Int. J. Adv. Appl. Math. and Mech. 8(2) (2020) 54 – 63 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

On Exact Solutions of Klein-Gordon Equations using the Semi Analytic Iterative Method

Research Article

Christian Kasumo *

Department of Science and Mathematics, Mulungushi University, School of Science, Engineering and Technology, P O Box 80415, Kabwe, Zambia

Received 02 November 2020; accepted (in revised version) 26 November 2020

Abstract: We solve linear and nonlinear Klein-Gordon equations using the semi analytic iterative method and compare the results with those from other methods used in the literature. Numerical examples are given and the results obtained underscore the high accuracy and efficiency of the method since it produces exact to near-exact solutions.

MSC: 74J30 • 81Q05

Keywords: Klein-Gordon equations • Semi-analytic iterative method • Adomian decomposition method • Variational iteration method • Homotopy analysis transform method

1. Introduction

Most nonlinear phenomena that arise in a wide variety of scientific fields such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics can be modelled by partial differential equations (PDEs). This paper is concerned with finding solutions to linear and nonlinear PDEs using the semi analytic iterative method (SAIM) which was first proposed by Temimi and Ansari [1]. This method has been used for solving all kinds of linear and nonlinear ordinary differential equations, PDEs and higher-order integrodifferential equations [1–3]. More recently it has been applied to solution of the KdV equation [4].

In this paper we propose to use the SAIM to solve linear and nonlinear Klein-Gordon equations (KGEs) which occur in relativistic quantum mechanics and field theory. Consider the KGE

$$u_{tt} + \alpha u_{xx} + \beta u + \delta F(u) = G(x,t), \tag{1}$$

subject to the initial conditions

$$u(x,0) = f(x),\ u_t(x,0) = g(x),$$

where $u = u(x,t)$ represents the wave displacement at position x and time t, $F(u)$ is the nonlinear force such that $\dfrac{\partial F}{\partial u} \geq 0$, $G(x,t)$, $f(x)$ and $g(x)$ are known analytic functions and α, β, δ are constants. $F(u)$ usually takes many forms which include u, u^2, u^3, $\sin(u)$, $\sin(u)+\sin(2u)$, $\sinh(u)+\sinh(2u)$ and e^u which characterises the linear KGE, the

* *E-mail address(es):* ckasumo@mu.ac.zm

nonlinear KGE with quadratic and cubic nonlinearity, respectively, the sine-Gordon, the double sine-Gordon, the double sinh-Gordon and Liouville equations, respectively [5].

The KGE in its linear and nonlinear forms has been the subject of much interest from researchers in studying solitons and finding soliton solutions. The KGE is a relativistic wave equation, i.e., it is a quantised version of the energy-momentum relation and has been solved using a variety of methods, e.g., Adomian decomposition method [6], variational iteration method [7], homotopy perturbation method [8], homotopy analysis transform method [9], reduced differential transform method [10], multi-quadric quasi-interpolation scheme [5] and perturbation iteration transform method [11], to name but a few.

The outline of the rest of the paper is as follows: Section 2 reviews the semi analytic iterative method. In Section 3 we apply the SAIM to the solution of five test problems in order to show the validity and efficiency of the method and Section 4 gives some conclusions.

2. Review of the Semi Analytic Iterative Method

The SAIM was used by Yassein [3] to solve higher order integro-differential equations and by Yassein and Aswhad [4] to solve KdV equations. This method uses an iterative approach together with analytical computations to provide a solution of a modified reformulated linear problem. The SAIM was inspired by the homotopy analysis method (HAM) which is a general approximate analytical approach for obtaining convergent series solutions of strongly nonlinear problems [2]. The SAIM offers several advantages over existing methods such as SAM and ADM. The SAIM is very easy to implement in that it avoids the calculation of Adomian polynomials for the nonlinear term in the ADM or Lagrange multipliers in the VIM, thus demanding less computational work [12]. In this paper we propose to use the SAIM to solve linear and nonlinear KGEs. Consider the KGE in (1) with $\alpha = -1$, i.e.,

$$u_{tt} - u_{xx} + \beta u + \delta F(u) = G(x,t), \tag{2}$$

subject to the initial conditions

$$u(x,0) = f(x),\ u_t(x,0) = g(x). \tag{3}$$

If $\delta = 0$, then (2) is linear, otherwise it is nonlinear. Equation (2) can be written as

$$Lu + Nu = G(x,t), \tag{4}$$

with the condition $C\left(u, \frac{\partial u}{\partial t}\right) = 0$, where $Lu = u_{tt}$, $Nu = -u_{xx} + \beta u + \delta F(u)$ and $G(x,t)$ is the source term. The first step of the SAIM is to find the initial approximation by solving

$$L[u_0(x,t)] - G(x,t) = 0 \text{ with } C\left(u_0, \frac{\partial u_0}{\partial t}\right) = 0. \tag{5}$$

The next iteration to the solution can be obtained by solving

$$L[u_1(x,t)] + N[u_0(x,t)] - G(x,t) = 0 \text{ with } C\left(u_1, \frac{\partial u_1}{\partial t}\right) = 0. \tag{6}$$

After several iterations we obtain the general form of the SAIM solution which is

$$L[u_{n+1}(x,t)] + N[u_n(x,t)] - G(x,t) = 0 \text{ with } C\left(u_{n+1}, \frac{\partial u_{n+1}}{\partial t}\right) = 0, \tag{7}$$

from which the general iterative formula for solving the KGE (2) is

$$u_{n+1}(x,t) = u_{n+1}(x,0) + L^{-1}\left[-N[u_n(x,t)] + G(x,t)\right], \tag{8}$$

where $L^{-1} = \int_0^t \int_0^t (\cdot) ds ds$. Each iteration of the function $u_n(x,t)$ effectively represents a complete solution for equation (4). For the homogeneous KGE, $G(x,t) = 0$.

3. Numerical Examples

In this section we present some numerical examples illustrating the applicability of the SAIM for solving linear KGEs as well as nonlinear KGEs with quadratic nonlinearity. All the computations associated with these examples were performed using a Samsung Series 3 PC with an Intel Celeron CPU 847 at 1.10GHz and 6.0GB internal memory. A fixed $t = 0.1$ was used throughout and the figures were constructed using MATLAB R2016a.

Example 3.1.
Consider the linear homogeneous KGE (Kumar et al. [9]; Khalid et al. [11]):

$$u_{tt} - u_{xx} + u = 0, \; u(x,0), \; u_t(x,0) = x, \tag{9}$$

with exact solution $u(x,t) = x\sin t$.

In order to use the SAIM, we need to rewrite (9) in the form

$$Lu + Nu = 0,$$

where $Lu = u_{tt}$ and $Nu = -u_{xx} + u$. The primary problem is to find the initial approximation by solving

$$L[u_0(x,t)] = 0, \text{ with } u_0(x,0) = 0, \; u_{0_t}(x,0) = x. \tag{10}$$

Using the initial conditions, the solution of the primary problem is

$$u_0(x,t) = u(x,0) + tu_t(x,0) = xt.$$

The general recursive relation for solving (9) is

$$L[u_{n+1}(x,t)] = -N[u_n(x,t)], \text{ with } u_{n+1}(x,0), \; u_{n_t}(x,0) = x, \tag{11}$$

i.e.,

$$u_{n+1}(x,t) = u_{n+1}(x,0) + \int_0^t \int_0^t \left[u_{n_{xx}} - u_n\right] dsds. \tag{12}$$

Using this recursive relation, we have the approximations

$$\begin{aligned}
u_0(x,t) &= xt,\\
u_1(x,t) &= xt + \int_0^t \int_0^t \left[u_{0_{xx}} - u_0\right] dsds = x\left(t - \frac{1}{6}t^3\right),\\
u_2(x,t) &= xt + \int_0^t \int_0^t \left[u_{1_{xx}} - u_1\right] dsds = x\left(t - \frac{1}{6}t^3 + \frac{1}{24}t^4\right),\\
u_3(x,t) &= xt + \int_0^t \int_0^t \left[u_{2_{xx}} - u_2\right] dsds = x\left(t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{720}t^6\right),\\
u_4(x,t) &= xt + \int_0^t \int_0^t \left[u_{3_{xx}} - u_3\right] dsds = x\left(t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \frac{1}{40320}t^8\right),
\end{aligned}$$

and so on. Thus, as $n \to \infty$ we obtain the solution

$$u(x,t) = x\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots\right) = x\sin t,$$

which is the exact solution of the given KGE. The results are the same as those obtained using the perturbation iteration transform method of Khalid et al. [11] and are shown in Table 1. Fig. 1 compares the exact and SAIM solutions.

Table 1. Comparison of approximate and exact solutions from SAIM and PITM for Example 3.1 ($t = 0.1$)

x	$u(x,t)$	$u_{\text{SAIM}}(x,t)$	$u_{\text{PITM}}(x,t)$
0	0	0	0
0.2	0.019966683	0.019966683	0.019966683
0.4	0.039933367	0.039933367	0.039933367
0.6	0.059900050	0.059900050	0.059900050
0.8	0.079866733	0.079866733	0.079866733
1.0	0.099833417	0.099833417	0.099833417

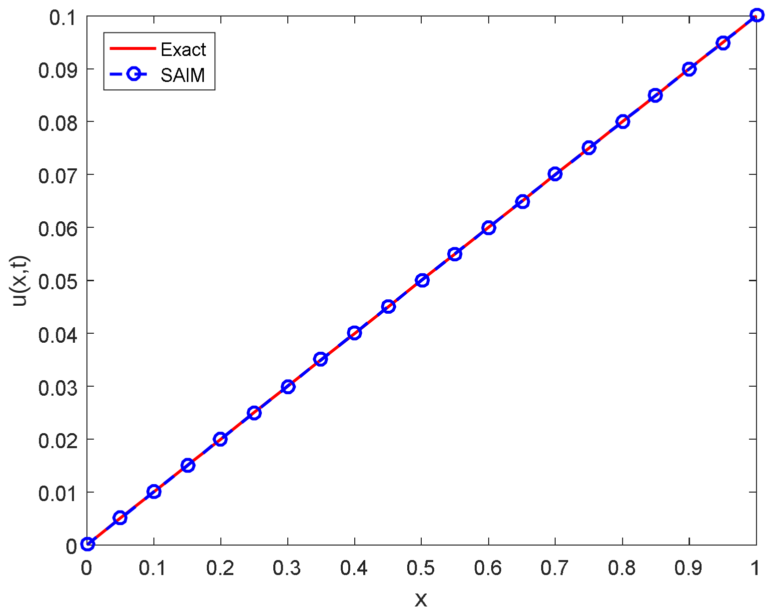

Fig. 1. Comparison of approximate and exact solutions for the linear KGE in Example 3.1 for fixed $t = 0.1$

Example 3.2.
Consider the linear homogeneous KGE (Yusufoğlu [7]):

$$u_{tt} - u_{xx} = u,\ u(x,0) = 1 + \sin x,\ u_t(x,t) = 0 \tag{13}$$

whose exact solution is $u(x,t) = \sin x + \cosh t$. Rewriting (13) as

$$Lu + Nu = 0, \tag{14}$$

where $Lu = u_{tt}$ and $Nu = -u_{xx} - u$, and the general recursive relation

$$L[u_{n+1}(x,t)] = -N[u_n(x,t)],\ \text{with}\ u_{n+1}(x,0) = 1 + \sin x,\ u_{(n+1)_t}(x,0) = 0, \tag{15}$$

we use the iteration

$$u_{n+1}(x,t) = u_{n+1}(x,0) + \int_0^t \int_0^t \left[u_{n_{xx}} + u_n\right] dsds \tag{16}$$

to obtain the successive approximations

$$\begin{aligned}
u_0(x,t) &= 1 + \sin x,\\
u_1(x,t) &= 1 + \sin x + \int_0^t \int_0^t \left[u_{0_{xx}} + u_0\right] dsds = 1 + \sin x + \frac{1}{2}t^2,\\
u_2(x,t) &= 1 + \sin x + \int_0^t \int_0^t \left[u_{1_{xx}} + u_1\right] dsds = 1 + \sin x + \frac{1}{2}t^2 + \frac{1}{24}t^4,\\
u_3(x,t) &= 1 + \sin x + \int_0^t \int_0^t \left[u_{2_{xx}} + u_2\right] dsds = 1 + \sin x + \frac{1}{2}t^2 + \frac{1}{24}t^4 + \frac{1}{720}t^6,
\end{aligned}$$

and so on, i.e.,

$$u(x,t) = \sin x + 1 + \frac{1}{2}t^2 + \frac{1}{24}t^4 + \frac{1}{720}t^6 + \cdots = \sin x + \cosh t,$$

the exact solution. This is the same result obtained using the Adomian decomposition method and variational iteration method [7] and variational homotopy perturbation method [13]. The results are shown in Table 2 and Fig. 2.

Table 2. Comparison of approximate and exact solutions from SAIM and VIM for Example 3.1 ($t = 0.1$)

x	$u(x,t)$	$u_{SAIM}(x,t)$	$u_{VIM}(x,t)$
0	1.005004168	1.005004168	1.005004168
0.2	1.203673499	1.203673499	1.203673499
0.4	1.394422510	1.394422510	1.394422510
0.6	1.569646641	1.569646641	1.569646641
0.8	1.722360259	1.722360259	1.722360259
1.0	1.846475153	1.846475153	1.846475153

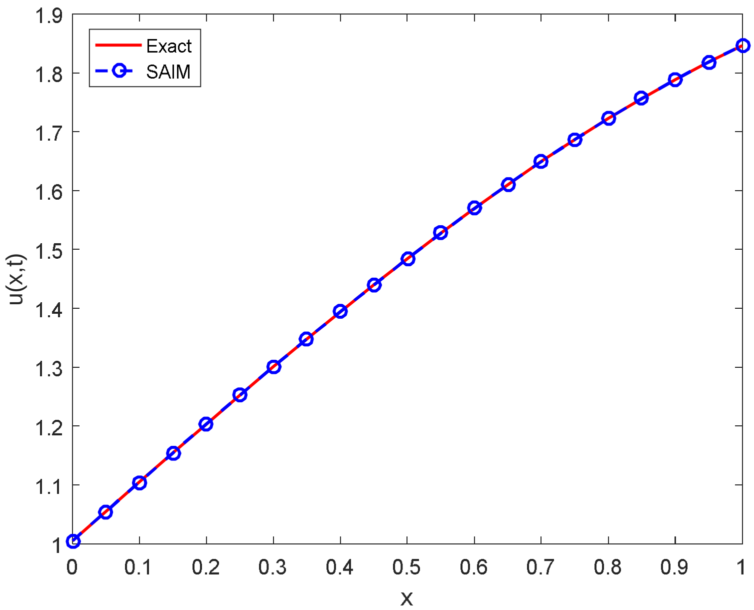

Fig. 2. Comparison of approximate and exact solutions for the linear KGE in Example 3.2 for fixed $t = 0.1$

Example 3.3.
Consider the nonlinear nonhomogeneous KGE with homogeneous initial conditions (Kumar et al. [9]; Sarboland and Aminataei [5]; Khalid et al. [11]):

$$u_{tt} - u_{xx} + u^2 = 6xt(x^2 - t^2) + x^6t^6, \; u(x,0) = 0, \; u_t(x,0) = 0. \tag{17}$$

The exact solution is $u(x,t) = x^3t^3$. Here, $Lu = u_{tt}$, $Nu = u_{xx} - u^2$ and $h(x,t) = 6xt(x^2 - t^2) + x^6t^6$. Knowing that the primary problem $Lu_0 = 0$, with $u_0(x,0) = u_t(x,0) = 0$, has a solution $u_0(x,t) = 0$, equation (17) can be solved using the general iterative scheme

$$u_{n+1}(x,t) = \int_0^t \int_0^t \left[u_{n_{xx}} - u_n^2 + 6xs(x^2 - s^2) + x^6s^6\right] ds\,ds. \tag{18}$$

Thus, the first three approximations are

$$\begin{aligned}
u_0(x,t) &= 0,\\
u_1(x,t) &= \int_0^t \int_0^t \left[u_{0_{xx}} - u_0^2 + 6xs(x^2 - s^2) + x^6s^6\right] ds\,ds,\\
&= x^3t^3 - \frac{1}{10}xt^5 + \frac{1}{56}x^6t^8,\\
u_2(x,t) &= \int_0^t \int_0^t \left[u_{1_{xx}} - u_1^2 + 6xs(x^2 - s^2) + x^6s^6\right] ds\,ds,\\
&= x^3t^3 + \frac{29}{2100}x^5t^{10} - \frac{3}{4400}x^2t^{12} - \frac{1}{4368}x^9t^{13} + \frac{1}{19600}x^7t^{15} - \frac{1}{959616}x^{12}t^{18}.
\end{aligned}$$

This third iteration approximates the exact solution $u(x,t) = x^3t^3$ which was also obtained by Kumar et al. [9], Sarboland and Aminataei [5] and Khalid et al. [11]. Table 3 compares the results from the SAIM and PITM. The exact and SAIM solutions are compared in Fig. 3.

Table 3. Comparison of approximate and exact solutions from SAIM and PITM for Example 3.5 ($t = 0.1$)

x	$u(x,t)$	$u_{SAIM}(x,t)$	$u_{PITM}(x,t)$
0	0	0	0
0.2	0.000008	0.000008	0.000008
0.4	0.000064	0.000064	0.000064
0.6	0.000216	0.000216	0.000216
0.8	0.000512	0.000512	0.000512
1.0	0.001000	0.001000	0.001000

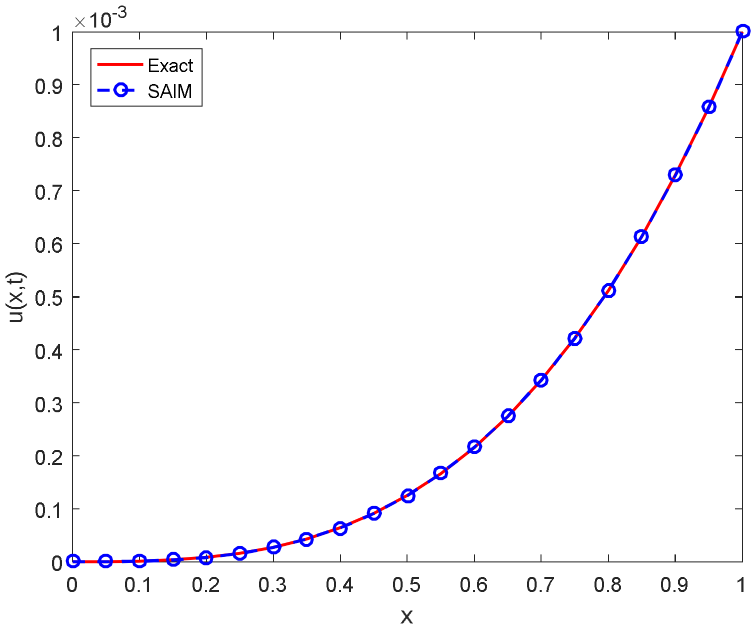

Fig. 3. Comparison of approximate and exact solutions for the nonlinear KGE in Example 3.3 for fixed $t = 0.1$

Example 3.4.
Consider the following nonlinear homogeneous KGE (Yusufoğlu [7]; Yousif and Mahmood [13]):

$$u_{tt} - u_{xx} = -u^2, \; u(x,0) = 1 + \sin x, \; u_t(x,0) = 0. \tag{19}$$

We rewrite this equation as

$$Lu + Nu = 0,$$

with $Lu = u_{tt}$ and $Nu = -u_{xx} + u^2$. With the initial problem yielding the solution $u_0(x,t) = 1 + \sin x$, the first three iterations give the approximations

$$
\begin{aligned}
u_0(x,t) &= 1+\sin x \\
u_1(x,t) &= 1+\sin x + \int_0^t\int_0^t \left[u_{0_{xx}} - u_0^2\right] ds ds, \\
&= 1+\sin x - \frac{1}{2}t^2\left(1+3\sin x+\sin^2 x\right), \\
u_2(x,t) &= \int_0^t\int_0^t \left[u_{1_{xx}} - u_1^2\right] ds ds, \\
&= 1+\sin x - \frac{1}{2}t^2 + \frac{1}{12}t^4\left(1-\cos(2x)\right) - \frac{1}{2}t^2\left(3-\frac{11}{12}t^2+\frac{1}{10}t^4\right)\sin x, \\
&- \frac{1}{2}t^2\left(1-\frac{2}{3}t^2+\frac{11}{60}t^4\right)\sin^2 x + \frac{1}{2}t^4\left(\frac{1}{6}-\frac{1}{10}t^2\right)\sin^3 x - \frac{1}{120}t^6\left(1+\sin^4 x\right).
\end{aligned}
$$

The solution is given by $u(x,t) \approx u_2(x,t)$ which can be compared with the ADM and VIM solutions obtained by Yusufoğlu [7] and the variational homotopy perturbation method solution by Yousif and Mahmood [13]. These results are compared in Table 4 and Fig. 4. The exact solution for this problem is not known. However, because the SAIM gives exact to near-exact solutions it can be taken that the exact solution is approximated by the SAIM solution.

Table 4. Comparison of approximate solutions from SAIM, ADM, VIM and VHPM for Example 3.4 ($t = 0.1$)

x	$u_{SAIM}(x,t)$	$u_{ADM}(x,t)$	$u_{VIM}(x,t)$	$u_{VHPM}(x,t)$
0	0.994999992	0.994999986	0.995000024	0.995002750
0.1	1.093291150	1.093291132	1.093291179	1.093318762
0.2	1.190503066	1.190502988	1.190503087	1.190560110
0.3	1.285668836	1.285668610	1.285668848	1.285759573
0.4	1.377844712	1.377844211	1.377844710	1.377972870
0.5	1.466119236	1.466118315	1.466119219	1.466287819
0.6	1.549621976	1.549620480	1.549621939	1.549833071
0.7	1.627531752	1.627529538	1.627531694	1.627786380
0.8	1.699084324	1.699081273	1.699084244	1.699382315
0.9	1.763579458	1.763575490	1.763579356	1.763919383
1.0	1.820387340	1.820382425	1.820387216	1.820766487

Example 3.5.
Consider the nonlinear nonhomogeneous KGE (Kumar et al. [9]):

$$
u_{tt} - u_{xx} + u^2 = x^2t^2,\ u(x,0) = 0,\ u_t(x,0) = x, \tag{20}
$$

with exact solution $u(x,t) = xt$. We rewrite this equation as

$$
Lu + Nu = h(x,t),
$$

where $Lu = u_{tt}$, $Nu = -u_{xx} + u^2$ and $h(x,t) = x^2t^2$. Using the SAIM, the iterations are

$$
\begin{aligned}
u_0(x,t) &= xt, \\
u_1(x,t) &= xt + \int_0^t\int_0^t \left[u_{0_{xx}} - u_0^2 + x^2t^2\right] ds ds = xt, \\
&\vdots \\
u_{n+1}(x,t) &= xt,\ n \geq 0,
\end{aligned}
$$

i.e., $u(x,t) = xt$, the exact solution also obtained by Kumar et al. [9] using the HATM. However, the SAIM achieves much faster convergence than the HATM. The results are shown in Table 5 and Fig. 5.

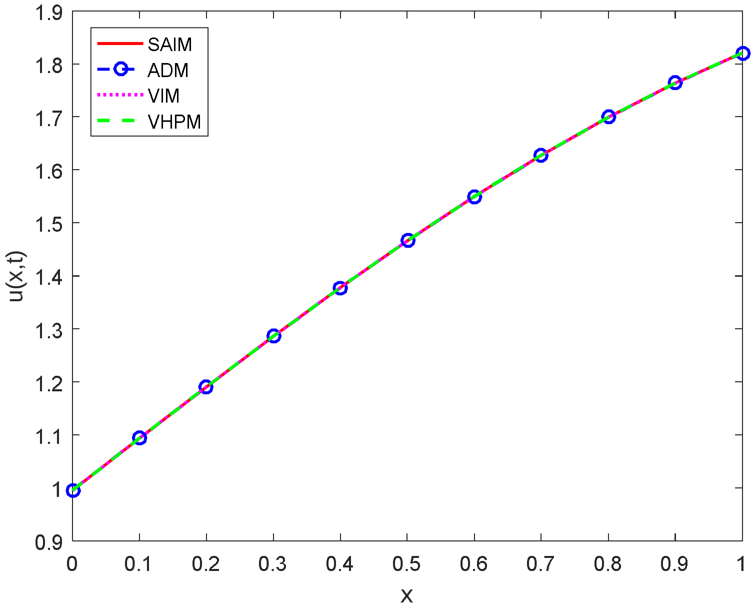

Fig. 4. Comparison of approximate and exact solutions for the nonlinear KGE in Example 3.4 for fixed $t = 0.1$

Table 5. Comparison of approximate and exact solutions from SAIM and HATM for Example 3.5 ($t = 0.1$)

x	$u(x,t)$	$u_{\text{SAIM}}(x,t)$	$u_{\text{HATM}}(x,t)$
0	0	0	0
0.2	0.02	0.02	0.02
0.4	0.04	0.04	0.04
0.6	0.06	0.06	0.06
0.8	0.08	0.08	0.08
1.0	0.1	0.1	0.1

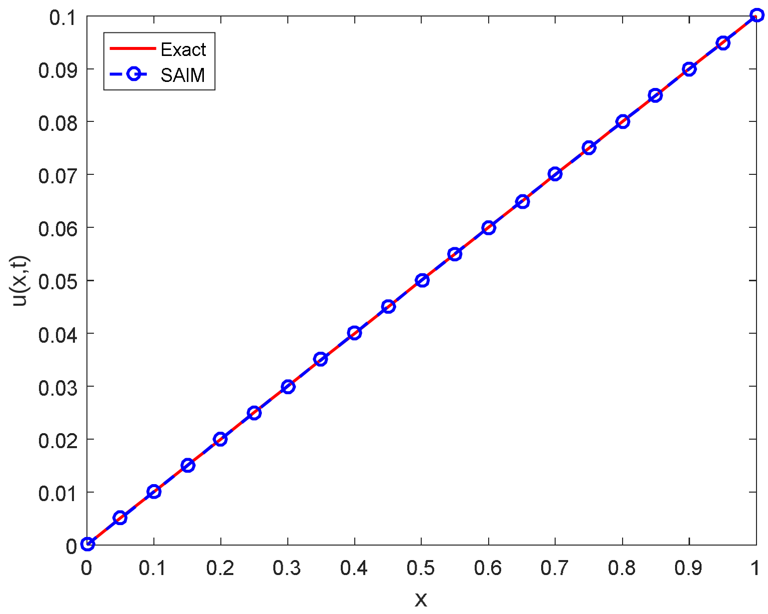

Fig. 5. Comparison of approximate and exact solutions for the nonlinear KGE in Example 3.5 for fixed $t = 0.1$

4. Conclusion

In this work the semi analytic iterative method has been applied to the solution of linear and nonlinear KGEs. The paper has confirmed the suitability of this method for solving these types of PDEs. The method compares favourably with, and in some cases performs better than, other analytical methods used in the literature such as the VIM, ADM, PITM, HPM and HATM. Possible future work includes the use of this method for solving initial and boundary value ODEs and many other linear and nonlinear PDEs.

Acknowledgements

The author gratefully acknowledges the support of Mulungushi University and thanks the Editor and anonymous referees whose valuable comments resulted in significant improvements in the paper.

References

[1] H. Temimi, A. Ansari, A semi analytical iterative technique for solving nonlinear problems, Computers and Mathematics with Applications 61(2) (2011) 203-210.
[2] H. Temimi, A. Ansari, A computational iteration method for solving nonlinear ordinary differential equations, LMS J. Comput. Math. 18(1) (2015) 730-753.
[3] S.M. Yassein, Application of iterative method for solving higher order integro-differential equations, Ibn Al Haitham J. Pure Appl. Sci. 32(2) (2019) 51-61.
[4] S.M. Yassein, A.A. Aswhad, Efficient iterative method for solving Korteweg-de Vries equations, Iraqi Journal of Science 60(7) (2019) 1575-1583.
[5] M. Sarboland, A. Aminataei, Numerical solution of the nonlinear Kelin-Gordon equation using multiquadric quasi-interpolation scheme, Universal J. Appl. Math. 3(3) (2015) 40-49.
[6] M. El-Sayed, The decomposition method for studying the Klein-Gordon equation, J. Chaos Solitons Fractals 18 (2003) 1025-1030.
[7] E. Yusufoğlu, The variational iteration method for studying the Klein-Gordon equation, J. Appl. Math. Letters 21 (2008) 669-674.

[8] M.S.H. Chowdhury, Application of homotopy perturbation method to Klein-Gordon and sine-Gordon equations, J. Chaos Solitons Fractals 39 (2009) 1928-1935.

[9] D. Kumar, J. Singh, S. Kumar, Sushila, Numerical computation of Klein-Gordon equations arising in quantum field theory by using homotopy analysis transform method, Alexandria Eng. J. 53 (2014) 469-474.

[10] A. Kumar, R. Arora, Solutions of the coupled system of Burgers' equations and coupled Klein-Gordon equations by RDT method, Int. J. Adv. Apl. Math. Mech. 1(2) (2013) 133-145.

[11] M. Khalid, M. Sultana, F. Zaidi, A. Uroosa, Solving linear and nonlinear Klein-Gordon equations by new perturbation iteration transform method, TWMS J. Appl. Eng. Math. 6(1) (2016) 115-125.

[12] B. Latif, M.S. Selamat, A.N. Rosli, A.I. Yusoff, N.M. Hasan, The semi analytics iterative method for solving Newell-Whitehead-Segel equation, Math. Stat. 8(2) (2020) 89-94.

[13] M.A. Yousif, B. A. Mahmood, Approximate solutions for solving the Klein-Gordon and sine-Gordon equations, Assoc. Arab. Univ. for Basic Appl. Sci. 22 (2017) 83-90.