q_c$.
We remark that 2-cycle exists for $\{q_+, q_-\} \subset per_2 Q_c$. that is for $c < -3/4$.

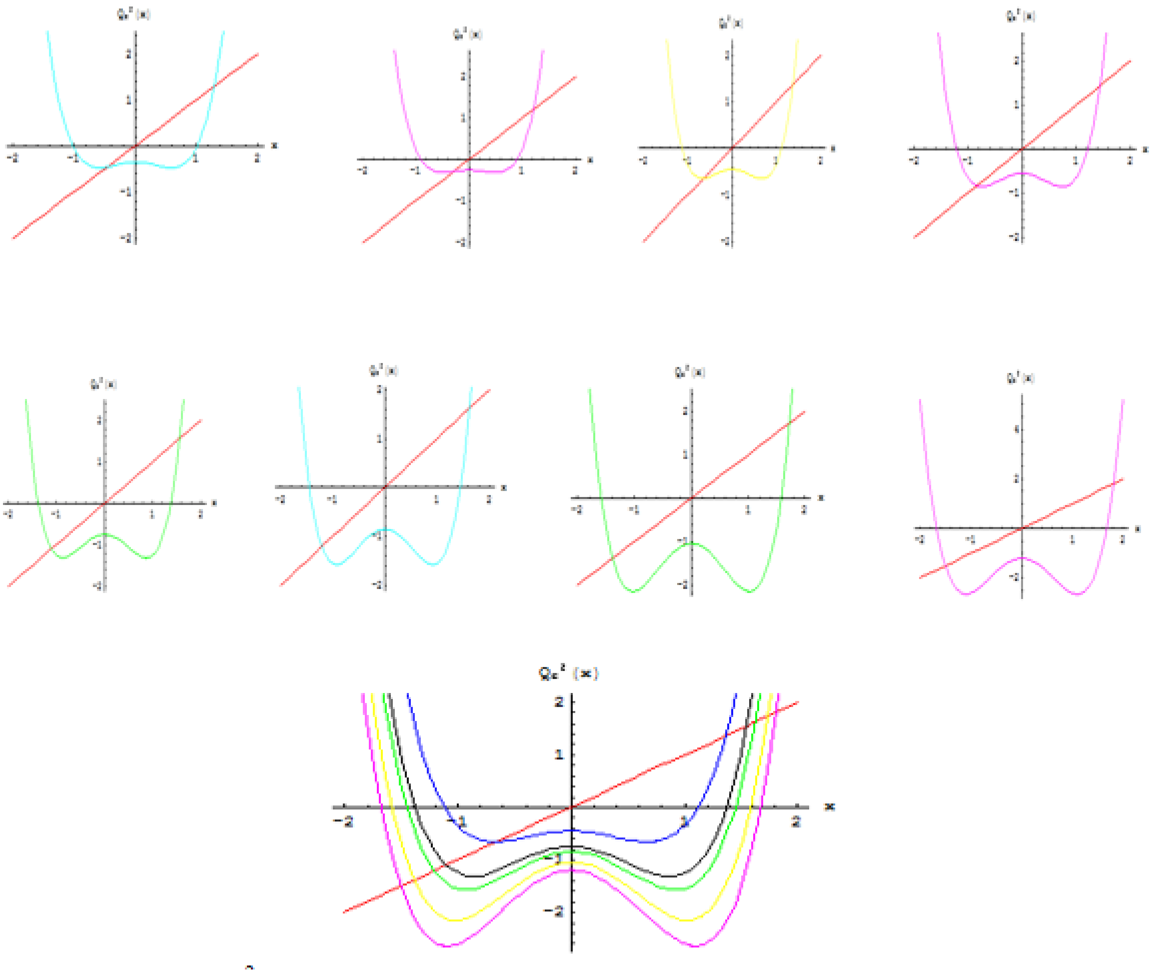


Fig. 5. The graph of $Q_c{}^2$ near the period doubling-bifurcation when $c > -0.75$, $c = -0.75$, $c < -0.75$.

Proposition 5.2.
The cycle of period 2 conferred by $q_\pm$ *is attracting for* $-5/4 < c < -3/4$

First of all, say that $Q'_c(x) = 2x$we have that

$$\begin{aligned}&(Q_c^2)'(q_+) = Q'_c(q_+).Q'_c(q_-)\\&= \left(-1+\sqrt{-4c-3}\right).\left(-1-\sqrt{-4c-3}\right)\\&= 1-(-4c-3)\\&= 4c+4\end{aligned} \tag{5.1}$$

To seek out the parameter values for which this 2-cycle is attracting, we ought to solve

$$\begin{aligned}&-1 < 4c+4 < 1 \Rightarrow -5 < 4c < -3\\&\Rightarrow -5/4 < c < -3/4.\end{aligned}$$

Proposition 5.3.
The cycle is neutral for $c = -5/4$

We apply equation 5.1, when $c = -5/4$, $4c+4 = -1$ and hence; the 2-cycle is neutral for this particular c-value

Proposition 5.4.
The cycle is repelling for $c < -5/4$

When $c < -5/4$, we have $4c+4 < -1$from equation 5.1. Hence, this period 2 orbit is repelling for $c < -5/4$,

6. Propositions

Let $Q_c(x) = x^2 + c$ be a quadratic family of functions. Then we have the following propositions:

6.1 The formulas for the fixed points $p_{\pm}$ and the 2-cycle. $q_{\pm}$

6.2 The cycle of period 2 given by $q_{\pm}$ is attracting for $-5/4 < c < -3/4$.

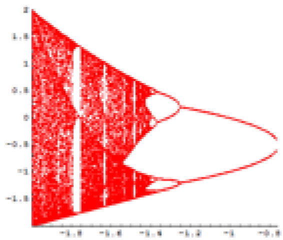
6.3 The cycle is neutral for $c = -5/4$.

6.4 The cycle is repelling for $c < -5/4$.

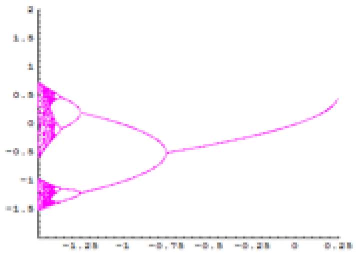
7. Bifurcation Diagrams of Quadratic Map, Q_c for Different Parameter Values

The bifurcation diagrams of Q_c are displayed for isolated intervals of c and isolated iterations. The dynamics of the entire family as well as an idea of how Q_c makes the transition to chaos. We have described the full orbit diagram of Q_c . From the above figure it is evidently visible that period-doubling bifurcation take place for an attracting 3-cycle. Other windows visible in all of the figures. In the chaotic band there are white soprano perforations of windows of periodic dynamics. The individual attracting fixed point bifurcates continually and then befalls chaotic.

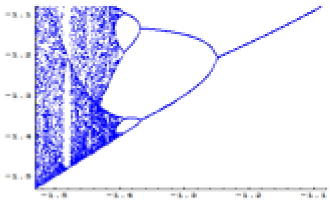
A bifurcation diagram is a schematic representation of the reduplicative points of a function of the parameter. The remarkable structures and patterns that occur in the following orbit diagrams.



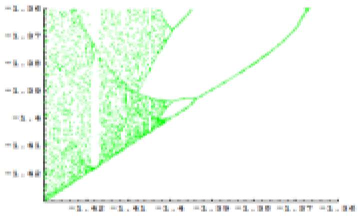
(a) Bifurcation when $-2 \le c \le -0.76$



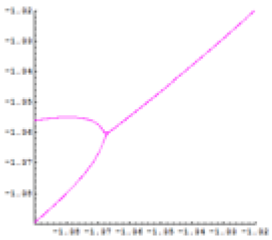
(b) Bifurcation when $-1.5 \le c \le 0.25$



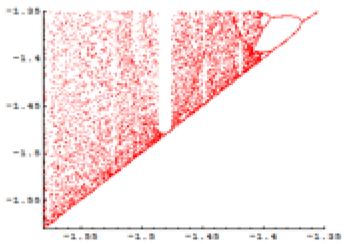
(c) Bifurcation when $-1.53 \le c \le -1.08$



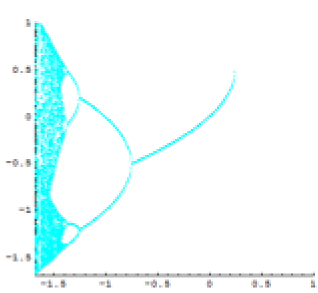
(d) Bifurcation when $-1.43 \le c \le -1.36$



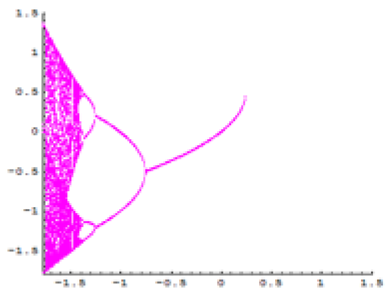
(e) Bifurcation when $-1.39 \le c \le -1.32$



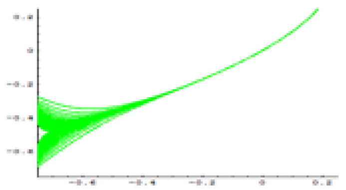
(f) Bifurcation when $-1.58 \le c \le -1.35$



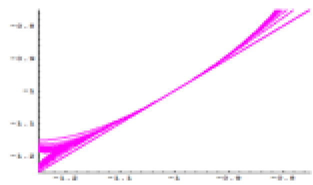
(g) Bifurcation when $-1.68 \le c \le 1$



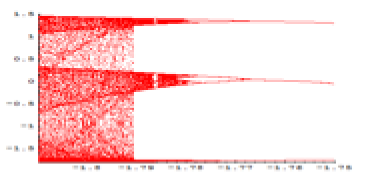
(h) Bifurcation when $-1.78 \le c \le 1.5$



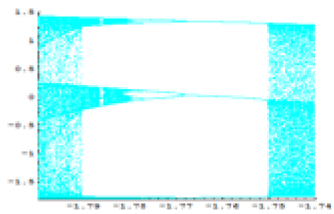
(i) Bifurcation when $-0.75 \le c \le 0.25$



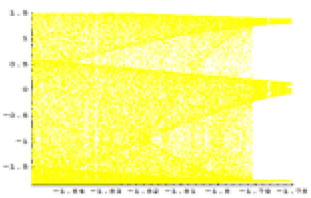
(j) Bifurcation when $-1.25 \le c \le -0.75$



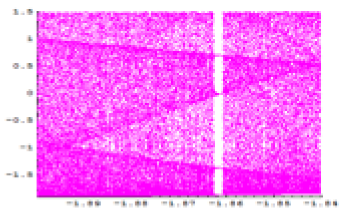
(k)Bifurcation when $-1.81 \le c \le -1.75$.



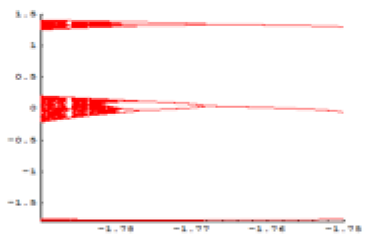
(l) Bifurcation when $-1.8 \le c \le -1.74$



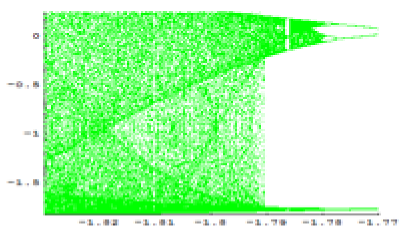
(m) Bifurcation when $-1.85 \le c \le -1.78$



(n) Bifurcation when $-1.9 \le c \le -1.84$



(o) Bifurcation when $-1.79 \le c \le -1.84$



(p) Bifurcation when $-1.83 \le c \le -1.77$

8. Concluion

We have investigated some basic convictions of bifurcation and period doubling bifurcation. Some Propositions of Quadratic map, animation of a point “rolling” along the logistic curve, bifurcation diagram on quadratic map for several iterations are illustrated. The orbit diagram yields empirical precedent that some of the c-values abaft the period-doubling reign propulsion to chaotic. At it is seen that a branching point due to period doubling bifurcation. It is observed that a continuation of period doubling bifurcations ascertaining what was narrated in this paper. The outcome of this research is delineated in the above figures. We motivated good graphics competence to pursue in circumstantial other, smaller part in the orbit diagram. It is accomplished that some segments of the orbit diagram as c exacerbate endures another sequence of period doublings bifurcations. The bifurcation diagram for the quadratic families is attached in the upwards figures. Finally, we analyzed the bifurcation diagram of quadratic map and their chaotic and non-chaotic ambiences for isolated parameter values.

Acknowledgements

Thanks all who have helped to complete this research works.

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