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# Continuous Functions via D-Preopen Sets in D-Metric Spaces

Research Article

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**Abstract:** The purpose of this paper is to introduce and investigate weak form of D-continuous function in D-metric spaces, namely D-precontinuous functions. The relationships among this form with the other known functions are introduced. We give the notions of contra and almost D-precontinuous functions via D-preopen sets.

**MSC:** 54A05 • 54E35

**Keywords:** Open set • Metric spaces • continuous functions

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## 1. Introduction

In 1984, Dhage, [3], introduced a new notion of a new structure of D-metric space which is a natural generalization of the notion of ordinary metric space to higher dimensional metric spaces, [4]. In 2000, Dhage, [2], introduced some results in D-metric spaces are obtained and the notion of open and closed balls. In 2017, Ali Fora, Massadeh and Bataineh, [1], introdused and a new topological of D-closed set, D-continuous and D-Fixed point property discussed of its properties, some result for this subject are also established, structure of D-closed set. In 2021, Hussain and Saif, [5], introduce and investigate weak form of D-open sets in D-metric spaces, namely D-preopen sets. The relationships among this form with the other known sets are introduced. They give introduce the notions of the interior operator, the closure operator and frontier operator via D-preopen sets.

This paper is organized as follows: Section 2 is devoted to some preliminaries. Section 3 introduces the concept of D-precontinuous function by utilizing the D-preopen sets. Furthermore, the relationship with the other known sets will be studied. In Section 4 we introduce the concepts of contra and almost D-precontinuous functions via D-preopen sets.

## 2. Preliminaries

**Definition 2.1.**
[1] . Let $f:(X,D)\longrightarrow(Y,P)$ be a function between two D-metric spaces $(X,D)$ and $(Y,P)$. Then $f$ is said to be $D_P$-continuous at $p\in X$ provided that for any sequence $\{x_n\}$ converging in $(X,D)$ to p, then {f($x_n$)} must converge in $(Y,P)$ to $f(p)$. A function $f:(X,D)\longrightarrow(Y,P)$ is called $D_P$-continuous if f is $D_P$-continuous at each p in $X$.

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By $O^D_\varepsilon(x)$, we mean the D-open ball with center $x$ and radius $\varepsilon > 0$, that is,

$$O^D_\varepsilon(x) = \{y \in X : d(x, y, y) < \varepsilon\}.$$

By $C^D_\varepsilon(x)$, we mean the D-closed ball with center $x$ and radius $\varepsilon > 0$, that is,

$$C^D_\varepsilon(x) = \{y \in X : d(x, y, y) \le \varepsilon\}.$$

The set $G \subseteq \bar{X}$ is called an D-open set in D-metric space $(X,D)$ if for every $x \in G$, there is $\varepsilon > 0$ such that $O^D_\varepsilon(x) \subseteq G$. The set $G$ is called D-closed set in D-metric space $(X,D)$ if $X - G$ is an D-open set in D-metric space $(X,D)$. For D-metric space $(X,D)$ and $G \subseteq X$, the interior set of $G$ is denoted by $Int_D(G)$ and the clouser set of $G$ is denoted by $Cl_D(G)$.
For D-metric space $(X,D)$ and $G \subseteq X$ is called a D-preopen set, [5], in D-metric space $(X,D)$ if for every $x \in G$, there is $\delta > 0$ such that for every $y \in O^D_\delta(x)$, $O^D_\varepsilon(y) \cap G \neq \emptyset$ for every $\varepsilon > 0$. The set $G \subseteq X$ is called a D-preclosed set in D-metric space $(X,D)$ if $X - G$ is a D-preopen set in D-metric space $(X,D)$. Recall, [5], that they introduced the interior operator, the closure operator and frontier operator via D-preopen sets. The set $G \subseteq X$, the $D_P$-interior set of $G$ is denoted by $Int^D_P(G)$ and the $D_P$-clouser set of $G$ is denoted by $Cl^D_P(G)$. For a subset $G$ of D-metric space $(X,D)$, the D-frontier operator of G is defined by

$$\Gamma^D_P(G) = Cl^D_P(G) - Int^D_P(G).$$

**Definition 2.2.**
[1]. Let $f : (X,D) \longrightarrow (Y,P)$ be a function between two D-metric spaces $(X,D)$ and $(Y,P)$. Then $f$ is said to be $D_P$-weakly continuous at $p \in X$ provided that for any P-open set $H$ in $Y$ containing $f(p)$, there exists a D-open set $U$ in $X$ containing p such that $f(U) \subseteq H$. A function $f : (X,D) \longrightarrow (Y,P)$ is called $D_P$-weakly continuous if f is $D_P$−weakly continuous at each $p$ in $X$.

**Theorem 2.1.**
*[1]. Let $f : (X,D) \longrightarrow (Y,P)$ be a function between two D-metric spaces $(X,D)$ and $(Y,P)$. If $f$ is $D_P$-continuous function then $f$ is $D_P$−weakly continuous.*

**Theorem 2.2.**
*[2]. The followings are equivalent for the function $f : (X,D) \longrightarrow (Y,P)$ between two D-metric spaces $(X,D)$ and $(Y,P)$.*

1. *$f$ is $D_P$-weakly continuous.*
2. *For any P-open set $H$ in $Y$, $f^{-1}(H)$ is D-open set in $X$.*
3. *For any P-closed set $M$ in $Y$, $f^{-1}(M)$ is D-closed set in $X$.*

Recall, [5], every D-open set is a D-preopen set. For a D-metric space $(X,D)$ and $G \subseteq X$, $Cl^D_P(G)$ is a D-preclosed set and $Int^D_P(G)$ is D-preopen set in D-metric space $(X,D)$. For a D-metric space $(X,D)$ and $G \subseteq X$, $Cl^D_P(G)$ and $Int^D_P(G)$ that $G \subseteq Cl^D_P(G)$ and $Int^D_P(G) \subseteq G$.

**Theorem 2.3.**
*[5]. For a D-metric space $(X,D)$ and $G \subseteq X$, $Int^D_P(G) = G$ if and only if G is a D-preopen set.*

**Theorem 2.4.**
*[5]. For a D-metric space $(X,D)$ and $G \subseteq X$, $Cl^D_P(G) = G$ if and only if G is a D-preclosed set.*

## 3. D-precontinuous functions

**Definition 3.1.**
Let $f : (X,D) \longrightarrow (Y,D')$ be a function between two D-metric spaces $(X,D)$ and $(Y,D')$ is called D-precontinuous if $f^{-1}(U)$ is a D-preopen set in $(X,D)$ for every $D'$-open set $U$ in $Y$.

**Example 3.1.**
Let $f : (R,D) \longrightarrow (R,D)$, be a function between two D-metric spaces $(X,D)$ and $(Y,D')$, where

$$D(x, y, z) = max\{d(x, y), d(y, z), d(z, x)\},$$

and $(R,d)$ is an usual metric space. The function given by

$$f(x)=\begin{cases}1, & x\in Q,\\ 0, & x\notin Q,\end{cases} \tag{1}$$

is D-precontinuous, since for any D-open set in $(R,D)$ we have the following:

$0,1\notin G$, implies that, $f^{-1}(G)=\emptyset$, or $0\in G$, $1\notin G$, implies that, $f^{-1}(G)=IR$, or $1\in G$, $0\notin G$, implies that, $f^{-1}(G)=Q$, or $0,1\in G$, implies that, $f^{-1}(G)=R$.

**Theorem 3.1.**
*Let $f:(X,D)\longrightarrow(Y,D')$ be a function between two D-metric spaces $(X,D)$ and $(Y,D')$ is D-precontinuous if and only if $f^{-1}(F)$ is a D-preclosed set in $(X,D)$ for every $D'$-closed set $F$ in $Y$.*

*Proof.* Let $f:(X,D)\longrightarrow(Y,D')$ be a D-precontinuous and $F$ be any $D'$-closed set in $Y$. Then $f^{-1}(Y-F)=X-f^{-1}(F)$ is a D-preopen set in $(X,D)$, that is, $f^{-1}(F)$ is D-preclosed set in $(X,D)$.
Conversely, suppose that $f^{-1}(F)$ is a D-preclosed set in $(X,D)$ for every $D'$-closed set $F$ in $Y$. Let $U$ be any $D'$-open set in $Y$. Then by the hypothesis, $f^{-1}(Y-U)=X-f^{-1}(U)$ is a D-preclosed set in $(X,D)$, that is, $f^{-1}(U)$ is a D-preopen set in $(X,D)$. Hence $f$ is a D-precontinuous. □

**Theorem 3.2.**
Let $f:(X,D)\longrightarrow(Y,D')$ be a function between two D-metric spaces $(X,D)$ and $(Y,D')$ is a D-precontinuous function if and only if for each $x\in X$ and each $D'$-open set $U$ in $Y$ with $f(x)\in U$, there exists a D-preopen set $V$ in $(X,D)$ such that $x\in V$ and $f(V)\subseteq U$.

*Proof.* Suppose function f is a D-precontinuous function. Let $x\in X$ and $U$ be any $D'$-open set in $Y$ containing $f(x)$. Put $V=f^{-1}(U)$. Since $f$ is a D-precontinuous then $V$ is a D-preopen set in $(X,D)$ such that $x\in V$ and $f(V)\subseteq U$.
Conversely, Let $U$ be any $D'$-open set in $Y$ and $x\in f^{-1}(U)$. Then $f(x)\in U$ and hence by the hypothesis, there exists a D-preopen set $V$ in $(X,D)$ such that $x\in V$ and $f(V)\subseteq U$. Hence $x\in V\subseteq f^{-1}(U)$ and $f^{-1}(U)$ is a D-preopen set in $(X,D)$. That is, $f$ is a D-precontinuous. □

**Theorem 3.3.**
Every D-continuous function is D-precontinuous function.

*Proof.* Let $f:(X,D)\longrightarrow(Y,D')$ is a D-continuous function and $U$ be any $D'$-open set in $Y$. Then $f^{-1}(U)$ is a D-open set in $(X,D)$ and hence $f^{-1}(U)$ is a D-preopen set in $(X,D)$. That is, $f$ is a D-precontinuous function. □

The converse of above theorem need not be true.

**Example 3.2.**
In Example (3.1) f is D-precontinuous but not D-continuous. An open intrival (0,2) is D-open set in $(R,D)$ but $f^{-1}(0,2)=Q$ is not D-open set in $(R,D)$.

**Theorem 3.4.**
Let $f:(X,D)\longrightarrow(Y,D')$ be a function between two D-metric spaces $(X,D)$ and $(Y,D')$. Then $f$ is a D-precontinuous if and only if $f[Cl_P^D(A)]\subseteq Cl_{D'}[f(A)]$ for all $A\subseteq X$.

*Proof.* Let $f$ be a D-precontinuous and $A$ be any subset of $X$. Then $Cl_{D'}[f(A)]$ is a $D'$-closed set in $Y$. Since $f$ is a D-precontinuous then by Theorem (3.1), $f^{-1}Cl_{D'}[f(A)]$ is a D-preclosed set in $(X,D)$. That is,

$$Cl_P^D[f^{-1}[Cl_{D'}[f(A)]]]=f^{-1}[Cl_{D'}[f(A)]].$$

Since $f(A)\subseteq Cl_{D'}[f(A)]$ then $A\subseteq f^{-1}[Cl_{D'}[f(A)]$. This implies,

$$Cl_P^D(A)\subseteq Cl_P^D[f^{-1}[Cl_{D'}[f(A)]]]=f^{-1}[Cl_{D'}[f(A)]].$$

Hence $f[Cl_P^D(A)]\subseteq Cl_{D'}[f(A)]$.
Conversely, let $H$ be any $D'$-closed set in $Y$, that is, $Cl_{D'}(H)=H$. Since $f^{-1}(H)\subseteq X$. Then by the hypothesis,

$$f[Cl_P^D[f^{-1}(H)]]\subseteq Cl_{D'}[f(f^{-1}(H))]\subseteq Cl_{D'}(H)=H.$$

This implies, $Cl_P^D[f^{-1}(H)]\subseteq f^{-1}(H)$. Hence $Cl_P^D[f^{-1}(H)]=f^{-1}(H)$, that is, $f^{-1}(H)$ is a D-preclosed set in $(X,D)$. Therefore $f$ is a D-precontinuous. □

**Theorem 3.5.**
Let $f:(X,D)\longrightarrow(Y,D')$ be a function between two D-metric spaces $(X,D)$ and $(Y,D')$. Then f is D-precontinuous if and only if $Cl_P^D[f^{-1}(B)]\subseteq f^{-1}[Cl_{D'}(B)]$ for all $B\subseteq Y$.

*Proof.* Let f be a D-precontinuous and $B$ be any subset of $Y$. Then $Cl_{D'}(B)$ is a $D'$-closed set in $Y$. Since $f$ is a D-precontinuous then by Theorem(3.1), $f^{-1}[Cl_{D'}(B)]$ is a D-preclosed set in $(X,D)$. That is,

$$Cl_P^D[f^{-1}[Cl_{D'}(B)]]=f^{-1}[Cl_{D'}(B)].$$

Since $B\subseteq Cl_{D'}(B)$ then $f^{-1}(B)\subseteq f^{-1}[Cl_{D'}(B)]$ This implies,

$$Cl_P^D[f^{-1}(B)]\subseteq Cl_P^D[f^{-1}[Cl_{D'}(B)]]=f^{-1}[Cl_{D'}(B)].$$

Hence $Cl_P^D[f^{-1}(B)]\subseteq f^{-1}[Cl_{D'}(B)]$.
Conversely, Let $H$ be any $D'$-closed set in $Y$, that is, $Cl_{D'}(H)=H$. Since $H\subseteq Y$. Then by the hypothesis,

$$Cl_P^D[f^{-1}(H)]\subseteq f^{-1}[Cl_{D'}(H)]=f^{-1}(H).$$

This implies, $Cl_P^D[f^{-1}(H)]\subseteq f^{-1}(H)$. Hence $Cl_P^D[f^{-1}(H)]=f^{-1}(H)$, that is, $f^{-1}(H)$ is a D-preclosed set in $(X,D)$. Hence $f$ is a D-precontinuous. □

**Theorem 3.6.**
Let $f:(X,D)\longrightarrow(Y,D')$ be a function between two D-metric spaces $(X,D)$ and $(Y,D')$. Then $f$ is D-precontinuous if and only if $f^{-1}(Int_{D'}(B))\subseteq Int_P^D[f^{-1}(B)]$ for all $B\subseteq Y$.

*Proof.* Let $f$ be a D-precontinuous and $B$ be any subset of $Y$. Then $Int_{D'}(B)$ is and $D'$-open set in $Y$. Since $f$ is a D-precontinuous then $f^{-1}(Int_{D'}(B))$ is a D-preopen set in $(X,D)$. That is,

$$Int_P^D[f^{-1}[Int_{D'}(B)]]=f^{-1}[Int_{D'}(B)].$$

Since $Int_{D'}(B)\subseteq B$ then $f^{-1}[Int_{D'}(B)]\subseteq f^{-1}(B)$. This implies,

$$f^{-1}[Int_{D'}(B)]=Int_P^D[f^{-1}[Int_{D'}(B)]]\subseteq Int_P^D[f^{-1}[Int_{D'}(B)].$$

Hence $f^{-1}[Int_{D'}(B)]\subseteq Int_P^D[f^{-1}(B)]$. Conversely, let $U$ be any $D'$-open set in $Y$ , that is, $Int_{D'}(U)=U$. Since $U\subseteq Y$. Then by the hypothesis,

$$f^{-1}(U)=f^{-1}[Int_{D'}(U)]\subseteq Int_P^D[f^{-1}(U)].$$

This implies, $f^{-1}(U)\subseteq Int_P^D[f^{-1}(U)]$. Hence $f^{-1}(U)=Int_P^D[f^{-1}(U)]$, that is, $f^{-1}(U)$ is a D-preopen set in $(X,D)$. Hence $f$ is D-precontinuous. □

**Definition 3.2.**
Let $f:(X,D)\longrightarrow(Y,D')$ be a function between two D-metric spaces $(X,D)$ and $(Y,D')$ is called a D-preclosed function if $f(G)$ is a $D'$-closed set in $(Y,D')$ for every D-preclosed set $G$ in $(X,D)$.

**Theorem 3.7.**
Let $f:(X,D)\longrightarrow(Y,D')$ be a function between two D-metric spaces $(X,D)$ and $(Y,D')$ is a D-preclosed function if and only if $Cl_{D'}[f(A)]\subseteq f[Cl_P^D(A)]$ for all $A\subseteq X$.

*Proof.* Suppose that $f$ is a D-preclosed function and $A$ be any subset of $X$. Since $Cl_P^D(A)$ is a D-preclosed set in $(X,D)$ and $f$ is a D-preclosed then $f[Cl_P^D(A)]$ is a $D'$-closed set in $Y$. That is,

$$Cl_{D'}[f[Cl_P^D(A)]]=f[Cl_P^D(A)].$$

Since $A\subseteq Cl_P^D(A)$ then $f(A)\subseteq f[Cl_P^D(A)]$. This implies,

$$Cl_{D'}[f(A)]\subseteq Cl_{D'}[f[Cl_P^D(A)]]=f[Cl_P^D(A)].$$

Hence $Cl_{D'}[f(A)]\subseteq f[Cl_P^D(A)]$.
Conversely, let $F$ be any D-preclosed set in $(X,D)$, that is, $Cl_P^D(F)=F$. Since $F\subseteq X$. Then by the hypothesis,

$$Cl_{D'}[f(F)]\subseteq f[Cl_P^D(F)]=f(F).$$

This implies, $Cl_{D'}[f(F)]\subseteq f(F)$. Hence $Cl_{D'}[f(F)]=f(F)$, that is, $f(F)$ is a $D'$-closed set in $Y$. Hence $f$ is a D-preclosed. □

**Definition 3.3.**
Let $f : (X, D) \longrightarrow (Y, D')$ be a function between two D-metric spaces $(X, D)$ and $(Y, D')$ is called a D-preopen function if $f(G)$ is a $D'$-open set in $(Y, D')$ for every D-preopen set $G$ in $(X, D)$.

**Theorem 3.8.**
Let $f : (X, D) \longrightarrow (Y, D')$ be a function between two D-metric spaces $(X, D)$ and $(Y, D')$ is a D-preopen function if and only if $f[Int_P^D(A)] \subseteq Int_{D'}[f(A)]$ for all $A \subseteq X$.

*Proof.* Suppose that $f$ is a D-preopen function and $A$ be any subset of $X$. Since $Int_P^D(A)$ is a D-preopen set in $(X, D)$ and $f$ is a D-preopen then $f[Int_P^D(A)]$ is a $D'$-open set in $Y$. That is,

$$Int_{D'}[f[Int_P^D(A)]] = f[Int_P^D(A)].$$

Since $Int_P^D(A) \subseteq A$ then $f[Int_P^D(A)] \subseteq f(A)$. This implies,

$$Int_{D'}[f[Int_P^D(A)]] \subseteq Int_{D'}[f(A)] = f[Int_P^D(A)].$$

Hence $f[Int_P^D(A)] \subseteq Int_{D'}[f(A)]$.
Conversely, let $F$ be any D-preopen set in $(X, D)$, that is, $Int_P^D(F) = F$. Since $F \subseteq X$. Then by the hypothesis,

$$f[Int_P^D(F)] \subseteq Int_{D'}[f(F)] = f(F).$$

This implies, $f(F) \subseteq Int_{D'}[f(F)]$. Hence $Int_{D'}[f(F)] = f(F)$, that is, $f(F)$ is a $D'$-open set in $Y$. Hence $f$ is a D-preopen. □

## 4. Contra and almost D-functions

**Definition 4.1.**
Let $f : (X, D) \longrightarrow (Y, D')$ be a function between two D-metric spaces $(X, D)$ and $(Y, D')$ is called contra D-precontinuous function if $f^{-1}(V)$ is a D-preclosed set in $(X, D)$ for every $D'$-open set $V$ in $Y$.

**Theorem 4.1.**
Let $f : (X, D) \longrightarrow (Y, D')$ be a function between two D-metric spaces $(X, D)$ and $(Y, D')$ is contra D-precontinuous if and only if $f^{-1}(F)$ is a D-preopen set in $(X, D)$ for every $D'$-closed set $F$ in $Y$.

*Proof.* Suppose that $f$ is contra D-precontinuous. Let $G$ be any $D'$-closed set in $Y$. Then $Y - G$ is an $D'$-open set in $Y$. Since $f$ is contra D-precontinuous then $X - f^{-1}(G) = f^{-1}(Y - G)$ is a D-preclosed set in $(X, D)$. That is, $f^{-1}(G)$ is a D-preopen set in $(X, D)$. Conversely, Let $G$ be any $D'$-open set in $Y$. Then $Y - G$ is an $D'$-closed set in $Y$. Then by the hypothesis, $f^{-1}(Y - G) = X - f^{-1}(G)$ is a D-preopen set in $(X, D)$. That is, $f^{-1}(G)$ is a D-preclosed set in $(X, D)$. Hence $f$ is contra D-precontinuous. □

**Theorem 4.2.**
Let $f : (X, D) \longrightarrow (Y, D')$ be a function between two D-metric spaces $(X, D)$ and $(Y, D')$ is contra D-precontinuous if and only if for each $x \in X$ and each $D'$-closed set $G$ in $Y$ containing $f(x)$, there is a D-preopen set $U$ in $(X, D)$ containing $x$ such that $f(U) \subseteq G$.

*Proof.* Suppose that $f$ is contra D-precontinuous. Let $x \in X$ and G be a $D'$-closed set in $Y$ containing $f(x)$. Then by the last theorem, $U = f^{-1}(G)$ is a D-preopen set in $(X, D)$. Since $f(x) \in G$ then $x \in f^{-1}(G) = U$ and $f(U) = f(f^{-1}(G)) \subseteq G$.
Conversely, Let $G$ be a $D'$-closed set in $Y$. For each $x \in f^{-1}(G)$, $f(x) \in G$. Then by the hypothesis, there is a D-preopen set $U_x$ in $(X, D)$ containing $x$ such that $f(U_x) \subseteq G$. Therefore, we obtain

$$f^{-1}(G) = \cup\{U_x : x \in f^{-1}(G)\}.$$

Then $f^{-1}(G)$ is a D-preopen set in $(X, D)$. Hence by the last theorem, $f$ is a contra D-precontinuous. □

**Theorem 4.3.**
The set of all points $x$ in $X$ at which $f : (X, D) \longrightarrow (Y, D')$ be a function between two D-metric spaces $(X, D)$ and $(Y, D')$ is not a contra D-precontinuous is identical with the union of the D-frontier of the inverse images of $D'$-closed sets of $Y$ containing $f(x)$.

*Proof.* Suppose that $f$ is not contra D-precontinuous at $x \in X$. Then by Theorem (4.2), there is a $D'$-closed set $G$ in $Y$ containing $f(x)$ such that $f(U) \nsubseteq G$ for all D-preopen set $U$ in $(X,D)$. containing $x$. That is, for all D-preopen set $U$ in $(X,D)$. containing $x$, $f(U) \cap (Y-G) \neq \emptyset$ and this implies

$$U \cap f^{-1}(Y-G) \neq \emptyset.$$

Therefore we have

$$x \in Cl_P^D[f^{-1}(Y-G)] = Cl_P^D[X - f^{-1}(G)].$$

However, since $f(x) \in G$, then

$$x \in f^{-1}(G) \subseteq Cl_P^D[f^{-1}(G)].$$

Then

$$x \in Cl_P^D[X - f^{-1}(G)] \cap Cl_P^D[f^{-1}(G)] = \Gamma_P^D[f^{-1}(G)].$$

Conversely, Suppose that $x \in X$ and $x \in \Gamma_P^D[f^{-1}(G)]$ for some $D'$-closed sets $G$ in $Y$ containing $f(x)$. If $f$ is a contra D-precontinuous at $x$ then there is D-preopen set $U$ in $(X,D)$. containing $x$ such that $f(U) \subseteq G$. Therefore we have $x \in U \subseteq f^{-1}(G)$. That is,

$$x \in Int_P^D[f^{-1}(G)] \subseteq \Gamma_P^D[f^{-1}(G)].$$

This is a contradiction. Hence by Theorem (4.2), $f$ is not contra D-precontinuous at $x$. □

**Definition 4.2.**
The kernel of a subset $A$ of a D-metric space $(X,D)$ is denoted by $Ker(A)$ and $Ker(A) = \cap\{U : A \subseteq U$ and $U$ is a D-open set in $X\}$.

**Lemma 4.1.**
*Let $A$ and $B$ be a subset of a D-metric space $(X,D)$. The following hold*

1. *$x \in$ Ker(A) if and only if $A \cap U \neq \emptyset$ for any D-closed set $U$ containing $x$.*
2. *$A \subseteq$ Ker(A) and A=Ker(A) if $A$ is a D-open set in $X$.*
3. *If $A \subseteq B$ then Ker(A) $\subseteq$ Ker(B).*

*Proof.*
1. Let V be any D-closed subset of X, containing x. Suppose that $A \cap V = \emptyset$. Then $A \subseteq X - V$. Sence $X - V$ is D-open set, then Ker(A)$\subseteq X - V$. Hence $x \in X - V$ but this is controdiction. Therefore $A \cap V \neq \emptyset$. Conversely, Suppose that $x \notin$Ker(A). Then there is at least D-open set containg A and $x \notin V$. Then $X - V$ is D-closed set containing $x$ and $A \cap (X - V) = \emptyset$. This is contradiction with the hypothesis. Hence $x \in$Ker(A).
2. By the definition of Ker(A), we get that $A \subseteq$Ker(A). If $A$ is D-open set, then A is the smallest D-open set containing $A$. That is, Ker(A)= $A$.
3. Let $x \in$Ker(A). Then by the part (1), $A \cap V \neq \emptyset$ for any D-closed set $V$ containg $x$. Sence $A \subseteq B$, then $B \cap V \neq \emptyset$. Hence $x \in$ Ker(B). That is, Ker(A)$\subseteq$ Ker(B).

□

**Theorem 4.4.**
Let $f : (X,D) \longrightarrow (Y,D')$ be a function between two D-metric spaces $(X,D)$ and $(Y,D')$ is a contra D-precontinuous if and only if $f[Cl_P^D[(A)]] \subseteq Ker[f(A)]$ for every subset $A$ of $X$.

*Proof.* Suppose that f is a contra D-precontinuous. Let $A$ be any subset of $X$. Let $y \notin Ker[f(A)]$. Then by Lemma (4.1), there is a $D'$-closed set $F$ in $Y$ containing $y$ such that $f(A) \cap F = \emptyset$. Then $A \cap f^{-1}(F) = \emptyset$. Since $F$ is a $D'$-closed set in $Y$ and $f$ is a contra D-precontinuous then by Theorem (4.1), $f^{-1}(F)$ is a D-preopen set in $(X,D)$. Then $X - f^{-1}(F)$ is a D-preclosed set in $(X,D)$, that is,

$$Cl_P^D[X - f^{-1}(F)] = X - f^{-1}(F).$$

Since $A \cap f^{-1}(F) = \emptyset$. then $A \subseteq X - f^{-1}(F)$, this implies,

$$Cl_P^D(A) \subseteq Cl_P^D[X - f^{-1}(F)] = X - f^{-1}(F).$$

Hence $Cl_P^D(A) \cap f^{-1}(F) = \emptyset$. Then $f[Cl_P^D(A)] \cap F = \emptyset$. Hence $y \notin f[Cl_P^D(A)]$. Therefore $f[Cl_P^D(A)] \subseteq Ker[f(A)]$.
Conversely, Suppose that $f[Cl_P^D(A)] \subseteq Ker[f(A)]$ for every subset A of $X$. Let $V$ be any $D'$-open subset of $Y$. Then $f^{-1}(V) \subseteq X$. Then by the hypothesis, $f[Cl_P^D(f^{-1}(V))] \subseteq Ker[f(f^{-1}(V))]$. Since $V$ is a $D'$-open in $Y$ then by Lemma (4.1),

$$f[Cl_P^D(f^{-1}(V))] \subseteq Ker[f(f^{-1}(V))] \subseteq Ker(V) = V.$$

This implies, $Cl_P^D(f^{-1}(V)) \subseteq f^{-1}(V)$, that is, $f^{-1}(V)$ is a D-preclosed in $(X,D)$. Hence $f$ is contra D-precontinuous. □

**Theorem 4.5.**
Let $f : (X,D) \longrightarrow (Y,D')$ be a function between two D-metric spaces $(X,D)$ and $(Y,D')$ is a contra D-precontinuous if and only if $Cl_P^D[f^{-1}(B)] \subseteq f^{-1}[Ker(B)]$ for every subset $B$ of $Y$.

*Proof.* Suppose that $f$ is a contra D-precontinuous. Let $B$ be any subset of $Y$. Since $f^{-1}(B)$ is a subset of $X$ and $f$ is a contra D-precontinuous then by the last theorem, $f[Cl_P^D[f^{-1}(B)]] \subseteq Ker[f(f^{-1}(B))]$. This implies, then

$$f[Cl_P^D[f^{-1}(B)]] \subseteq Ker[f(f^{-1}(B))] \subseteq Ker(B).$$

Hence $Cl_P^D[f^{-1}(B)] \subseteq f^{-1}[Ker(B)]$.
Conversely, Suppose that $Cl_P^D[f^{-1}(B)] \subseteq f^{-1}[Ker(B)]$ for every subset $B$ of $Y$. Let $V$ be any open subset of $Y$. By the hypothesis and Lemma (4.1),

$$Cl_P^D[f^{-1}(V)] \subseteq f^{-1}[Ker(V)] \subseteq f^{-1}(V).$$

That is, $Cl_P^D[f^{-1}(V)] = f^{-1}(V)$ and so $f^{-1}(V)$ is a D-preclosed in $(X,D)$. Therefore $f$ is a contra D-precontinuous. □

**Definition 4.3.**
A subset A of a D-metric space $(X,D)$ is called r-open set if $A = Int_D(Cl_D(A))$. The complement of r-open set called r-closed set. A subset of a D-metric space is called a D-preclopen set if it is both D-preopen and D-preclosed set.

**Definition 4.4.**
Let $f : (X,D) \longrightarrow (Y,D')$ be a function between two D-metric spaces $(X,D)$ and $(Y,D')$ is called:

1. almost D-precontinuous if for each $x \in X$ and each open set $V$ in $Y$ containing $f(x)$, there is a D-preopen set $U$ in $(X,D)$ containing $x$ such that $f(U) \subseteq Int_{D'}[Cl_{D'}[f(U)]]$.
2. almost contra D-precontinuous function if $f^{-1}(V)$ is a D-preclosed set in $(X,D)$ for every r-open set $V$ in $Y$.
3. weakly D-precontinuous function, if for each $x \in X$ and each $D'$-open set $V$ in $Y$ containing $f(x)$, there is a D-preopen set $U$ in $(X,D)$ containing $x$ such that $f(U) \subseteq Cl_{D'}(V)$.

**Theorem 4.6.**
Let $f : (X,D) \longrightarrow (Y,D')$ be a function between two D-metric spaces $(X,D)$ and $(Y,D')$ is almost contra D-precontinuous if and only if $f^{-1}(F)$ is a D-preopen set in $(X,D)$ for every r-closed set $F$ in $Y$.

*Proof.* It is clear. □

**Theorem 4.7.**
Let $f : (X,D) \longrightarrow (Y,D')$ be a function between two D-metric spaces $(X,D)$ and $(Y,D')$ is almost contra D-precontinuous if and only if for each $x \in X$ and each r-closed set $F$ in $Y$ containing $f(x)$, there is a D-preopen set $U$ in $(X,D)$ containing $x$ such that $f(U) \subseteq F$.

*Proof.* Suppose that $f$ is almost contra D-precontinuous. Let $x \in X$ and $F$ be a r-closed set in $Y$ containing $f(x)$. Then by the last theorem, $U = f^{-1}(F)$ is a D-preopen set in $(X,D)$. Since $f(x) \in F$ then $x \in f^{-1}(F) = U$ and $f(U) = f(f^{-1}(F)) \subseteq F$.
Conversely, Let $F$ be a r-closed set in $Y$. For each $x \in f^{-1}(F)$, $f(x) \in F$. Then by the hypothesis, there is a D-preopen set $U_x$ in $(X,D)$ containing $x$ such that $f(U_x) \subseteq F$. Therefore, we obtain

$$f^{-1}(F) = \{U_x : x \in f^{-1}(F)\}.$$

Then $f^{-1}(F)$ is a D-preopen set in $(X,D)$. Hence by the last theorem, f is an almost contra D-precontinuous. □

It is clear that every contra D-precontinuous function is almost contra D-precontinuous, since every r-preopen set is open.

**Theorem 4.8.**
Every almost contra D-precontinuous function is a weakly D-precontinuous.

*Proof.* Let $f:(X,D)\longrightarrow(Y,D')$ be almost contra D-precontinuous. Let $x\in X$ be any point in $(X,D)$ and $V$ be any $D'$-open set in $Y$ containing $f(x)$. Then

$$Cl_{D'}(V)=Cl_{D'}[Int_{D'}(V)]\subseteq Cl_{D'}[Int_{D'}(Cl_{D'}(V)).$$

$$Cl_{D'}[Int_{D'}(Cl_{D'}(V))]\subseteq Cl_{D'}[Cl_{D'}(V)]=Cl_{D'}(V),$$

this implies, $Cl_{D'}(V)=Cl_{D'}[Int_{D'}(Cl_{D'}(V))]$. That is, $Cl_{D'}(V)$ is r-closed set in $Y$ containing $f(x)$. Since $f$ is almost contra D-precontinuous then by Theorem (4.7), there is a D-preopen set $U$ in $(X,D)$ containing $x$ such that $f(U)\subseteq Cl_{D'}(V)$. That is, $f$ is a weakly D-precontinuous. □

The converse of the last theorem need not be true.

**Theorem 4.9.**
Let $f:(X,D)\longrightarrow(Y,D')$ be a function between two D-metric spaces $(X,D)$ and $(Y,D')$ is almost contra D-precontinuous if and only if for each $x\in X$ and each r-open set $V$ in $Y$ non-containing $f(x)$, there is a D-preclosed set $U$ in $(X,D)$ non-containing $x$ such that $f^{-1}(V)\subseteq U$.

*Proof.* Suppose that $f$ is almost contra D-precontinuous. Let $x\in X$ and $V$ be a r-open set in $Y$ non-containing $f(x)$. Then $Y-V$ is a r-closed set in $Y$ containing $f(x)$. Then by Theorem (4.7), there is a D-preopen set $G$ in $(X,D)$ containing $x$ such that $f(G)\subseteq Y-V$. That is, $U=X-G$ is a D-preclosed set in $(X,D)$ non-containing $x$ and so

$$G\subseteq f^{-1}(X-G)\subseteq f^{-1}(Y-V)=X-f^{-1}(V).$$

Hence $f^{-1}(V)\subseteq X-G=U$.
Conversely, Let $x\in X$ any point in $X$ and $F$ be any r-closed set in $Y$ containing $f(x)$. Y-F is r-open set in $Y$ non-containing $f(x)$. Then by the hypothesis, there is a D-preclosed set $G$ in $(X,D)$ non-containing $x$ such that $f^{-1}(Y-F)\subseteq G$. Hence $U=X-G$ is a D-preopen set in $(X,D)$ containing $x$ and

$$f(U)=f(X-G)\subseteq f[X-f^{-1}(Y-F))]=f[f^{-1}(F)]\subseteq F.$$

Then by Theorem (4.7), f is an almost contra D-precontinuous □

**Theorem 4.10.**
If a function $f:(X,D)\longrightarrow(Y,D')$ is a D-precontinuous and contra D-precontinuous then $f$ is an almost D-precontinuous.

*Proof.* Let $x\in X$ be any point in $X$ and $V$ be any $D'$-open set in $Y$ containing $f(x)$. Since $f$ ia contra D-precontinuous and $Cl_{D'}(V)$ be a $D'$-closed set in $Y$ containing $f(x)$ then by Theorem (4.7), there is a D-preopen set $U$ in $(X,D)$ containing $x$ such that $f(U)\subseteq Cl_{D'}(V)$. Since $f$ is a D-preopen and $U$ is a D-preopen set in $(X,D)$ then $f(U)$ is $D'$-open set in $Y$ and

$$f(U)=Int_{D'}[f(U)]\subseteq Int_{D'}[Cl_{D'}(f(U))]\subseteq Int_{D'}[Cl_{D'}(V)].$$

This shows that $f$ is an almost D-precontinuous. □

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