Int. J. Adv. Appl. Math. and Mech. 8(3) (2021) 24 – 40 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

Properties of Generalized Sixth Order Jacobsthal Sequence

Research Article

Yüksel Soykan*, **Evren Eyican Polatlı**

Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey

Received 22 February 2021; accepted (in revised version) 04 March 2021

Abstract: In this paper, we investigate the generalized sixth order Jacobsthal sequences and we deal with, in detail, six special cases which we call them as sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas sequences.

MSC: 11B37 • 11B39 • 11B83

Keywords: Jacobsthal numbers • sixth order Jacobsthal numbers • sixth order Jacobsthal-Lucas numbers • Hexanacci numbers.

1. Introduction

The Jacobsthal sequence (sequence A001045 in [25]) $\{J_n\}$ is defined recursively by the equation, for $n \geq 0$

$$J_{n+2} = J_{n+1} + 2J_n$$

with the initial conditions $J_0 = 0$ and $J_1 = 1$.This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1],[2],[3],[10],[11],[14],[15],[16],[17],[18],[20],[21],[22]]. For higher order Jacobsthal sequences, see [4],[5],[6],[7],[8],[9],[13],[31],[32].

In this paper, we introduce the generalized sixth order Jacobsthal sequences and we investigate, in detail, six special cases which we call them sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas sequences. First we recall the definition and some properties of a generalized Hexanacci sequence.

The generalized Hexanacci sequence (or the generalized (r,s,t,u,v,y) sequence or 6-step Fibonacci sequence) $\{W_n(W_0,W_1,W_2,W_3,W_4,W_5;r,s,t,u,v,y)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined by the sixth order recurrence relation

$$\begin{aligned} W_n &= rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}+vW_{n-5}+yW_{n-6}, \\ W_0 &= c_0, W_1=c_1, W_2=c_2,\ W_3=c_3, W_4=c_4, W_5=c_5,\ n\geq 6 \end{aligned} \tag{1}$$

where W_0,W_1,W_2,W_3,W_4,W_5 are arbitrary real or complex numbers and r,s,t,u,v,y are real numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{v}{y}W_{-n+1}-\frac{u}{y}W_{-n+2}-\frac{t}{y}W_{-n+3}-\frac{s}{y}W_{-n+4}-\frac{r}{y}W_{-n+5}+\frac{1}{y}W_{-n+6}$$

* Corresponding author.

E-mail address(es): yuksel_soykan@hotmail.com (Yüksel Soykan), evreneyican@hotmail.com (Evren Eyican Polatl)

for $n = 1, 2, 3, \ldots$ when $y \neq 0$. Therefore, recurrence (1) holds for all integers n. Generalized Hexanacci sequence has been studied by many authors, see for example [23],[24],[28],[29],[30].

As $\{W_n\}$ is a sixth order recurrence sequence (difference equation), it's characteristic equation is

$$x^6 - rx^5 - sx^4 - tx^3 - ux^2 - vx - y = 0 \tag{2}$$

whose roots are $\alpha, \beta, \gamma, \delta, \lambda, \mu$.

Note that we have the following identities:

$\alpha + \beta + \gamma + \delta + \lambda + \mu = r$,

$\alpha\beta + \alpha\lambda + \alpha\gamma + \alpha\mu + \beta\lambda + \alpha\delta + \beta\gamma + \beta\mu + \lambda\gamma + \lambda\mu + \beta\delta + \lambda\delta + \gamma\mu + \gamma\delta + \mu\delta = -s$,

$\alpha\beta\lambda + \alpha\beta\gamma + \alpha\beta\mu + \alpha\lambda\gamma + \alpha\lambda\mu + \alpha\beta\delta + \alpha\lambda\delta + \alpha\gamma\mu + \beta\lambda\gamma + \beta\lambda\mu + \alpha\gamma\delta + \alpha\mu\delta + \beta\lambda\delta + \beta\gamma\mu + \lambda\gamma\mu + \beta\gamma\delta + \beta\mu\delta + \lambda\gamma\delta + \lambda\mu\delta + \gamma\mu\delta = t$,

$\alpha\beta\lambda\gamma + \alpha\beta\lambda\mu + \alpha\beta\lambda\delta + \alpha\beta\gamma\mu + \alpha\lambda\gamma\mu + \alpha\beta\gamma\delta + \alpha\beta\mu\delta + \alpha\lambda\gamma\delta + \alpha\lambda\mu\delta + \beta\lambda\gamma\mu + \alpha\gamma\mu\delta + \beta\lambda\gamma\delta + \beta\lambda\mu\delta + \beta\gamma\mu\delta + \lambda\gamma\mu\delta = -u$

$\alpha\beta\lambda\gamma\mu + \alpha\beta\lambda\gamma\delta + \alpha\beta\lambda\mu\delta + \alpha\beta\gamma\mu\delta + \alpha\lambda\gamma\mu\delta + \beta\lambda\gamma\mu\delta = v$,

$\alpha\beta\lambda\gamma\mu\delta = -y$.

Throughout the paper, we use the following notations interchangeable.

$$r = r_1, s = r_2, t = r_3, u = r_4, v = r_5, y = r_6$$

and

$$\alpha = \alpha_1, \beta = \alpha_2, \gamma = \alpha_3, \delta = \alpha_4, \lambda = \alpha_5, \mu = \alpha_6.$$

So (1) and (2) can be written as follows:

$$W_n = r_1 W_{n-1} + r_2 W_{n-2} + r_3 W_{n-3} + r_4 W_{n-4} + r_5 W_{n-5} + r_6 W_{n-6}$$

and

$$x^6 - r_1 x^5 - r_2 x^4 - r_3 x^3 - r_4 x^2 - r_5 x - r_6 = 0$$

respectively.

Generalized Hexanacci numbers can be expressed, for all integers n, using Binet's formula.

Theorem 1.1.
[[28]](Binet's formula of generalized (r, s, t, u, v, y) numbers (generalized Hexanacci numbers))

$$\begin{aligned} W_n = {} & \frac{p_1 \alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)} + \frac{p_2 \beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\ & + \frac{p_3 \gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)} + \frac{p_4 \delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\ & + \frac{p_5 \lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)} + \frac{p_6 \mu^n}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)} \end{aligned} \tag{3}$$

where

$$\begin{aligned} p_1 = {} & W_5 - (\beta+\gamma+\delta+\lambda+\mu)W_4 + (\beta\lambda+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\beta\delta+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3 \\ & -(\beta\lambda\gamma+\beta\lambda\mu+\beta\lambda\delta+\beta\gamma\mu+\lambda\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta)W_2 \\ & +(\beta\lambda\gamma\mu+\beta\lambda\gamma\delta+\beta\lambda\mu\delta+\beta\gamma\mu\delta+\lambda\gamma\mu\delta)W_1 - \beta\lambda\gamma\mu\delta W_0, \end{aligned}$$

$$\begin{aligned} p_2 = {} & W_5 - (\alpha+\gamma+\delta+\lambda+\mu)W_4 + (\alpha\lambda+\alpha\gamma+\alpha\mu+\alpha\delta+\lambda\gamma+\lambda\mu+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3 \\ & -(\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\lambda\delta+\alpha\gamma\mu+\alpha\gamma\delta+\alpha\mu\delta+\lambda\gamma\mu+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta)W_2 \\ & +(\alpha\lambda\gamma\mu+\alpha\lambda\gamma\delta+\alpha\lambda\mu\delta+\alpha\gamma\mu\delta+\lambda\gamma\mu\delta)W_1 - \alpha\lambda\gamma\mu\delta W_0, \end{aligned}$$

$$\begin{aligned} p_3 = {} & W_5 - (\alpha+\beta+\delta+\lambda+\mu)W_4 + (\alpha\beta+\alpha\lambda+\alpha\mu+\beta\lambda+\alpha\delta+\beta\mu+\lambda\mu+\beta\delta+\lambda\delta+\mu\delta)W_3 \\ & -(\alpha\beta\lambda+\alpha\beta\mu+\alpha\lambda\mu+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\mu+\alpha\mu\delta+\beta\lambda\delta+\beta\mu\delta+\lambda\mu\delta)W_2 \\ & +(\alpha\beta\lambda\mu+\alpha\beta\lambda\delta+\alpha\beta\mu\delta+\alpha\lambda\mu\delta+\beta\lambda\mu\delta)W_1 - \alpha\beta\lambda\mu\delta W_0, \end{aligned}$$

$$\begin{aligned} p_4 &= W_5-(\alpha+\beta+\gamma+\lambda+\mu)W_4+(\alpha\beta+\alpha\lambda+\alpha\gamma+\alpha\mu+\beta\lambda+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\gamma\mu)W_3 \\ &\quad -(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\beta\mu+\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\gamma\mu+\beta\lambda\gamma+\beta\lambda\mu+\beta\gamma\mu+\lambda\gamma\mu)W_2 \\ &\quad +(\alpha\beta\lambda\gamma+\alpha\beta\lambda\mu+\alpha\beta\gamma\mu+\alpha\lambda\gamma\mu+\beta\lambda\gamma\mu)W_1-\alpha\beta\lambda\gamma\mu W_0, \end{aligned}$$

$$\begin{aligned} p_5 &= W_5-(\alpha+\beta+\gamma+\delta+\mu)W_4+(\alpha\beta+\alpha\gamma+\alpha\mu+\alpha\delta+\beta\gamma+\beta\mu+\beta\delta+\gamma\mu+\gamma\delta+\mu\delta)W_3 \\ &\quad -(\alpha\beta\gamma+\alpha\beta\mu+\alpha\beta\delta+\alpha\gamma\mu+\alpha\gamma\delta+\alpha\mu\delta+\beta\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\gamma\mu\delta)W_2 \\ &\quad +(\alpha\beta\gamma\mu+\alpha\beta\gamma\delta+\alpha\beta\mu\delta+\alpha\gamma\mu\delta+\beta\gamma\mu\delta)W_1-\alpha\beta\gamma\mu\delta W_0, \end{aligned}$$

$$\begin{aligned} p_6 &= W_5-(\alpha+\beta+\gamma+\delta+\lambda)W_4+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)W_3 \\ &\quad -(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)W_2 \\ &\quad +(\alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta)W_1-\alpha\beta\lambda\gamma\delta W_0. \end{aligned}$$

Usually, it is customary to choose $r_1, r_2, r_3, r_4, r_5, r_6$ so that the Equ. (2) has at least one real (say α_1) solutions.

Note that the Binet form of a sequence satisfying (2) for non-negative integers is valid for all integers n, (see [19], this result of Howard and Saidak [19] is even true in the case of higher-order recurrence relations).

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 1.1.
[[28]]Suppose that $f_{W_n}(x)=\sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized (r,s,t,u,v,y) sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by

$$\sum_{n=0}^{\infty} W_n x^n=\frac{\Lambda}{1-rx-sx^2-tx^3-ux^4-vx^5-yx^6}=\frac{\Lambda}{1-r_1x-r_2x^2-r_3x^3-r_4x^4-r_5x^5-r_6x^6} \tag{4}$$

where

$$\begin{aligned} \Lambda &= W_0+(W_1-rW_0)x+(W_2-rW_1-sW_0)x^2+(W_3-rW_2-sW_1-tW_0)x^3 \\ &\quad +(W_4-rW_3-sW_2-tW_1-uW_0)x^4+(W_5-rW_4-sW_3-tW_2-uW_1-vW_0)x^5 \end{aligned}$$

or (in short formula)

$$\begin{aligned} \Lambda &= W_0+(W_1-r_1W_0)x+(W_2-r_1W_1-r_2W_0)x^2+(W_3-r_1W_2-r_2W_1-r_3W_0)x^3 \\ &\quad +(W_4-r_1W_3-r_2W_2-r_3W_1-r_4W_0)x^4+(W_5-r_1W_4-r_2W_3-r_3W_2-r_4W_1-r_5W_0)x^5 \\ &= W_0+\sum_{i=1}^{6-1} x^i\left(W_i-\sum_{j=1}^{i} r_j W_{i-j}\right). \end{aligned}$$

We next give another form of Binet's formula of generalized (r,s,t,u,v,y) numbers $\{W_n\}$ by the use of generating function for W_n.

Theorem 1.2.
[[28]](Binet's formula of generalized (r,s,t,u,v,y) numbers)

$$W_n=\sum_{k=1}^{6}\frac{q_k\alpha_k^n}{\prod_{\substack{j=1\\ k\neq j}}^{6}(\alpha_k-\alpha_j)} \tag{5}$$

where

$$q_l=W_0\alpha_l^{6-1}+\sum_{i=1}^{6-1}\alpha_l^{6-1-i}\left[W_i-\sum_{j=1}^{i} r_j W_{i-j}\right],\quad 1\leq l\leq m=6.$$

In this paper we consider the case $r_1 = r_2 = r_3 = r_4 = r_5 = 1, r_6 = 2$ and in this case we write $V_n = W_n$. A generalized sixth order Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4, V_5)\}_{n\geq 0}$ is defined by the sixth order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} + V_{n-5} + 2V_{n-6} \tag{6}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3, V_4 = c_4, V_5 = c_5$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} - \frac{1}{2}V_{-(n-2)} - \frac{1}{2}V_{-(n-3)} - \frac{1}{2}V_{-(n-4)} - \frac{1}{2}V_{-(n-5)} + \frac{1}{2}V_{-(n-6)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (6) holds for all integer n.

As $\{V_n\}$ is a sixth order recurrence sequence (difference equation), it's characteristic equation is

$$x^6 - x^5 - x^4 - x^3 - x^2 - x - 2 = (x-2)(x+1)(x^4 + x^2 + 1) = 0. \tag{7}$$

The roots $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and θ_6 of Equation (7) are given by

$$\begin{aligned} \theta_1 &= \alpha = 2, \\ \theta_2 &= \beta = -1, \\ \theta_3 &= \gamma = \frac{1 + i\sqrt{3}}{2}, \\ \theta_4 &= \delta = \frac{1 - i\sqrt{3}}{2}, \\ \theta_5 &= \lambda = \frac{-1 + i\sqrt{3}}{2}, \\ \theta_6 &= \mu = \frac{-1 - i\sqrt{3}}{2}. \end{aligned}$$

Note that we have the following identities:

$\alpha + \beta + \gamma + \delta + \lambda + \mu = 1$,

$\alpha\beta + \alpha\lambda + \alpha\gamma + \alpha\mu + \beta\lambda + \alpha\delta + \beta\gamma + \beta\mu + \lambda\gamma + \lambda\mu + \beta\delta + \lambda\delta + \gamma\mu + \gamma\delta + \mu\delta = -1$,

$\alpha\beta\lambda + \alpha\beta\gamma + \alpha\beta\mu + \alpha\lambda\gamma + \alpha\lambda\mu + \alpha\beta\delta + \alpha\lambda\delta + \alpha\gamma\mu + \beta\lambda\gamma + \beta\lambda\mu + \alpha\gamma\delta + \alpha\mu\delta + \beta\lambda\delta + \beta\gamma\mu + \lambda\gamma\mu + \beta\gamma\delta + \beta\mu\delta + \lambda\gamma\delta + \lambda\mu\delta + \gamma\mu\delta = 1$,

$\alpha\beta\lambda\gamma + \alpha\beta\lambda\mu + \alpha\beta\lambda\delta + \alpha\beta\gamma\mu + \alpha\lambda\gamma\mu + \alpha\beta\gamma\delta + \alpha\beta\mu\delta + \alpha\lambda\gamma\delta + \alpha\lambda\mu\delta + \beta\lambda\gamma\mu + \alpha\gamma\mu\delta + \beta\lambda\gamma\delta + \beta\lambda\mu\delta + \beta\gamma\mu\delta + \lambda\gamma\mu\delta = -1$

$\alpha\beta\lambda\gamma\mu + \alpha\beta\lambda\gamma\delta + \alpha\beta\lambda\mu\delta + \alpha\beta\gamma\mu\delta + \alpha\lambda\gamma\mu\delta + \beta\lambda\gamma\mu\delta = 1$,

$\alpha\beta\lambda\gamma\mu\delta = -2$.

The first few generalized sixth order Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 2.

Table 1. A few generalized sixth order Jacobsthal numbers

n	V_n	V_{-n}
0	V_0	...
1	V_1	$\frac{1}{2}V_5 - \frac{1}{2}V_1 - \frac{1}{2}V_2 - \frac{1}{2}V_3 - \frac{1}{2}V_4 - \frac{1}{2}V_0$
2	V_2	$\frac{3}{4}V_4 - \frac{1}{4}V_1 - \frac{1}{4}V_2 - \frac{1}{4}V_3 - \frac{1}{4}V_0 - \frac{1}{4}V_5$
3	V_3	$\frac{7}{8}V_3 - \frac{1}{8}V_1 - \frac{1}{8}V_2 - \frac{1}{8}V_0 - \frac{1}{8}V_4 - \frac{1}{8}V_5$
4	V_4	$\frac{15}{16}V_2 - \frac{1}{16}V_1 - \frac{1}{16}V_0 - \frac{1}{16}V_3 - \frac{1}{16}V_4 - \frac{1}{16}V_5$
5	V_5	$\frac{31}{32}V_1 - \frac{1}{32}V_0 - \frac{1}{32}V_2 - \frac{1}{32}V_3 - \frac{1}{32}V_4 - \frac{1}{32}V_5$
6	$2V_0 + V_1 + V_2 + V_3 + V_4 + V_5$	$\frac{63}{64}V_0 - \frac{1}{64}V_1 - \frac{1}{64}V_2 - \frac{1}{64}V_3 - \frac{1}{64}V_4 - \frac{1}{64}V_5$
7	$2V_0 + 3V_1 + 2V_2 + 2V_3 + 2V_4 + 2V_5$	$\frac{63}{128}V_5 - \frac{65}{128}V_1 - \frac{65}{128}V_2 - \frac{65}{128}V_3 - \frac{65}{128}V_4 - \frac{65}{128}V_0$
8	$4V_0 + 4V_1 + 5V_2 + 4V_3 + 4V_4 + 4V_5$	$\frac{191}{256}V_4 - \frac{65}{256}V_1 - \frac{65}{256}V_2 - \frac{65}{256}V_3 - \frac{65}{256}V_0 - \frac{65}{256}V_5$
9	$8V_0 + 8V_1 + 8V_2 + 9V_3 + 8V_4 + 8V_5$	$\frac{447}{512}V_3 - \frac{65}{512}V_1 - \frac{65}{512}V_2 - \frac{65}{512}V_0 - \frac{65}{512}V_4 - \frac{65}{512}V_5$
10	$16V_0 + 16V_1 + 16V_2 + 16V_3 + 17V_4 + 16V_5$	$\frac{959}{1024}V_2 - \frac{65}{1024}V_1 - \frac{65}{1024}V_0 - \frac{65}{1024}V_3 - \frac{65}{1024}V_4 - \frac{65}{1024}V_5$
11	$32V_0 + 32V_1 + 32V_2 + 32V_3 + 32V_4 + 33V_5$	$\frac{1983}{2048}V_1 - \frac{65}{2048}V_0 - \frac{65}{2048}V_2 - \frac{65}{2048}V_3 - \frac{65}{2048}V_4 - \frac{65}{2048}V_5$
12	$66V_0 + 65V_1 + 65V_2 + 65V_3 + 65V_4 + 65V_5$	$\frac{4031}{4096}V_0 - \frac{65}{4096}V_1 - \frac{65}{4096}V_2 - \frac{65}{4096}V_3 - \frac{65}{4096}V_4 - \frac{65}{4096}V_5$
13	$130V_0 + 131V_1 + 130V_2 + 130V_3 + 130V_4 + 130V_5$	$\frac{4031}{8192}V_5 - \frac{4161}{8192}V_1 - \frac{4161}{8192}V_2 - \frac{4161}{8192}V_3 - \frac{4161}{8192}V_4 - \frac{4161}{8192}V_0$

Now we define six special case of the sequence $\{V_n\}$. Sixth-order Jacobsthal sequence $\{J_n^{(6)}\}_{n\geq 0}$, sixth order Jacobsthal-Lucas sequence $\{j_n^{(6)}\}_{n\geq 0}$, modified sixth order Jacobsthal sequence $\{K_n^{(6)}\}_{n\geq 0}$, sixth order Jacobsthal Perrin sequence $\{Q_n^{(6)}\}_{n\geq 0}$, adjusted sixth order Jacobsthal sequence $\{S_n^{(6)}\}_{n\geq 0}$ and modified sixth order Jacobsthal-Lucas sequence $\{R_n^{(6)}\}_{n\geq 0}$ are defined, respectively, by the sixth order recurrence relations

$$\begin{aligned} J_{n+6}^{(6)} &= J_{n+5}^{(6)} + J_{n+4}^{(6)} + J_{n+3}^{(6)} + J_{n+2}^{(6)} + J_{n+1}^{(6)} + 2J_n^{(6)}, \\ J_0^{(6)} &= 0, J_1^{(6)} = 1, J_2^{(6)} = 1, J_3^{(6)} = 1, J_4^{(6)} = 1, J_5^{(6)} = 1, \end{aligned} \tag{8}$$

$$\begin{aligned} j_{n+6}^{(6)} &= j_{n+5}^{(6)} + j_{n+4}^{(6)} + j_{n+3}^{(6)} + j_{n+2}^{(6)} + j_{n+1}^{(6)} + 2j_{n}^{(6)}, \\ j_{0}^{(6)} &= 2, j_{1}^{(6)} = 1, j_{2}^{(6)} = 5, j_{3}^{(6)} = 10, j_{4}^{(6)} = 20, j_{5}^{(6)} = 40, \end{aligned} \tag{9}$$

$$\begin{aligned} K_{n+6}^{(6)} &= K_{n+5}^{(6)} + K_{n+4}^{(6)} + K_{n+3}^{(6)} + K_{n+2}^{(6)} + K_{n+1}^{(6)} + 2K_{n}^{(6)}, \\ K_{0}^{(6)} &= 3, K_{1}^{(6)} = 1, K_{2}^{(6)} = 3, K_{3}^{(6)} = 10, K_{4}^{(6)} = 20, K_{5}^{(6)} = 40, \end{aligned} \tag{10}$$

$$\begin{aligned} Q_{n+6}^{(6)} &= Q_{n+5}^{(6)} + Q_{n+4}^{(6)} + Q_{n+3}^{(6)} + Q_{n+2}^{(6)} + Q_{n+1}^{(6)} + 2Q_{n}^{(6)}, \\ Q_{0}^{(6)} &= 3, Q_{1}^{(6)} = 0, Q_{2}^{(6)} = 2, Q_{3}^{(6)} = 8, Q_{4}^{(6)} = 16, Q_{5}^{(6)} = 32, \end{aligned} \tag{11}$$

$$\begin{aligned} S_{n+6}^{(6)} &= S_{n+5}^{(6)} + S_{n+4}^{(6)} + S_{n+3}^{(6)} + S_{n+2}^{(6)} + S_{n+1}^{(6)} + 2S_{n}^{(6)}, \\ S_{0}^{(6)} &= 0, S_{1}^{(6)} = 1, S_{2}^{(6)} = 1, S_{3}^{(6)} = 2, S_{4}^{(6)} = 4, S_{5}^{(6)} = 8, \end{aligned} \tag{12}$$

$$\begin{aligned} R_{n+6}^{(6)} &= R_{n+5}^{(6)} + R_{n+4}^{(6)} + R_{n+3}^{(6)} + R_{n+2}^{(6)} + R_{n+1}^{(6)} + 2R_{n}^{(6)}, \\ R_{0}^{(6)} &= 6, R_{1}^{(6)} = 1, R_{2}^{(6)} = 3, R_{3}^{(6)} = 7, R_{4}^{(6)} = 15, R_{5}^{(6)} = 31. \end{aligned} \tag{13}$$

The sequences $\{J_n^{(6)}\}_{n\geq 0}$, $\{j_n^{(6)}\}_{n\geq 0}$,$\{K_n^{(6)}\}_{n\geq 0}$, $Q_n^{(6)}\}_{n\geq 0}$, $\{S_n^{(6)}\}_{n\geq 0}$ and $\{R_n^{(6)}\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
J_{-n}^{(6)} &= -\frac{1}{2}J_{-(n-1)}^{(6)} - \frac{1}{2}J_{-(n-2)}^{(6)} - \frac{1}{2}J_{-(n-3)}^{(6)} - \frac{1}{2}J_{-(n-4)}^{(6)} - \frac{1}{2}J_{-(n-5)}^{(6)} + \frac{1}{2}J_{-(n-6)}^{(6)} \\
j_{-n}^{(6)} &= -\frac{1}{2}j_{-(n-1)}^{(6)} - \frac{1}{2}j_{-(n-2)}^{(6)} - \frac{1}{2}j_{-(n-3)}^{(6)} - \frac{1}{2}j_{-(n-4)}^{(6)} - \frac{1}{2}j_{-(n-5)}^{(6)} + \frac{1}{2}j_{-(n-6)}^{(6)} \\
K_{-n}^{(6)} &= -\frac{1}{2}K_{-(n-1)}^{(6)} - \frac{1}{2}K_{-(n-2)}^{(6)} - \frac{1}{2}K_{-(n-3)}^{(6)} - \frac{1}{2}K_{-(n-4)}^{(6)} - \frac{1}{2}K_{-(n-5)}^{(6)} + \frac{1}{2}K_{-(n-6)}^{(6)} \\
Q_{-n}^{(6)} &= -\frac{1}{2}Q_{-(n-1)}^{(6)} - \frac{1}{2}Q_{-(n-2)}^{(6)} - \frac{1}{2}Q_{-(n-3)}^{(6)} - \frac{1}{2}Q_{-(n-4)}^{(6)} - \frac{1}{2}Q_{-(n-5)}^{(6)} + \frac{1}{2}Q_{-(n-6)}^{(6)} \\
S_{-n}^{(6)} &= -\frac{1}{2}S_{-(n-1)}^{(6)} - \frac{1}{2}S_{-(n-2)}^{(6)} - \frac{1}{2}S_{-(n-3)}^{(6)} - \frac{1}{2}S_{-(n-4)}^{(6)} - \frac{1}{2}S_{-(n-5)}^{(6)} + \frac{1}{2}S_{-(n-6)}^{(6)} \\
R_{-n}^{(6)} &= -\frac{1}{2}R_{-(n-1)}^{(6)} - \frac{1}{2}R_{-(n-2)}^{(6)} - \frac{1}{2}R_{-(n-3)}^{(6)} - \frac{1}{2}R_{-(n-4)}^{(6)} - \frac{1}{2}R_{-(n-5)}^{(6)} + \frac{1}{2}R_{-(n-6)}^{(6)}
\end{aligned}$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences (8)- (13) hold for all integer n.

Next, we present the first few values of the sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special sixth order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$J_n^{(6)}$	0	1	1	1	1	1	5	11	21	41	81	161	325	651
$J_{-n}^{(6)}$		$-\frac{3}{2}$	$-\frac{1}{4}$	$\frac{3}{8}$	$\frac{11}{16}$	$\frac{27}{32}$	$-\frac{5}{64}$	$-\frac{197}{128}$	$-\frac{69}{256}$	$\frac{187}{512}$	$\frac{699}{1024}$	$\frac{1723}{2048}$	$-\frac{325}{4096}$	$-\frac{12613}{8192}$
$j_n^{(6)}$	2	1	5	10	20	40	80	157	317	634	1268	2536	5072	10141
$j_{-n}^{(6)}$		1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{23}{16}$	$\frac{25}{32}$	$\frac{25}{64}$	$\frac{25}{128}$	$\frac{25}{256}$	$\frac{25}{512}$	$-\frac{1511}{1024}$	$\frac{1561}{2048}$	$\frac{1561}{4096}$
$K_n^{(6)}$	3	1	3	10	20	40	80	155	311	626	1252	2504	5008	10011
$K_{-n}^{(6)}$		$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$-\frac{29}{16}$	$-\frac{45}{32}$	$\frac{115}{64}$	$\frac{115}{128}$	$\frac{115}{256}$	$\frac{115}{512}$	$-\frac{1933}{1024}$	$-\frac{2957}{2048}$	$\frac{7283}{4096}$	$\frac{7283}{8192}$
$Q_n^{(6)}$	3	0	2	8	16	32	64	122	246	496	992	1984	3968	7930
$Q_{-n}^{(6)}$		$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$-\frac{29}{16}$	$-\frac{61}{32}$	$\frac{131}{64}$	$\frac{131}{128}$	$\frac{131}{256}$	$\frac{131}{512}$	$-\frac{1917}{1024}$	$-\frac{3965}{2048}$	$\frac{8323}{4096}$	$\frac{8323}{8192}$
$S_n^{(6)}$	0	1	1	2	4	8	16	33	65	130	260	520	1040	2081
$S_{-n}^{(6)}$		0	0	0	0	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$-\frac{1}{32}$	$-\frac{1}{64}$	$\frac{63}{128}$	$-\frac{65}{256}$	$-\frac{65}{512}$
$R_n^{(6)}$	6	1	3	7	15	31	69	127	255	511	1023	2047	4101	8191
$R_{-n}^{(6)}$		$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	$-\frac{15}{16}$	$-\frac{31}{32}$	$\frac{321}{64}$	$-\frac{127}{128}$	$-\frac{255}{256}$	$-\frac{511}{512}$	$-\frac{1023}{1024}$	$-\frac{2047}{2048}$	$\frac{20481}{4096}$	$-\frac{8191}{8192}$

In the rest of the paper, for easy writing, we drop the superscripts and write J_n, j_n, K_n, Q_n, S_n and R_n for $J_n^{(6)}, j_n^{(6)}$, $K_n^{(6)}, Q_n^{(6)}, S_n^{(6)}$ and $R_n^{(6)}$, respectively.

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 1.2.
Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ *is the ordinary generating function of the generalized sixth order Jacobsthal sequence* $\{V_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} V_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{\Lambda}{1-x-x^2-x^3-x^4-x^5-2x^6} \tag{14}$$

where

$$\begin{aligned}\Lambda &= V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2 + (V_3 - V_2 - V_1 - V_0)x^3 \\ &\quad + (V_4 - V_3 - V_2 - V_1 - V_0)x^4 + (V_5 - V_4 - V_3 - V_2 - V_1 - V_0)x^5 \\ &= V_0 + \sum_{i=1}^{6-1} x^i \left(V_i - \sum_{j=1}^{i} V_{i-j}\right).\end{aligned}$$

Proof. Take $r = s = t = u = v = 1, y = 2$ in Lemma 1.1. □
The previous Lemma gives the following results as particular examples.

Corollary 1.1.
Generated functions of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas numbers are

$$\begin{aligned}\sum_{n=0}^{\infty} J_n x^n &= \frac{x - x^3 - 2x^4 - 3x^5}{1-x-x^2-x^3-x^4-x^5-2x^6}, \\ \sum_{n=0}^{\infty} j_n x^n &= \frac{2 - x + 2x^2 + 2x^3 + 2x^4 + 2x^5}{1-x-x^2-x^3-x^4-x^5-2x^6}, \\ \sum_{n=0}^{\infty} K_n x^n &= \frac{3 - 2x - x^2 + 3x^3 + 3x^4 + 3x^5}{1-x-x^2-x^3-x^4-x^5-2x^6}, \\ \sum_{n=0}^{\infty} Q_n x^n &= \frac{3 - 3x - x^2 + 3x^3 + 3x^4 + 3x^5}{1-x-x^2-x^3-x^4-x^5-2x^6}, \\ \sum_{n=0}^{\infty} S_n x^n &= \frac{x}{1-x-x^2-x^3-x^4-x^5-2x^6}, \\ \sum_{n=0}^{\infty} R_n x^n &= \frac{6 - 5x - 4x^2 - 3x^3 - 2x^4 - x^5}{1-x-x^2-x^3-x^4-x^5-2x^6},\end{aligned}$$

respectively.

We next give Binet formula of generalized sixth order Jacobsthal numbers $\{V_n\}$ by the use of Theorems 1.1 and 1.2.

Theorem 1.3.
(Binet formula of generalized sixth order Jacobsthal numbers)

$$\begin{aligned}V_n &= \frac{c_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)} + \frac{c_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\ &\quad + \frac{c_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)} + \frac{c_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\ &\quad + \frac{c_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)} + \frac{c_6\mu^n}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)}\end{aligned}$$

where

$$\begin{aligned}c_1 &= V_5 - (\beta+\gamma+\delta+\lambda+\mu)V_4 + (\beta\lambda+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\beta\delta+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta)V_3 \\ &\quad - (\beta\lambda\gamma+\beta\lambda\mu+\beta\lambda\delta+\beta\gamma\mu+\lambda\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta)V_2 \\ &\quad + (\beta\lambda\gamma\mu+\beta\lambda\gamma\delta+\beta\lambda\mu\delta+\beta\gamma\mu\delta+\lambda\gamma\mu\delta)V_1 - \beta\lambda\gamma\mu\delta V_0 \\ &= V_0\alpha^5 + (V_1 - V_0)\alpha^4 + (V_2 - V_1 - V_0)\alpha^3 + (V_3 - V_2 - V_1 - V_0)\alpha^2 \\ &\quad + (V_4 - V_3 - V_2 - V_1 - V_0)\alpha + (V_5 - V_4 - V_3 - V_2 - V_1 - V_0),\end{aligned}$$

$$\begin{aligned}
c_2 &= V_5-(\alpha+\gamma+\delta+\lambda+\mu)V_4+(\alpha\lambda+\alpha\gamma+\alpha\mu+\alpha\delta+\lambda\gamma+\lambda\mu+\lambda\delta+\gamma\mu+\gamma\delta+\mu\delta)V_3 \\
&\quad -(\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\lambda\delta+\alpha\gamma\mu+\alpha\gamma\delta+\alpha\mu\delta+\lambda\gamma\mu+\lambda\gamma\delta+\lambda\mu\delta+\gamma\mu\delta)V_2 \\
&\quad +(\alpha\lambda\gamma\mu+\alpha\lambda\gamma\delta+\alpha\lambda\mu\delta+\alpha\gamma\mu\delta+\lambda\gamma\mu\delta)V_1-\alpha\lambda\gamma\mu\delta V_0 \\
&= V_0\beta^5+(V_1-V_0)\beta^4+(V_2-V_1-V_0)\beta^3+(V_3-V_2-V_1-V_0)\beta^2 \\
&\quad +(V_4-V_3-V_2-V_1-V_0)\beta+(V_5-V_4-V_3-V_2-V_1-V_0),
\end{aligned}$$

$$\begin{aligned}
c_3 &= V_5-(\alpha+\beta+\delta+\lambda+\mu)V_4+(\alpha\beta+\alpha\lambda+\alpha\mu+\beta\lambda+\alpha\delta+\beta\mu+\lambda\mu+\beta\delta+\lambda\delta+\mu\delta)V_3 \\
&\quad -(\alpha\beta\lambda+\alpha\beta\mu+\alpha\lambda\mu+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\mu+\alpha\mu\delta+\beta\lambda\delta+\beta\mu\delta+\lambda\mu\delta)V_2 \\
&\quad +(\alpha\beta\lambda\mu+\alpha\beta\lambda\delta+\alpha\beta\mu\delta+\alpha\lambda\mu\delta+\beta\lambda\mu\delta)V_1-\alpha\beta\lambda\mu\delta V_0 \\
&= V_0\gamma^5+(V_1-V_0)\gamma^4+(V_2-V_1-V_0)\gamma^3+(V_3-V_2-V_1-V_0)\gamma^2 \\
&\quad +(V_4-V_3-V_2-V_1-V_0)\gamma+(V_5-V_4-V_3-V_2-V_1-V_0),
\end{aligned}$$

$$\begin{aligned}
c_4 &= V_5-(\alpha+\beta+\gamma+\lambda+\mu)V_4+(\alpha\beta+\alpha\lambda+\alpha\gamma+\alpha\mu+\beta\lambda+\beta\gamma+\beta\mu+\lambda\gamma+\lambda\mu+\gamma\mu)V_3 \\
&\quad -(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\beta\mu+\alpha\lambda\gamma+\alpha\lambda\mu+\alpha\gamma\mu+\beta\lambda\gamma+\beta\lambda\mu+\beta\gamma\mu+\lambda\gamma\mu)V_2 \\
&\quad +(\alpha\beta\lambda\gamma+\alpha\beta\lambda\mu+\alpha\beta\gamma\mu+\alpha\lambda\gamma\mu+\beta\lambda\gamma\mu)V_1-\alpha\beta\lambda\gamma\mu V_0 \\
&= V_0\delta^5+(V_1-V_0)\delta^4+(V_2-V_1-V_0)\delta^3+(V_3-V_2-V_1-V_0)\delta^2 \\
&\quad +(V_4-V_3-V_2-V_1-V_0)\delta+(V_5-V_4-V_3-V_2-V_1-V_0),
\end{aligned}$$

$$\begin{aligned}
c_5 &= V_5-(\alpha+\beta+\gamma+\delta+\mu)V_4+(\alpha\beta+\alpha\gamma+\alpha\mu+\alpha\delta+\beta\gamma+\beta\mu+\beta\delta+\gamma\mu+\gamma\delta+\mu\delta)V_3 \\
&\quad -(\alpha\beta\gamma+\alpha\beta\mu+\alpha\beta\delta+\alpha\gamma\mu+\alpha\gamma\delta+\alpha\mu\delta+\beta\gamma\mu+\beta\gamma\delta+\beta\mu\delta+\gamma\mu\delta)V_2 \\
&\quad +(\alpha\beta\gamma\mu+\alpha\beta\gamma\delta+\alpha\beta\mu\delta+\alpha\gamma\mu\delta+\beta\gamma\mu\delta)V_1-\alpha\beta\gamma\mu\delta V_0 \\
&= V_0\lambda^5+(V_1-V_0)\lambda^4+(V_2-V_1-V_0)\lambda^3+(V_3-V_2-V_1-V_0)\lambda^2 \\
&\quad +(V_4-V_3-V_2-V_1-V_0)\lambda+(V_5-V_4-V_3-V_2-V_1-V_0),
\end{aligned}$$

$$\begin{aligned}
c_6 &= V_5-(\alpha+\beta+\gamma+\delta+\lambda)V_4+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)V_3 \\
&\quad -(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)V_2 \\
&\quad +(\alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta)V_1-\alpha\beta\lambda\gamma\delta V_0 \\
&= V_0\mu^5+(V_1-V_0)\mu^4+(V_2-V_1-V_0)\mu^3+(V_3-V_2-V_1-V_0)\mu^2 \\
&\quad +(V_4-V_3-V_2-V_1-V_0)\mu+(V_5-V_4-V_3-V_2-V_1-V_0).
\end{aligned}$$

Next, using Theorem 1.3, we present the Binet formulas of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas sequences.

Corollary 1.2.
Binet formulas of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas sequences are

$$\begin{aligned}
J_n &= \frac{(\alpha^4-\alpha^2-2\alpha-3)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}+\frac{(\beta^4-\beta^2-2\beta-3)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\
&\quad +\frac{(\gamma^4-\gamma^2-2\gamma-3)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}+\frac{(\delta^4-\delta^2-2\delta-3)\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\
&\quad +\frac{(\lambda^4-\lambda^2-2\lambda-3)\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}+\frac{(\mu^4-\mu^2-2\mu-3)\mu^n}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)},
\end{aligned}$$

$$\begin{aligned}
j_n &= \frac{(2\alpha^5-\alpha^4+2\alpha^3+2\alpha^2+2\alpha+2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}+\frac{(2\beta^5-\beta^4+2\beta^3+2\beta^2+2\beta+2)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\
&\quad +\frac{(2\gamma^5-\gamma^4+2\gamma^3+2\gamma^2+2\gamma+2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}+\frac{(2\delta^5-\delta^4+2\delta^3+2\delta^2+2\delta+2)\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\
&\quad +\frac{(2\lambda^5-\lambda^4+2\lambda^3+2\lambda^2+2\lambda+2)\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}+\frac{(2\mu^5-\mu^4+2\mu^3+2\mu^2+2\mu+2)\mu^n}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)},
\end{aligned}$$

$$\begin{aligned} K_n &= \frac{(3\alpha^5-2\alpha^4-\alpha^3+3\alpha^2+3\alpha+3)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}+\frac{(3\beta^5-2\beta^4-\beta^3+3\beta^2+3\beta+3)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\ &+\frac{(3\gamma^5-2\gamma^4-\gamma^3+3\gamma^2+3\gamma+3)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}+\frac{(3\delta^5-2\delta^4-\delta^3+3\delta^2+3\delta+3)\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\ &+\frac{(3\lambda^5-2\lambda^4-\lambda^3+3\lambda^2+3\lambda+3)\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}+\frac{(3\mu^5-2\mu^4-\mu^3+3\mu^2+3\mu+3)\mu^n}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)}, \end{aligned}$$

$$\begin{aligned} Q_n &= \frac{(3\alpha^5-3\alpha^4-\alpha^3+3\alpha^2+3\alpha+3)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}+\frac{(3\beta^5-3\beta^4-\beta^3+3\beta^2+3\beta+3)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\ &+\frac{(3\gamma^5-3\gamma^4-\gamma^3+3\gamma^2+3\gamma+3)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}+\frac{(3\delta^5-3\delta^4-\delta^3+3\delta^2+3\delta+3)\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\ &+\frac{(3\lambda^5-3\lambda^4-\lambda^3+3\lambda^2+3\lambda+3)\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}+\frac{(3\mu^5-3\mu^4-\mu^3+3\mu^2+3\mu+3)\mu^n}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)}, \end{aligned}$$

$$\begin{aligned} S_n &= \frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)(\alpha-\mu)}+\frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)(\beta-\mu)} \\ &+\frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)(\gamma-\mu)}+\frac{\delta^{n+4}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)(\delta-\mu)} \\ &+\frac{\lambda^{n+4}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)(\lambda-\mu)}+\frac{\mu^{n+4}}{(\mu-\alpha)(\mu-\beta)(\mu-\gamma)(\mu-\delta)(\mu-\lambda)}, \end{aligned}$$

$$R_n=\alpha^n+\beta^n+\gamma^n+\delta^n+\lambda^n+\mu^n,$$

respectively.

Binet formulas of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas sequences can be given in the followinf forms:

$$\begin{aligned} J_n &= \frac{5}{63}\alpha^n+\frac{1}{9}\beta^n+(-\frac{5}{18}i\sqrt{3}-\frac{1}{6})\gamma^n+(\frac{5}{18}i\sqrt{3}-\frac{1}{6})\delta^n+(\frac{1}{14}-\frac{5}{42}i\sqrt{3})\lambda^n+(\frac{5}{42}i\sqrt{3}+\frac{1}{14})\mu^n, \\ j_n &= \frac{26}{21}\alpha^n+\frac{1}{3}\beta^n+\frac{1}{6}i\sqrt{3}\gamma^n-\frac{1}{6}i\sqrt{3}\delta^n+(\frac{1}{7}i\sqrt{3}+\frac{3}{14})\lambda^n+(\frac{3}{14}-\frac{1}{7}i\sqrt{3})\mu^n, \\ K_n &= \frac{11}{9}\alpha^n+\frac{1}{9}\beta^n+(\frac{7}{18}i\sqrt{3}+\frac{1}{3})\gamma^n+(\frac{1}{3}-\frac{7}{18}i\sqrt{3})\delta^n+\frac{1}{2}\lambda^n+\frac{1}{2}\mu^n, \\ Q_n &= \frac{61}{63}\alpha^n+\frac{2}{9}\beta^n+(\frac{4}{9}i\sqrt{3}+\frac{1}{3})\gamma^n+(\frac{1}{3}-\frac{4}{9}i\sqrt{3})\delta^n+(\frac{1}{21}i\sqrt{3}+\frac{4}{7})\lambda^n+(\frac{4}{7}-\frac{1}{21}i\sqrt{3})\mu^n, \\ S_n &= \frac{1}{63}\alpha^{n+4}-\frac{1}{9}\beta^{n+4}+(\frac{1}{36}i\sqrt{3}+\frac{1}{12})\gamma^{n+4}+(\frac{1}{12}-\frac{1}{36}i\sqrt{3})\delta^{n+4} \\ &\quad +(\frac{5}{84}i\sqrt{3}-\frac{1}{28})\lambda^{n+4}+(-\frac{5}{84}i\sqrt{3}-\frac{1}{28})\mu^{n+4} \\ &= \frac{16}{63}\alpha^n-\frac{1}{9}\beta^n-\frac{1}{18}i\sqrt{3}\gamma^n+\frac{1}{18}i\sqrt{3}\delta^n+(-\frac{1}{21}i\sqrt{3}-\frac{1}{14})\lambda^n+(\frac{1}{21}i\sqrt{3}-\frac{1}{14})\mu^n, \\ R_n &= \alpha^n+\beta^n+\gamma^n+\delta^n+\lambda^n+\mu^n. \end{aligned}$$

2. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1}-F_n^2=(-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix}=(-1)^n.$$

The following Theorem gives generalization of this result to the generalized Hexanacci sequence $\{W_n\}$.

Theorem 2.1 (Simson Formula of Generalized Hexanacci Numbers).
For all integers n we have

$$\begin{vmatrix} W_{n+5} & W_{n+4} & W_{n+3} & W_{n+2} & W_{n+1} & W_{n} \\ W_{n+4} & W_{n+3} & W_{n+2} & W_{n+1} & W_{n} & W_{n-1} \\ W_{n+3} & W_{n+2} & W_{n+1} & W_{n} & W_{n-1} & W_{n-2} \\ W_{n+2} & W_{n+1} & W_{n} & W_{n-1} & W_{n-2} & W_{n-3} \\ W_{n+1} & W_{n} & W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} \\ W_{n} & W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} & W_{n-5} \end{vmatrix} = (-1)^n y^6 \begin{vmatrix} W_5 & W_4 & W_3 & W_2 & W_1 & W_0 \\ W_4 & W_3 & W_2 & W_1 & W_0 & W_{-1} \\ W_3 & W_2 & W_1 & W_0 & W_{-1} & W_{-2} \\ W_2 & W_1 & W_0 & W_{-1} & W_{-2} & W_{-3} \\ W_1 & W_0 & W_{-1} & W_{-2} & W_{-3} & W_{-4} \\ W_0 & W_{-1} & W_{-2} & W_{-3} & W_{-4} & W_{-5} \end{vmatrix}. \quad (15)$$

Proof. (15) is given in Soykan [26]. □

A special case of the above theorem is the following Theorem which gives Simson formula of the generalized sixth order Jacobsthal sequence $\{V_n\}$.

Theorem 2.2 (Simson Formula of Generalized Sixth-Order Jacobsthal Numbers).
For all integers n we have

$$\begin{vmatrix} V_{n+5} & V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_{n} \\ V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_{n} & V_{n-1} \\ V_{n+3} & V_{n+2} & V_{n+1} & V_{n} & V_{n-1} & V_{n-2} \\ V_{n+2} & V_{n+1} & V_{n} & V_{n-1} & V_{n-2} & V_{n-3} \\ V_{n+1} & V_{n} & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} \\ V_{n} & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} & V_{n-5} \end{vmatrix} = (-1)^n \times 2^n \begin{vmatrix} V_5 & V_4 & V_3 & V_2 & V_1 & V_0 \\ V_4 & V_3 & V_2 & V_1 & V_0 & V_{-1} \\ V_3 & V_2 & V_1 & V_0 & V_{-1} & V_{-2} \\ V_2 & V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} \\ V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} \\ V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} & V_{-5} \end{vmatrix}.$$

The previous Theorem gives the following results as particular examples.

Corollary 2.1.
Simson's formulas of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas numbers are given as

$$\begin{vmatrix} J_{n+5} & J_{n+4} & J_{n+3} & J_{n+2} & J_{n+1} & J_{n} \\ J_{n+4} & J_{n+3} & J_{n+2} & J_{n+1} & J_{n} & J_{n-1} \\ J_{n+3} & J_{n+2} & J_{n+1} & J_{n} & J_{n-1} & J_{n-2} \\ J_{n+2} & J_{n+1} & J_{n} & J_{n-1} & J_{n-2} & J_{n-3} \\ J_{n+1} & J_{n} & J_{n-1} & J_{n-2} & J_{n-3} & J_{n-4} \\ J_{n} & J_{n-1} & J_{n-2} & J_{n-3} & J_{n-4} & J_{n-5} \end{vmatrix} = 35(-1)^{n+1} \times 2^{n-1},$$

$$\begin{vmatrix} j_{n+5} & j_{n+4} & j_{n+3} & j_{n+2} & j_{n+1} & j_{n} \\ j_{n+4} & j_{n+3} & j_{n+2} & j_{n+1} & j_{n} & j_{n-1} \\ j_{n+3} & j_{n+2} & j_{n+1} & j_{n} & j_{n-1} & j_{n-2} \\ j_{n+2} & j_{n+1} & j_{n} & j_{n-1} & j_{n-2} & j_{n-3} \\ j_{n+1} & j_{n} & j_{n-1} & j_{n-2} & j_{n-3} & j_{n-4} \\ j_{n} & j_{n-1} & j_{n-2} & j_{n-3} & j_{n-4} & j_{n-5} \end{vmatrix} = 9477(-1)^{n+1} \times 2^{n-4},$$

$$\begin{vmatrix} K_{n+5} & K_{n+4} & K_{n+3} & K_{n+2} & K_{n+1} & K_{n} \\ K_{n+4} & K_{n+3} & K_{n+2} & K_{n+1} & K_{n} & K_{n-1} \\ K_{n+3} & K_{n+2} & K_{n+1} & K_{n} & K_{n-1} & K_{n-2} \\ K_{n+2} & K_{n+1} & K_{n} & K_{n-1} & K_{n-2} & K_{n-3} \\ K_{n+1} & K_{n} & K_{n-1} & K_{n-2} & K_{n-3} & K_{n-4} \\ K_{n} & K_{n-1} & K_{n-2} & K_{n-3} & K_{n-4} & K_{n-5} \end{vmatrix} = 98637(-1)^{n+1} \times 2^{n-5},$$

$$\begin{vmatrix} Q_{n+5} & Q_{n+4} & Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_{n} \\ Q_{n+4} & Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_{n} & Q_{n-1} \\ Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_{n} & Q_{n-1} & Q_{n-2} \\ Q_{n+2} & Q_{n+1} & Q_{n} & Q_{n-1} & Q_{n-2} & Q_{n-3} \\ Q_{n+1} & Q_{n} & Q_{n-1} & Q_{n-2} & Q_{n-3} & Q_{n-4} \\ Q_{n} & Q_{n-1} & Q_{n-2} & Q_{n-3} & Q_{n-4} & Q_{n-5} \end{vmatrix} = 8113(-1)^{n+1} \times 2^{n},$$

$$\begin{vmatrix} S_{n+5} & S_{n+4} & S_{n+3} & S_{n+2} & S_{n+1} & S_n \\ S_{n+4} & S_{n+3} & S_{n+2} & S_{n+1} & S_n & S_{n-1} \\ S_{n+3} & S_{n+2} & S_{n+1} & S_n & S_{n-1} & S_{n-2} \\ S_{n+2} & S_{n+1} & S_n & S_{n-1} & S_{n-2} & S_{n-3} \\ S_{n+1} & S_n & S_{n-1} & S_{n-2} & S_{n-3} & S_{n-4} \\ S_n & S_{n-1} & S_{n-2} & S_{n-3} & S_{n-4} & S_{n-5} \end{vmatrix} = (-1)^n \times 2^{n-1},$$

$$\begin{vmatrix} R_{n+5} & R_{n+4} & R_{n+3} & R_{n+2} & R_{n+1} & R_n \\ R_{n+4} & R_{n+3} & R_{n+2} & R_{n+1} & R_n & R_{n-1} \\ R_{n+3} & R_{n+2} & R_{n+1} & R_n & R_{n-1} & R_{n-2} \\ R_{n+2} & R_{n+1} & R_n & R_{n-1} & R_{n-2} & R_{n-3} \\ R_{n+1} & R_n & R_{n-1} & R_{n-2} & R_{n-3} & R_{n-4} \\ R_n & R_{n-1} & R_{n-2} & R_{n-3} & R_{n-4} & R_{n-5} \end{vmatrix} = 321489(-1)^{n+1} \times 2^{n-1}.$$

3. Some Identities

In this section, we obtain some identities of sixth order Jacobsthal, sixth order Jacobsthal-Lucas and modified sixth order Jacobsthal numbers. First, we can give a few basic relations between $\{J_n\}$ and $\{j_n\}$.

Lemma 3.1.
The following equalities are true:

$$\begin{aligned} 234J_n &= -53j_{n+5}+25j_{n+4}+103j_{n+3}+181j_{n+2}-53j_{n+1}-131j_n, \qquad (16) \\ 117J_n &= -14j_{n+4}+25j_{n+3}+64j_{n+2}-53j_{n+1}-92j_n-53j_{n-1}, \\ 117J_n &= 11j_{n+3}+50j_{n+2}-67j_{n+1}-106j_n-67j_{n-1}-28j_{n-2}, \\ 117J_n &= 61j_{n+2}-56j_{n+1}-95j_n-56j_{n-1}-17j_{n-2}+22j_{n-3}, \\ 117J_n &= 5j_{n+1}-34j_n+5j_{n-1}+44j_{n-2}+83j_{n-3}+122j_{n-4}, \end{aligned}$$

and

$$\begin{aligned} 70j_n &= -26J_{n+5}+109J_{n+4}-11J_{n+3}+49J_{n+2}+19J_{n+1}+34J_n, \\ 70j_n &= 83J_{n+4}-37J_{n+3}+23J_{n+2}-7J_{n+1}+8J_n-52J_{n-1}, \\ 70j_n &= 46J_{n+3}+106J_{n+2}+76J_{n+1}+91J_n+31J_{n-1}+166J_{n-2}, \\ 70j_n &= 152J_{n+2}+122J_{n+1}+137J_n+77J_{n-1}+212J_{n-2}+92J_{n-3}, \\ 70j_n &= 274J_{n+1}+289J_n+229J_{n-1}+364J_{n-2}+244J_{n-3}+304J_{n-4}. \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (16). To show (16), writing

$J_n = a\times j_{n+5}+b\times j_{n+4}+c\times j_{n+3}+d\times j_{n+2}+e\times j_{n+1}+f\times j_n$

and solving the system of equations

$$\begin{aligned} J_0 &= a\times j_5+b\times j_4+c\times j_3+d\times j_2+e\times j_1+f\times j_0 \\ J_1 &= a\times j_6+b\times j_5+c\times j_4+d\times j_3+e\times j_2+f\times j_1 \\ J_2 &= a\times j_7+b\times j_6+c\times j_5+d\times j_4+e\times j_3+f\times j_2 \\ J_3 &= a\times j_8+b\times j_7+c\times j_6+d\times j_5+e\times j_4+f\times j_3 \\ J_4 &= a\times j_9+b\times j_8+c\times j_7+d\times j_6+e\times j_5+f\times j_4 \\ J_5 &= a\times j_{10}+b\times j_9+c\times j_8+d\times j_7+e\times j_6+f\times j_5 \end{aligned}$$

we find that $a=-\frac{53}{234}, b=, \frac{25}{234} c=\frac{103}{234}, d=\frac{181}{234}, e=-\frac{53}{234}, f=-\frac{131}{234}$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we present a few basic relations between $\{J_n\}$ and $\{S_n\}$.

Lemma 3.2.
The following equalities are true:

$$\begin{aligned} 16J_n &= 11S_{n+5}-5S_{n+4}-21S_{n+3}-37S_{n+2}+11S_{n+1}+27S_n, \\ 8J_n &= 3S_{n+4}-5S_{n+3}-13S_{n+2}+11S_{n+1}+19S_n+11S_{n-1}, \\ 4J_n &= -S_{n+3}-5S_{n+2}+7S_{n+1}+11S_n+7S_{n-1}+3S_{n-2}, \\ 2J_n &= -3S_{n+2}+3S_{n+1}+5S_n+3S_{n-1}+S_{n-2}-S_{n-3}, \\ J_n &= S_n-S_{n-2}-2S_{n-3}-3S_{n-4}, \end{aligned}$$

and

$$\begin{aligned}
70S_n &= 18J_{n+5}-27J_{n+4}+13J_{n+3}-7J_{n+2}+3J_{n+1}-2J_n,\\
70S_n &= -9J_{n+4}+31J_{n+3}+11J_{n+2}+21J_{n+1}+16J_n+36J_{n-1},\\
70S_n &= 22J_{n+3}+2J_{n+2}+12J_{n+1}+7J_n+27J_{n-1}-18J_{n-2},\\
70S_n &= 24J_{n+2}+34J_{n+1}+29J_n+49J_{n-1}+4J_{n-2}+44J_{n-3},\\
70S_n &= 58J_{n+1}+53J_n+73J_{n-1}+28J_{n-2}+68J_{n-3}+48J_{n-4}.
\end{aligned}$$

Thirdly, we give a few basic relations between $\{J_n\}$ and $\{R_n\}$.

Lemma 3.3.
The following equalities are true:

$$\begin{aligned}
3969J_n &= 178R_{n+5}+73R_{n+4}-137R_{n+3}-557R_{n+2}-1397R_{n+1}-431R_n,\\
3969J_n &= 251R_{n+4}+41R_{n+3}-379R_{n+2}-1219R_{n+1}-253R_n+356R_{n-1},\\
3969J_n &= 292R_{n+3}-128R_{n+2}-968R_{n+1}-2R_n+607R_{n-1}+502R_{n-2},\\
3969J_n &= 164R_{n+2}-676R_{n+1}+290R_n+899R_{n-1}+794R_{n-2}+584R_{n-3},\\
3969J_n &= -512R_{n+1}+454R_n+1063R_{n-1}+958R_{n-2}+748R_{n-3}+328R_{n-4},
\end{aligned}$$

and

$$\begin{aligned}
35R_n &= -73J_{n+5}+127J_{n+4}+27J_{n+3}+77J_{n+2}+52J_{n+1}+117J_n,\\
35R_n &= 54J_{n+4}-46J_{n+3}+4J_{n+2}-21J_{n+1}+44J_n-146J_{n-1},\\
35R_n &= 8J_{n+3}+58J_{n+2}+33J_{n+1}+98J_n-92J_{n-1}+108J_{n-2},\\
35R_n &= 66J_{n+2}+41J_{n+1}+106J_n-84J_{n-1}+116J_{n-2}+16J_{n-3},\\
35R_n &= 107J_{n+1}+172J_n-18J_{n-1}+182J_{n-2}+82J_{n-3}+132J_{n-4}.
\end{aligned}$$

Next, we present a few basic relations between $\{j_n\}$ and $\{S_n\}$.

Lemma 3.4.
The following equalities are true:

$$\begin{aligned}
8j_n &= S_{n+5}+S_{n+4}+S_{n+3}+S_{n+2}+S_{n+1}-23S_n,\\
4j_n &= S_{n+4}+S_{n+3}+S_{n+2}+S_{n+1}-11S_n+S_{n-1},\\
2j_n &= S_{n+3}+S_{n+2}+S_{n+1}-5S_n+S_{n-1}+S_{n-2},\\
j_n &= S_{n+2}+S_{n+1}-2S_n+S_{n-1}+S_{n-2}+S_{n-3},\\
j_n &= 2S_{n+1}-S_n+2S_{n-1}+2S_{n-2}+2S_{n-3}+2S_{n-4},
\end{aligned}$$

and

$$\begin{aligned}
117S_n &= j_{n+5}+j_{n+4}+j_{n+3}+j_{n+2}+j_{n+1}-38j_n,\\
117S_n &= 2j_{n+4}+2j_{n+3}+2j_{n+2}+2j_{n+1}-37j_n+2j_{n-1},\\
117S_n &= 4j_{n+3}+4j_{n+2}+4j_{n+1}-35j_n+4j_{n-1}+4j_{n-2},\\
117S_n &= 8j_{n+2}+8j_{n+1}-31j_n+8j_{n-1}+8j_{n-2}+8j_{n-3},\\
117S_n &= 16j_{n+1}-23j_n+16j_{n-1}+16j_{n-2}+16j_{n-3}+16j_{n-4}.
\end{aligned}$$

Now, we give a few basic relations between $\{j_n\}$ and $\{R_n\}$.

Lemma 3.5.
The following equalities are true:

$$\begin{aligned}
2646j_n &= -295R_{n+5}+377R_{n+4}+398R_{n+3}+440R_{n+2}+524R_{n+1}+692R_n\\
2646j_n &= 82R_{n+4}+103R_{n+3}+145R_{n+2}+229R_{n+1}+397R_n-590R_{n-1}\\
2646j_n &= 185R_{n+3}+227R_{n+2}+311R_{n+1}+479R_n-508R_{n-1}+164R_{n-2}\\
2646j_n &= 412R_{n+2}+496R_{n+1}+664R_n-323R_{n-1}+349R_{n-2}+370R_{n-3}\\
2646j_n &= 908R_{n+1}+1076R_n+89R_{n-1}+761R_{n-2}+782R_{n-3}+824R_{n-4}
\end{aligned}$$

and

$$\begin{aligned}
78R_n &= 27j_{n+5}+j_{n+4}-25j_{n+3}-51j_{n+2}-233j_{n+1}+53j_n,\\
39R_n &= 14j_{n+4}+j_{n+3}-12j_{n+2}-103j_{n+1}+40j_n+27j_{n-1},\\
39R_n &= 15j_{n+3}+2j_{n+2}-89j_{n+1}+54j_n+41j_{n-1}+28j_{n-2},\\
39R_n &= 17j_{n+2}-74j_{n+1}+69j_n+56j_{n-1}+43j_{n-2}+30\times j_{n-3},\\
39R_n &= -57j_{n+1}+86j_n+73j_{n-1}+60j_{n-2}+47j_{n-3}+34j_{n-4}.
\end{aligned}$$

Next, we present a few basic relations between $\{S_n\}$ and $\{R_n\}$.

Lemma 3.6.
The following equalities are true:

$$\begin{aligned}
7938S_n &= 379R_{n+5}-293R_{n+4}-314R_{n+3}-356R_{n+2}-440R_{n+1}-608R_n,\\
7938S_n &= 86R_{n+4}+65R_{n+3}+23R_{n+2}-61R_{n+1}-229R_n+758R_{n-1},\\
7938S_n &= 151R_{n+3}+109R_{n+2}+25R_{n+1}-143R_n+844R_{n-1}+172R_{n-2},\\
7938S_n &= 260R_{n+2}+176R_{n+1}+8R_n+995R_{n-1}+323R_{n-2}+302R_{n-3},\\
7938S_n &= 436R_{n+1}+268R_n+1255R_{n-1}+583R_{n-2}+562R_{n-3}+520R_{n-4},
\end{aligned}$$

and

$$\begin{aligned}
16R_n &= -15S_{n+5}+S_{n+4}+17S_{n+3}+33S_{n+2}+145S_{n+1}-31S_n,\\
8R_n &= -7S_{n+4}+S_{n+3}+9S_{n+2}+65S_{n+1}-23S_n-15S_{n-1},\\
4R_n &= -3S_{n+3}+S_{n+2}+29S_{n+1}-15S_n-11S_{n-1}-7S_{n-2},\\
2R_n &= -S_{n+2}+13S_{n+1}-9S_n-7S_{n-1}-5S_{n-2}-3S_{n-3},\\
R_n &= 6S_{n+1}-5S_n-4S_{n-1}-3S_{n-2}-2S_{n-3}-S_{n-4}.
\end{aligned}$$

Note that we also can give some other identities which related to $\{K_n\}$ and $\{Q_n\}$. For example, we have

$$\begin{aligned}
16K_n &= -29S_{n+5}+35S_{n+4}+35S_{n+3}+35S_{n+2}+35S_{n+1}-45S_n,\\
16Q_n &= -29S_{n+5}+35S_{n+4}+35S_{n+3}+35S_{n+2}+35S_{n+1}-61S_n.
\end{aligned}$$

4. Linear Sums

4.1. Sums of Terms with Positive Subscripts:

The following proposition presents some formulas of generalized sixth order Jacobsthal numbers with positive subscripts.

Proposition 4.1.
For $n\geq 0$ we have the following formula:

$$\sum_{k=0}^{n} V_k=\frac{1}{6}(V_{n+6}-V_{n+4}-2V_{n+3}-3V_{n+2}-4V_{n+1}-V_5+V_3+2V_2+3V_1+4V_0)$$

Proof. Take $r=1, s=1, t=1, u=1, v=1, y=2$, in Theorem 2.1 in [27]. □

From the last proposition, we have the following corollary which gives sum formulas of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas, respectively.

Corollary 4.1.
For $n\geq 0$, we have the following formulas:

(a) $\sum_{k=0}^{n} J_k=\frac{1}{6}(J_{n+6}-J_{n+4}-2J_{n+3}-3J_{n+2}-4J_{n+1}+5)$.

(b) $\sum_{k=0}^{n} j_k=\frac{1}{6}(j_{n+6}-j_{n+4}-2j_{n+3}-3j_{n+2}-4j_{n+1}-9)$.

(c) $\sum_{k=0}^{n} K_k = \frac{1}{6}(K_{n+6} - K_{n+4} - 2K_{n+3} - 3K_{n+2} - 4K_{n+1} - 9).$

(d) $\sum_{k=0}^{n} Q_k = \frac{1}{6}(Q_{n+6} - Q_{n+4} - 2Q_{n+3} - 3Q_{n+2} - 4Q_{n+1} - 8).$

(e) $\sum_{k=0}^{n} S_k = \frac{1}{6}(S_{n+6} - S_{n+4} - 2S_{n+3} - 3S_{n+2} - 4S_{n+1} - 1).$

(f) $\sum_{k=0}^{n} R_k = \frac{1}{6}(R_{n+6} - R_{n+4} - 2R_{n+3} - 3R_{n+2} - 4R_{n+1} + 9).$

4.2. Sums of Terms with Negative Subscripts

The following proposition presents some formulas of generalized sixth order Jacobsthal numbers with negative subscripts.

Proposition 4.2.
For $n \geq 1$ we have the following formula:

$$\sum_{k=1}^{n} V_{-k} = \frac{1}{6}(-V_{-n+5} + V_{-n+3} + 2V_{-n+2} + 3V_{-n+1} + 4V_{-n} + V_5 - V_3 - 2V_2 - 3V_1 - 4V_0).$$

Proof. Take $r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$, in Theorem 3.1 in [27]. □

From the last proposition, we have the following corollary which gives sum formulas of sixth order Jacobsthal, sixth order Jacobsthal-Lucas, modified sixth order Jacobsthal, sixth order Jacobsthal Perrin, adjusted sixth order Jacobsthal and modified sixth order Jacobsthal-Lucas, respectively.

Corollary 4.2.
For $n \geq 1$, we have the following formulas:

(a) $\sum_{k=1}^{n} J_{-k} = \frac{1}{6}(-J_{-n+5} + J_{-n+3} + 2J_{-n+2} + 3J_{-n+1} + 4J_{-n} - 5).$

(b) $\sum_{k=1}^{n} j_{-k} = \frac{1}{6}(-j_{-n+5} + j_{-n+3} + 2j_{-n+2} + 3j_{-n+1} + 4j_{-n} + 9).$

(c) $\sum_{k=1}^{n} K_{-k} = \frac{1}{6}(-K_{-n+5} + K_{-n+3} + 2K_{-n+2} + 3K_{-n+1} + 4K_{-n} + 9).$

(d) $\sum_{k=1}^{n} Q_{-k} = \frac{1}{6}(-Q_{-n+5} + Q_{-n+3} + 2Q_{-n+2} + 3Q_{-n+1} + 4Q_{-n} + 8).$

(e) $\sum_{k=1}^{n} S_{-k} = \frac{1}{6}(-S_{-n+5} + S_{-n+3} + 2S_{-n+2} + 3S_{-n+1} + 4S_{-n} + 1).$

(f) $\sum_{k=1}^{n} R_{-k} = \frac{1}{6}(-R_{-n+5} + R_{-n+3} + 2R_{-n+2} + 3R_{-n+1} + 4R_{-n} - 9).$

4.3. A Sum Formula

The formula in the following proposition can be used to calculate the sum of the generalized sixth order Jacobsthal numbers.

Proposition 4.3.
For all integers m and j with $R_m^4 - 4R_m^3 + 3R_{2m}^2 + 12R_m^2 - 6R_m^2R_{2m} + 8R_{3m}R_m + 12R_{2m}R_m - 6R_{4m} - 8R_{3m} - 12R_{2m} - 24R_m - 24(-2)^m(R_{-m} - 1) + 24 \neq 0$, we have

$$\sum_{k=0}^{n} V_{mk+j} = \frac{\Lambda + \Psi_1}{\Omega} \tag{17}$$

where

$\Lambda = -24V_{mn+5m+j} + 24(R_m - 1)V_{mn+4m+j} - 12(R_m^2 - 2R_m - R_{2m} + 2)V_{mn+3m+j} + 4(R_m^3 - 3R_m^2 - 3R_{2m}R_m + 3R_{2m} + 2R_{3m} + 6R_m - 6)V_{mn+2m+j} + (-R_m^4 + 4R_m^3 + 6R_m^2R_{2m} - 12R_m^2 - 3R_{2m}^2 - 12R_{2m}R_m - 8R_{3m}R_m + 6R_{4m} + 8R_{3m} + 12R_{2m} + 24R_m - 24)V_{mn+m+j} + 24(-2)^m V_{mn+j}$

$\Psi_1 = 24V_{5m+j} - 24(R_m - 1)V_{4m+j} + 12(R_m^2 - 2R_m - R_{2m} + 2)V_{3m+j} - 4(R_m^3 - 3R_m^2 - 3R_{2m}R_m + 3R_{2m} + 2R_{3m} + 6R_m - 6)V_{2m+j} - (-R_m^4 + 4R_m^3 + 6R_m^2R_{2m} - 12R_m^2 - 3R_{2m}^2 - 12R_{2m}R_m - 8R_{3m}R_m + 6R_{4m} + 8R_{3m} + 12R_{2m} + 24R_m - 24)V_{m+j} - (-R_m^4 + 4R_m^3 - 12R_m^2 + 6R_m^2R_{2m} - 3R_{2m}^2 - 8R_{3m}R_m - 12R_{2m}R_m + 6R_{4m} + 8R_{3m} + 12R_{2m} + 24R_m + 24(-2)^m R_{-m} - 24)V_j$

$\Omega = R_m^4 - 4R_m^3 + 3R_{2m}^2 + 12R_m^2 - 6R_m^2R_{2m} + 8R_{3m}R_m + 12R_{2m}R_m - 6R_{4m} - 8R_{3m} - 12R_{2m} - 24R_m - 24(-2)^m(R_{-m} - 1) + 24$

Proof. Take $r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ and $R_n = H_n$ in Soykan [28], Theorem 13. □

Note that (17) can be written in the following form:

$$\sum_{k=1}^{n} V_{mk+j} = \frac{\Lambda + \Psi_2}{\Omega}$$

where

$\Psi_2 = 24V_{5m+j} - 24(R_m - 1)V_{4m+j} + 12(R_m^2 - 2R_m - R_{2m} + 2)V_{3m+j} - 4(R_m^3 - 3R_m^2 - 3R_{2m}R_m + 3R_{2m} + 2R_{3m} + 6R_m - 6)V_{2m+j} - (-R_m^4 + 4R_m^3 + 6R_m^2 R_{2m} - 12R_m^2 - 3R_{2m}^2 - 12R_{2m}R_m - 8R_{3m}R_m + 6R_{4m} + 8R_{3m} + 12R_{2m} + 24R_m - 24)V_{m+j} - 24\left(-y\right)^m V_j$.

As special cases of the above proposition, we have the following identities.

Corollary 4.3.
The following identities hold:

1. $m = 1,\ j = 3$.

   (a) $\sum_{k=0}^{n} J_{k+3} = \frac{1}{6}(J_{n+8} - J_{n+6} - 2J_{n+5} - 3J_{n+4} + 2J_{n+3} - 7)$.

   (b) $\sum_{k=0}^{n} j_{k+3} = \frac{1}{6}(j_{n+8} - j_{n+6} - 2j_{n+5} - 3j_{n+4} + 2j_{n+3} - 57)$.

   (c) $\sum_{k=0}^{n} K_{k+3} = \frac{1}{6}(K_{n+8} - K_{n+6} - 2K_{n+5} - 3K_{n+4} + 2K_{n+3} - 51)$.

   (d) $\sum_{k=0}^{n} Q_{k+3} = \frac{1}{6}(Q_{n+8} - Q_{n+6} - 2Q_{n+5} - 3Q_{n+4} + 2Q_{n+3} - 38)$.

   (e) $\sum_{k=0}^{n} S_{k+3} = \frac{1}{6}(S_{n+8} - S_{n+6} - 2S_{n+5} - 3S_{n+4} + 2S_{n+3} - 13)$.

   (f) $\sum_{k=0}^{n} R_{k+3} = \frac{1}{6}(R_{n+8} - R_{n+6} - 2R_{n+5} - 3R_{n+4} + 2R_{n+3} - 51)$.

2. $m = -1,\ j = 0$.

   (a) $\sum_{k=0}^{n} J_{-k} = \frac{1}{6}\left(-J_{-n} - 6J_{-n-1} - 5J_{-n-2} - 4J_{-n-3} - 3J_{-n-4} - 2J_{-n-5} - 5\right)$.

   (b) $\sum_{k=0}^{n} j_{-k} = \frac{1}{6}\left(-j_{-n} - 6j_{-n-1} - 5j_{-n-2} - 4j_{-n-3} - 3j_{-n-4} - 2j_{-n-5} + 21\right)$.

   (c) $\sum_{k=0}^{n} K_{-k} = \frac{1}{6}\left(-K_{-n} - 6K_{-n-1} - 5K_{-n-2} - 4K_{-n-3} - 3K_{-n-4} - 2K_{-n-5} + 27\right)$.

   (d) $\sum_{k=0}^{n} Q_{-k} = \frac{1}{6}\left(-Q_{-n} - 6Q_{-n-1} - 5Q_{-n-2} - 4Q_{-n-3} - 3Q_{-n-4} - 2Q_{-n-5} + 26\right)$.

   (e) $\sum_{k=0}^{n} S_{-k} = \frac{1}{6}\left(-S_{-n} - 6S_{-n-1} - 5S_{-n-2} - 4S_{-n-3} - 3S_{-n-4} - 2S_{-n-5} + 1\right)$.

   (f) $\sum_{k=0}^{n} R_{-k} = \frac{1}{6}\left(-R_{-n} - 6R_{-n-1} - 5R_{-n-2} - 4R_{-n-3} - 3R_{-n-4} - 2R_{-n-5} + 27\right)$.

5. Matrices Related with Generalized Sixth-Order Jacobsthal numbers

We define the square matrix A of order 6 as:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = -2$. We also define

$$B_n = \begin{pmatrix} S_{n+1} & \sum\limits_{k=3}^{7} r_{k-1}S_{n+3-k} & \sum\limits_{k=3}^{6} r_k S_{n+3-k} & \sum\limits_{k=3}^{5} r_{k+1}S_{n+3-k} & \sum\limits_{k=3}^{4} r_{k+2}S_{n+3-k} & r_6 S_n \\ S_n & \sum\limits_{k=4}^{8} r_{k-2}S_{n+3-k} & \sum\limits_{k=4}^{7} r_{k-1}S_{n+3-k} & \sum\limits_{k=4}^{6} r_k S_{n+3-k} & \sum\limits_{k=4}^{5} r_{k+1}S_{n+3-k} & r_6 S_{n-1} \\ S_{n-1} & \sum\limits_{k=5}^{9} r_{k-3}S_{n+3-k} & \sum\limits_{k=5}^{8} r_{k-2}S_{n+3-k} & \sum\limits_{k=5}^{7} r_{k-1}S_{n+3-k} & \sum\limits_{k=5}^{6} r_k S_{n+3-k} & r_6 S_{n-2} \\ S_{n-2} & \sum\limits_{k=6}^{10} r_{k-4}S_{n+3-k} & \sum\limits_{k=6}^{9} r_{k-3}S_{n+3-k} & \sum\limits_{k=6}^{8} r_{k-2}S_{n+3-k} & \sum\limits_{k=6}^{7} r_{k-1}S_{n+3-k} & r_6 S_{n-3} \\ S_{n-3} & \sum\limits_{k=7}^{11} r_{k-5}S_{n+3-k} & \sum\limits_{k=7}^{10} r_{k-4}S_{n+3-k} & \sum\limits_{k=7}^{9} r_{k-3}S_{n+3-k} & \sum\limits_{k=7}^{8} r_{k-2}S_{n+3-k} & r_6 S_{n-4} \\ S_{n-4} & \sum\limits_{k=8}^{12} r_{k-6}S_{n+3-k} & \sum\limits_{k=8}^{11} r_{k-5}S_{n+3-k} & \sum\limits_{k=8}^{10} r_{k-4}S_{n+3-k} & \sum\limits_{k=8}^{9} r_{k-3}S_{n+3-k} & r_6 S_{n-5} \end{pmatrix}$$

and

$$C_n=\begin{pmatrix} V_{n+1} & \sum\limits_{k=3}^{7} r_{k-1}V_{n+3-k} & \sum\limits_{k=3}^{6} r_{k}V_{n+3-k} & \sum\limits_{k=3}^{5} r_{k+1}V_{n+3-k} & \sum\limits_{k=3}^{4} r_{k+2}V_{n+3-k} & r_6V_n \\ V_{n} & \sum\limits_{k=4}^{8} r_{k-2}V_{n+3-k} & \sum\limits_{k=4}^{7} r_{k-1}V_{n+3-k} & \sum\limits_{k=4}^{6} r_{k}V_{n+3-k} & \sum\limits_{k=4}^{5} r_{k+1}V_{n+3-k} & r_6V_{n-1} \\ V_{n-1} & \sum\limits_{k=5}^{9} r_{k-3}V_{n+3-k} & \sum\limits_{k=5}^{8} r_{k-2}V_{n+3-k} & \sum\limits_{k=5}^{7} r_{k-1}V_{n+3-k} & \sum\limits_{k=5}^{6} r_{k}V_{n+3-k} & r_6V_{n-2} \\ V_{n-2} & \sum\limits_{k=6}^{10} r_{k-4}V_{n+3-k} & \sum\limits_{k=6}^{9} r_{k-3}V_{n+3-k} & \sum\limits_{k=6}^{8} r_{k-2}V_{n+3-k} & \sum\limits_{k=6}^{7} r_{k-1}V_{n+3-k} & r_6V_{n-3} \\ V_{n-3} & \sum\limits_{k=7}^{11} r_{k-5}V_{n+3-k} & \sum\limits_{k=7}^{10} r_{k-4}V_{n+3-k} & \sum\limits_{k=7}^{9} r_{k-3}V_{n+3-k} & \sum\limits_{k=7}^{8} r_{k-2}V_{n+3-k} & r_6V_{n-4} \\ V_{n-4} & \sum\limits_{k=8}^{12} r_{k-6}V_{n+3-k} & \sum\limits_{k=8}^{11} r_{k-5}V_{n+3-k} & \sum\limits_{k=8}^{10} r_{k-4}V_{n+3-k} & \sum\limits_{k=8}^{9} r_{k-3}V_{n+3-k} & r_6V_{n-5} \end{pmatrix}$$

where

$$r_1=1, r_2=1, r_3=1, r_4=1, r_5=1, r_6=2.$$

Theorem 5.1.
For all integer $m,n\geq 0$, we have

(a) $B_n = A^n$

(b) $C_1A^n = A^nC_1$

(c) $C_{n+m} = C_nB_m = B_mC_n$.

Proof. Take $r=1, s=1, t=1, u=1, v=1, y=2$ and $S_n = G_n$ in Soykan [[28], Theorem 16]. □

Theorem 5.2.
For all integers m,n, we have

$$\begin{aligned} V_{n+m} &= V_nS_{m+1}+V_{n-1}(S_m+S_{m-1}+S_{m-2}+S_{m-3}+2S_{m-4}) \\ &\quad +V_{n-2}(S_m+S_{m-1}+S_{m-2}+2S_{m-3}) \\ &\quad +V_{n-3}(S_m+S_{m-1}+2S_{m-2})+V_{n-4}(S_m+2S_{m-1})+2S_mV_{n-5}. \end{aligned}$$

Proof. Take $r=1, s=1, t=1, u=1, v=1, y=2$ and $S_n = G_n$ in Soykan [[28], Theorem 17]. □

Corollary 5.1.
For all integers m,n, we have

$$\begin{aligned} J_{n+m} &= J_nS_{m+1}+J_{n-1}(S_m+S_{m-1}+S_{m-2}+S_{m-3}+2S_{m-4})+J_{n-2}(S_m+S_{m-1}+S_{m-2}+2S_{m-3}) \\ &\quad +J_{n-3}(S_m+S_{m-1}+2S_{m-2})+J_{n-4}(S_m+2S_{m-1})+2S_mJ_{n-5}, \\ j_{n+m} &= j_nS_{m+1}+j_{n-1}(S_m+S_{m-1}+S_{m-2}+S_{m-3}+2S_{m-4})+j_{n-2}(S_m+S_{m-1}+S_{m-2}+2S_{m-3}) \\ &\quad +j_{n-3}(S_m+S_{m-1}+2S_{m-2})+j_{n-4}(S_m+2S_{m-1})+2S_mj_{n-5}, \\ K_{n+m} &= K_nS_{m+1}+K_{n-1}(S_m+S_{m-1}+S_{m-2}+S_{m-3}+2S_{m-4})+K_{n-2}(S_m+S_{m-1}+S_{m-2}+2S_{m-3}) \\ &\quad +K_{n-3}(S_m+S_{m-1}+2S_{m-2})+K_{n-4}(S_m+2S_{m-1})+2S_mK_{n-5}, \\ Q_{n+m} &= Q_nS_{m+1}+Q_{n-1}(S_m+S_{m-1}+S_{m-2}+S_{m-3}+2S_{m-4})+Q_{n-2}(S_m+S_{m-1}+S_{m-2}+2S_{m-3}) \\ &\quad +Q_{n-3}(S_m+S_{m-1}+2S_{m-2})+Q_{n-4}(S_m+2S_{m-1})+2S_mQ_{n-5}, \\ S_{n+m} &= S_nS_{m+1}+S_{n-1}(S_m+S_{m-1}+S_{m-2}+S_{m-3}+2S_{m-4})+S_{n-2}(S_m+S_{m-1}+S_{m-2}+2S_{m-3}) \\ &\quad +S_{n-3}(S_m+S_{m-1}+2S_{m-2})+S_{n-4}(S_m+2S_{m-1})+2S_mS_{n-5}, \\ R_{n+m} &= R_nS_{m+1}+R_{n-1}(S_m+S_{m-1}+S_{m-2}+S_{m-3}+2S_{m-4})+R_{n-2}(S_m+S_{m-1}+S_{m-2}+2S_{m-3}) \\ &\quad +R_{n-3}(S_m+S_{m-1}+2S_{m-2})+R_{n-4}(S_m+2S_{m-1})+2S_mR_{n-5}. \end{aligned}$$

References

[1] Akbulak M., Öteleş A., On the Sum of Pell and Jacobsthal Numbers by Matrix Method, Bull. Iranian Mathematical Society. 40(4), 1017-1025, 2014.

[2] Aydın, F.T., On generalizations of the Jacobsthal sequence, Notes on Number Theory and Discrete Mathematics, 24(1), 120–135, 2018.

[3] Catarino, P., Vasco, P., Campos, A.P.A., Borges, A., New families of Jacobsthal and Jacobsthal-Lucas numbers, Algebra and Discrete Mathematics,20(1), 40–54, 2015.

[4] Cerda-Morales, G., A Note On Modified Third-Order Jacobsthal Numbers, arXiv:1905.00725v1, 2019.

[5] Cerda-Morales G., On fourth-order jacobsthal quaternions, Journal of Mathematical Sciences and Modelling, 1 (2), 73-79, 2018.

[6] Cerda-Morales G., On the Third-Order Jacobsthal and Third-Order Jacobsthal–Lucas Sequences and Their Matrix Representations, Mediterranean Journal of Mathematics, Mediterr. J. Math. (2019) 16:32 https://doi.org/10.1007/s00009-019-1319-9 1660-5446/19/020001-12, 2019.

[7] Cerda-Morales G., Third-Order Jacobsthal Generalized Quaternions, SCImago Journal & Country Rank, 50, 1-100, 2018.

[8] Cerda-Morales G., The Third Order Jacobsthal Octonions: Some Combinatorial Properties, An. St. Univ. Ovidius Constanta, 26(3), 57-71, 2018.

[9] Cerda-Morales G., Identities for Third Order Jacobsthal Quaternions, Advances in Applied Clifford Algebras, 27, 1043–1053, 2017.

[10] Čerin, Z., Formulae for sums of Jacobsthal–Lucas numbers, Int. Math. Forum, 2(40), 1969–1984, 2007.

[11] Čerin, Z., Sums of Squares and Products of Jacobsthal Numbers. Journal of Integer Sequences, 10, Article 07.2.5, 2007.

[12] Choi, E., Modular Tribonacci Numbers by Matrix Method, Journal of the Korean Society of Mathematical Education Series B: Pure and Applied. Mathematics. 20(3), 207–221, 2013.

[13] Cook C. K., Bacon, M. R., Some Identities for Jacobsthal and Jacobsthal-Lucas Numbers Satisfying Higher Order Recurrence Relations, Annales Mathematicae et Informaticae, 41, 27–39, 2013.

[14] Daşdemir, A., On The Jacobsthal Numbers By Matrix Method. SDU Journal of Science, 7(1), 69-76, 2012.

[15] Daşdemir, A., A study on the Jacobsthal and Jacobsthal–Lucas numbers by matrix method, DUFED Journal of Sciences, 3(1), 13–18, 2014.

[16] Gnanam, A., Anitha, B., Sums of Squares Jacobsthal Numbers. IOSR Journal of Mathematics, 11(6), 62-64. 2015.

[17] Horadam, A.F., Jacobsthal Representation Numbers, Fibonacci Quarterly, 34, 40-54, 1996.

[18] Horadam, A.F., Jacobsthal and Pell Curves, Fibonacci Quarterly, 26, 77-83, 1988.

[19] Howard, F.T., Saidak, F., Zhou's Theory of Constructing Identities, Congress Numer. 200, 225-237, 2010.

[20] Kocer, G.E., Circulant, Negacyclic and Semicirculant Matrices With the Modified Pell, Jacobsthal and Jacobsthal-Lucas Numbers, Hacettepe Journal of Mathematics and Statistics, 36(2). 133-142, 2007.

[21] Köken, F., Bozkurt, D., On the Jacobsthal numbers by matrix methods, Int. J. Contemp Math. Sciences, 3(13), 605–614, 2008.

[22] Mazorchuk V. New Families of Jacobsthal and Jacobsthal-Lucas Numbers. Algebra and Discrete Mathematics. Vol. 20. No 1. 40-54. 2015.

[23] Natividad, L. R., On Solving Fibonacci-Like Sequences of Fourth, Fifth and Sixth Order, International Journal of Mathematics and Computing, 3 (2), 2013.

[24] Rathore, G.P.S., Sikhwal, O., Choudhary, R., Formula for finding nth Term of Fibonacci-Like Sequence of Higher Order, International Journal of Mathematics And its Applications, 4 (2-D), 75-80, 2016.

[25] N.J.A. Sloane, The on-line encyclopedia of integer sequences, http://oeis.org/

[26] Soykan, Y., Simson Identity of Generalized m-step Fibonacci Numbers, International Journal of Advances in Applied Mathematics and Mechanics, 7(2), 45-56, 2019.

[27] Soykan, Y., A Study On Sum Formulas of Generalized Sixth-Order Linear Recurrence Sequences, Asian Journal of Advanced Research and Reports, 14(2), 36-48, 2020. DOI: 10.9734/AJARR/2020/v14i230329

[28] Soykan, Y., A Study On Generalized (r,s,t,u,v,y)-Numbers, Journal of Progressive Research in Mathematics, 17(1), 54-72, 2020.

[29] Soykan, Y., On Generalized Sixth-Order Pell Sequence, Journal of Scientific Perspectives, 4(1), 49-70, 2020, DOİ: https://doi.org/10.26900/jsp.4.005.

[30] Soykan, Y., Properties of Generalized 6-primes Numbers, Archives of Current Research International, 20(6), 12-30, 2020. DOI: 10.9734/ACRI/2020/v20i630199

[31] Srividhya, G., Ragunathan, T., k-Jacobsthal and k-Jacobsthal Lucas Numbers and their Associated Numbers, International Journal of Research in Advent Technology, 7(1), 2019.

[32] Uygun, S., The binomial transforms of the generalized (s,t)-Jacobsthal matrix sequence, Int. J. Adv. Appl. Math.

and Mech. 6(3) (2019) 14–20.