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The packing chromatic number of total graph of some standard graphs

Research Article

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Abstract: Let G be a connected graph and k be an integer, $k \geq 1$. A packing k-coloring of a graph G is a mapping $f : V(G) \rightarrow \{1,2,\ldots,k\}$ such that any two vertices of color i are at distance at least $i+1$. The packing chromatic number $\chi_\rho(G)$ is the smallest integer k for which G has packing k-coloring. In this paper, we compute the packing chromatic number in total graphs.

MSC: 05C15 • 05C75

Keywords: Coloring • Packing chromatic number • Total graph

1. Introduction

All the graphs $G = (V, E)$ considered here are simple, finite and undirected, where $|V| = p$ denotes number of vertices and $|E| = q$ denotes number of edges of G. In general we use $\langle X_i \rangle$ to denote subgraph induced by the set of vertices X and $N(v)$ and $N[v]$ denote open and closed neighborhood of a vertex v, respectively. Let $deg(v)$ be the degree of vertex v and usual $\delta(G)$ the minimum degree and $\Delta(G)$ the maximum degree of a graph G. A subgraph H of a graph G is called a component of G, if H is maximally connected sub graph of G. Any undefined term in this paper may be found in Harary [6].

A coloring of a graph G is an assignment of colors to its vertices. So, that no two adjacent vertices have the same color. The set of all vertices with any one color is independent and is called a color class. An k-coloring of a graph G uses k-colors. The chromatic number $\chi(G)$ is defined as the minimum k for which G has an k-coloring. For complete review on theory of coloring we refer [4] and [7].

Let G be a connected graph and k be a positive integer. A packing k-coloring of a graph G is a mapping $f : V(G) \rightarrow \{1,2,\ldots,k\}$ such that any two vertices of color i are at distance at least $i+1$. The packing chromatic number $\chi_\rho(G)$ of G is the smallest integer k for which G has packing k-coloring. The concept of packing coloring comes from the area of frequency assignment in wireless networks and was introduced by Goddard et al. [5] under the name broadcast coloring. It has several applications, such as, in resource placement and biological diversity. The term packing chromatic number was introduced by Bresar et al., [2]. For complete review on packing coloring, we refer to [9].

The total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G. The structural properties of total graph are investigate in [1] and [10].

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2. Results

Proposition 2.1.
Let P_p and C_p be a Path and Cycle with $p \geq 3$ vertices, respectively. Then

(i) $\chi_\rho(P_p) = \begin{cases} 2, if\ p < 3, \\ 3, if\ p \geq 4. \end{cases}$

(ii) $\chi_\rho(C_p) = \begin{cases} 3, if\ p = 4t; t \geq 1 \\ 4, otherwise. \end{cases}$

Proposition 2.2.
For any nontrivial graph G with no isolated vertices,

$$\chi_\rho(G) \leq \chi(T(G)) \leq \chi_\rho(T(G)).$$

Theorem 2.1.
Let $T(P_p)$ be the total graph of path of order p. Then

$$\chi_\rho(T(P_p)) = \begin{cases} 3, \text{ if } p = 2; \\ 4, \text{ if } p = 3; \\ 5, \text{ if } p = 4; \\ 6, otherwise. \end{cases}$$

Proof. Let $v_0, v_1, \ldots, v_{p-1}$ and $e_0 = (v_0; v_1), e_1 = (v_1; v_2), \ldots, e_{p-2} = (v_{p-2}; v_{p-1})$ are the vertices and edges of path P_p then $V(T(P_p)) = \{v_0, v_1, \ldots, v_{p-1}, e_0, e_1, \ldots, e_{p-2}\}$. Then, we have the following cases.

(i) First assume that $p = 2$. The graph $T(P_2)$ is a graph K_3 and the required packing coloring of $T(P_2)$ be $\{v_1, v_2\}$. Coloring the vertices with pair wise $i \geq 1$ and the sequence $12, 13, \ldots$ like wise and hence $\chi_\rho(T(P_2)) = 3$.

(ii) Next, assume that $p = 3$. The graph $K_{1,4}$ is a subgraph of $T(P_4)$ and the required packing coloring of $T(P_4)$, be $\{v_1, v_2, v_3\}$. Coloring the vertices with pair wise $i \geq 1$ and the sequence $12, 13, \ldots$ like wise and hence $\chi_\rho(T(P_3)) = 4$.

(iii) Now, assume that $p = 4$ and let the required packing coloring of $T(P_4)$ be $\{v_1, v_2, v_3, v_4\}$. Color the vertices of $T(P_4)$ in such a way that the corresponding color classes. Hence, $\chi_\rho(T(P_4)) = 5$.

(iv) For $p \geq 5$, in the graph $T(P_p)$, the close neighborhood of each e_i where $i = 1, \ldots, p-3$ contains three vertices with degree Δ. Hence by Proposition 2.1(i), $\chi_\rho(T(P_p)) = 6$. □

Illustration : The packing chromatic number of $T(P_8)$ as shown in Figure 1. Here, $\chi_\rho(T(P_8)) = 6$.

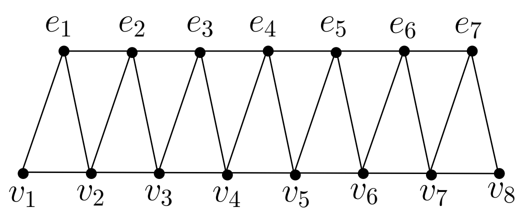

Fig. 1.

Theorem 2.2.
Let $T(C_p)$ be the total graph of cycle of order $p \geq 3$. Then

$$\chi_\rho(T(C_p)) = \left\lceil \frac{p+q}{2} \right\rceil + 3.$$

Proof. Let $v_0, v_1, \ldots, v_{p-1}$ and $e_0 = (v_0, v_1), e_1 = (v_1, v_2), \ldots, e_{p-1} = (v_{p-1}, v_0)$ are the vertices and edges of cycle C_p then $V(T(C_p)) = \{v_0, v_1, \ldots, v_{p-1}, e_0, e_1, \ldots, e_{p-1}\}$. The required packing coloring of $T(C_p)$ be $\{v_1, v_2, v_3, \ldots, v_p\}$. Coloring the vertices in a cyclic way the sequence $12, 13, \ldots$. Thus by the coloring procedure, the packing chromatic number of total graph of every cycle is $\lceil \frac{p+q}{2} \rceil + 3$. □

Illustration : The packing chromatic number of $T(C_5)$ as shown in in figure 2. Here, $\chi_\rho(T(C_5)) = 8$.

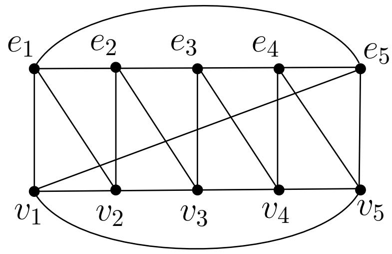

Fig. 2.

Theorem 2.3.
Total graph of a cycle C_p is 4 regular and has $2p$ vertices. More over, if G has $2p$ vertices and 4-regular then G is isomorphic to $T(C_p)$.

Theorem 2.4.
Let $T(K_p)$ be the total graph of complete graph of order $p \geq 3$. Then

$$\chi_\rho(T(K_p)) = p + q - 1.$$

Proof. Let $\{v_i : 1, 2, \ldots, p\}$ be the vertices and $\{e_i : i = 1, 2, \ldots, \frac{p(p-1)}{2}\}$ be the edges of a complete graph K_p. Then, $T(G)$ has the vertex set $\{u_1, u_2, \ldots, u_p, u_{p+1}, \frac{u_{p(p-1)}}{2}\}$. Consider a packing coloring of total graph of K_p as follows. For $1 \leq i \leq p$, assign the color i to v_i and for $2 \leq i \leq p-1$, assign the color i to e_i. Coloring the vertices in a cyclic way the sequence $12, 13, \ldots$. Thus by the coloring procedure, the packing chromatic number of total graph of every complete graph is $p + q - 1$. □

Illustration : The packing chromatic number of $T(K_4)$ as shown in Figure 3. Here, $\chi_\rho(T(K_4)) = 9$.

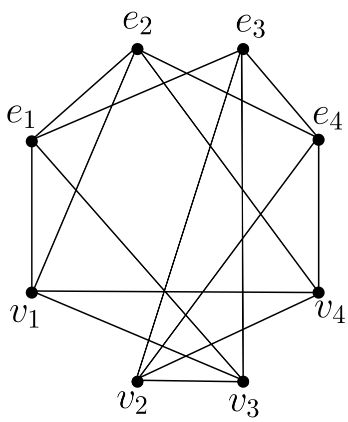

Fig. 3.

Theorem 2.5.
Let $T(K_{1,p})$ be the total graph of star graph of order $p \geq 3$. Then

$$\chi_\rho(T(K_{1,p})) = 2p + 1.$$

Proof. Let $v_1, v_2, \dots, v_p$ and $e_1 = (v_1, v_2), \dots, e_p = (v_1, v_p)$ are the vertices and edges of star $K_{1,p}$ then $V(T(K_{1,p})) = \{v_1, \dots, v_p, e_1, \dots, e_p\}$. The star $K_{1,2p}$ is a subgraph of $T(K_{1,p})$ and hence $\chi_p(T(K_{1,p})) \geq 2p+1$. Now define $f : V(T(K_{1,p})) \to \{1, 2, \dots, 2p+1\}$ as follows. The above defined function provides packing chromatic number for $T(K_{1,p})$ and hence $\chi_\rho(T(K_{1,p})) \leq 2p+1$. Thus, we have $\chi_\rho(T(K_{1,p})) = 2p+1$. □

Illustration : The packing chromatic number of $T(K_{1,3})$ as shown in Figure 4. Here, $\chi_\rho(T(K_{1,3})) = 7$.

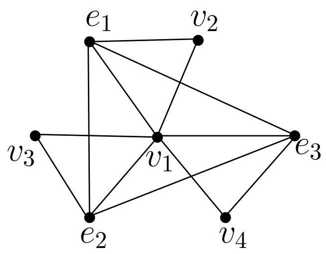

Fig. 4.

Theorem 2.6.
Let $T(K_{r,s})$ be the total graph of complete bipartite graph of order $p \geq 4$. Then

$$\chi_\rho(T(K_{r,s})) = \begin{cases} 2r+3, \text{ if } r = s \\ 3s+1, \text{ if } r < s \end{cases}$$

Proof. Consider the complete bipartite graph $K_{r,s}$ with bipartition $L = \{v_1, v_2, v_3, \dots, \dots, v_r\}$ and $M = \{u_1, u_2, u_3, \dots, u_s\}$. By the definition of Total graph, let v_{ij} be the newly introduced vertex in the edge connecting v_i and u_j in $T(K_{r,s})$. The vertex set of $V[T(K_{r,s})] = \{v_i / 1 \leq i \leq r\} \cup \{u_j / 1 \leq j \leq s\} \cup \{v_{ij} / 1 \leq i \leq r, 1 \leq j \leq s\}$. Here every vertex $v_i : 1 \leq i \leq r$ and $u_j : 1 \leq j \leq s$ are mutually adjacent with each other. Also every vertex v_i is incident with s edges and every vertex u_j is incident with r edges. Therefore degree of vertices in L is of degree $2s$ in $T(K_{r,s})$ and degree of vertices in M is of degree $2r$ in $T(K_{r,s})$. Also we find s disjoint cliques of order s in every $T(K_{r,s})$.

Case 1 : when $r = s$

Clearly $u_1, u_2, u_3, \dots, u_s$ and $v_1, v_2, v_3, \dots, v_r$ are vertices in v_i and u_j i.e. $|v_i| = r$ and $|u_j| = s$. Consider the color class $X = \{X_1, X_2, X_3, \dots, X_k, X_{k+1}, X_{k+2}\}$. Now assign a proper coloring to $T(K_{r,s})$ as follows. Assign the color X_1 to the vertex v_i for $i = 1, 2, 3, \dots, r$, assign the color X_{i+j} to v_{ij}'s when $i + j \leq s+1$ and assign the color X_{i+j-s} to v_{ij}'s when $i + j > s+1$. Here in $T(K_{r,s})$ the vertex v_i along with edges incident with v_i forms a clique of order $s+1$, also each $v_i (1 \leq i \leq r)$ is adjacent with u_j for $1 \leq j \leq s$. Due to this, the above coloring produces a packing coloring. Also we see that in $T(K_{r,s})$ each v_i receives one color different from the color class assigned to the clique introduced by v_{ij}'s for $1 \leq i \leq s$, $1 \leq j \leq r$. Next we assign the color X_{s+2} to u_j for $j = 1, 2, 3, \dots, s$, the vertices $v_i (1 \leq i \leq r)$ and $u_j (1 \leq j \leq s)$ realizes its own color, which produces a packing coloring. Thus by the coloring procedure, the above said coloring is maximum and packing chromatic. Therefore $\chi_\rho(T(K_{r,s})) = 2r+3$. When $r = s$.

Illustration : The packing chromatic number of $T(K_{2,2})$ as shown in Figure 5. Here, $\chi_\rho(T(K_{2,2})) = 7$.

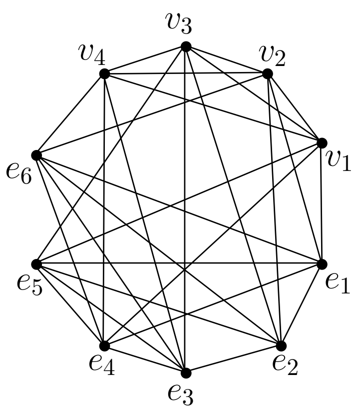

Fig. 5.

Case 2 : when $r < s$

Assign the color X_1 to the vertex v_i for $i = 1,2,3,\dots,s$, assign the color X_{i+j} to v_{ij}'s when $i + j \le s+1$ and assign the color X_{i+j-s} to remaining v_{ij} when $i + j > s+1$. Suppose if we assign the color X_{s+2} to the u_j for $j = 1,2,3,\dots,s$ (as in case 1), it contradicts the definition of packing coloring because all u_j for $j = 1,2,3,\dots,s$ are adjacent with v_i for $i = 1,2,3,\dots,r$ but $r < s$, due to this condition the vertex u_j realize the new color, which is the produce a packing coloring. Thus by the coloring procedure, the above said coloring is maximum and packing chromatic. Therefore $\chi_\rho(T(K_{r,s})) = 3s+1$. □

Illustration : The packing chromatic number of $T(K_{2,3})$ as shown in Figure 6. Here, $\chi_\rho(T(K_{2,3})) = 10$.

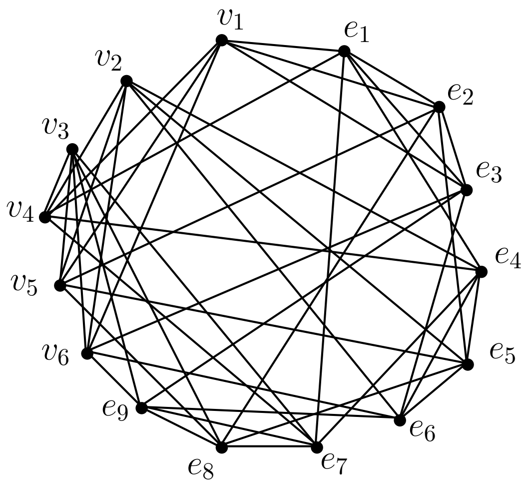

Fig. 6.

Corollary 2.1.

The Total graph of every $K_{s,s}$ is a $2s$ regular graph

Theorem 2.7.

Let $T(F_{1,p})$ be the total graph of fan graph of order $p \ge 3$. Then

$$\chi_\rho(T(F_{1,p})) = 2p+1.$$

Proof. Let (X,Y) be a bi-partition of $F_{1,p}$ with $|X| = 1$ and $|Y| = p$. Let $X = \{v\}$ and $Y = \{u_1, u_2, \dots, u_p\}$. In $T(F_{1,p})$ by the definition of Total graph each edge vu_i for $1 \le i \le p$ of $F_{1,p}$ is subdivided by new vertex v'_i and for $i = 1,2,3,\dots,p-1, u_i, u_{i+1}$ is subdivided by vertex w_i where $T = \{w_i/1 \le i \le p-1\}$. Clearly T is an independent set. The vertex set of total graph of $F_{1,p}$ is defined as $V[T(F_{1,p})] = \{v\} \cup \{v'_i/1 \le i \le p\} \cup \{w_i/1 \le i \le p-1\} \cup \{u_i/1 \le i \le p\}$. Here the vertices $v, v'_1, v'_2, v'_3, \dots, v'_p$ induces a clique of order $p+1$ in $T(F_{1,p})$. So that we say the packing chromatic number of Total graph of Fan graph will have more than or equal to $2p+1$ colors. Now we assign a proper coloring to these vertices as follows. Consider the color class $X = \{X_1, X_2, X_3, \dots, X_k\}$. Assign the color X_i to the vertex v'_i for the distance $i > 1$. Here the vertices assigning colors pairwise and distinct color which produces a packing coloring. Thus there is a possibility of assigning not more than $2p+1$ colors to every total graph of fan graph that is $\chi_\rho(T(F_{1,p})) \le 2p+1$. Therefore we have $\chi_\rho(T(F_{1,p})) = 2p+1$.

Thus by the coloring procedure the above said coloring produces a maximal and packing coloring. □

Illustration : The packing chromatic number of $T(F_{1,4})$ as shown in Figure 7. Here, $\chi_\rho(T(F_{1,4})) = 9$.

To prove our next result, we make use of the definition of bistar graph as follows.

Bistar $B_{n,n}$ is the graph obtained by joining the center (apex) vertices of two copies of $K_{1,p}$ by an edge. The vertex set of $B_{n,n}$ is $V(B_{n,n}) = \{u, v, u_i, v_i/1 \le i \le n\}$, where u, v are apex vertices and u_i, v_i are pendant vertices. The edge set of $B_{n,n}$ is $E(B_{n,n}) = \{uv, uu_i, vv_i/1 \le i \le n\}$. So, $|V(B_{n,n})| = 2n+2$ and $|E(B_{n,n})| = 2n+1$. For more details, we refer [3].

Theorem 2.8.

Let $T(B_{p,p})$ be the total graph of bistar graph of order p. Then

$$\chi_\rho(T(B_{p,p})) = 2p+3.$$

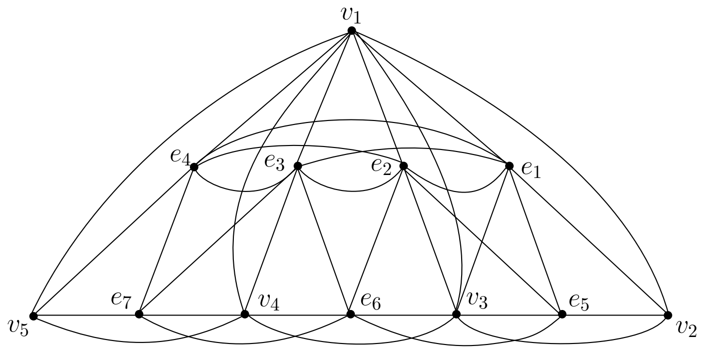

Fig. 7.

Proof. Consider the Bistar $B_{p,p}$. By definition of Bistar, let $u_1, u_2, \ldots, u_p$ be the p pendant edges attached to the vertex u and $v_1, v_2, \ldots, v_p$ be another p pendant edges attached to the vertex v. Consider the total graph of Bistar, by the definition of total graph each edge uu_i and vv_i is subdivided by the newly introduced vertices u'_i and v'_i for $i = 1,2,3,\ldots,p$. Let S be the newly introduced vertex in between the vertices u and v. In $T(B_{p,p})$, both vertex set and edge set of $B_{p,p}$ corresponds to the vertex set of $T(B_{p,p})$. That is $V[T(B_{p,p})] = \{v\} \cup \{u\} \cup \{u_i/1 \le i \le p\} \cup \{v_i/1 \le i \le p\} \cup \{S\} \cup \{u'_i/1 \le i \le p\} \cup \{v'_i/1 \le i \le p\}$. In $T(B_{p,p})$, $u'_1, u'_2, u'_3, \ldots, u'_p$ along with the vertex u and s induces a clique of order $p+2$. Also $v'_1, v'_2, \ldots, v'_p$ along with vertex v and s induces another clique of order $p+2$. Now assign a proper coloring to these vertices as follows. Consider the color class $X = \{X_1, X_2, X_3, \ldots, X_k, X_{k+1}, X_{k+2}, X_{k+3}\}$. First assign the color X_i to u'_i for $i > 1$ and assign the color X_{k+1} to v_i and X_{k+2} to S. Here the above assignment of coloring produces a packing coloring. Next assign the color X_{k+3} to the vertex v and X_{k+3+i} to the vertices u_i for $i > 1$, it does produce a packing coloring. Therefore, we have $\chi_\rho(T(B_{p,p})) = 2p+3$.

Thus by the coloring procedure the above said coloring produces maximum and packing coloring. □

Illustration : The packing chromatic number of $T(B_{5,5})$ as shown in Figure 8. Here, $\chi_\rho(T(B_{5,5})) = 13$.

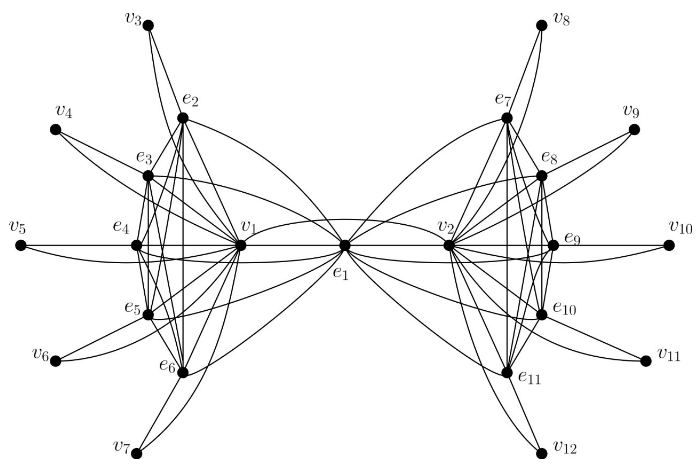

Fig. 8.

Theorem 2.9.
Let $T(K_{1,p,p})$ be the total graph of Double star graph of order p. Then

$$\chi_\rho(T(K_{1,p,p})) = 2p+1.$$

Proof. By the definition of total graph, we have $V(T(K_{1,p,p})) = \{v\} \cup \{v_i : 1 \le i \le p\} \cup \{u_i : 1 \le i \le p\} \cup \{e_i : 1 \le i \le p\} \cup \{s_i : 1 \le i \le p\}$ in which the vertices $v, e_1, e_2, \ldots, e_p$ induce a clique of order $p+1$. Clearly $\chi_\rho(T(K_{1,p,p})) > p+1$. Consider the colors $c_1, c_2, \ldots, c_{p+1}$. Assign χ_ρ-coloring as follows: Assign the colors c_{p+1} to v and assign the colors c_i to e_i, where $1 \le i \le p$. For $1 \le i \le p$, assign two distinct colors other than c_{p+1} and c_i to the vertices v_i and s_i. Finaly, assign the colors c_{p+1} to each $u_i (1 \le i \le p)$. Hence, $T(K_{1,p,p})$ is $\chi_\rho(T(K_{1,p,p})) = 2p+1$. □

Illustration : The packing chromatic number of $T(K_{1,4,4})$ as shown in Figure 9. Here, $\chi_\rho(T(K_{1,4,4})) = 9$.

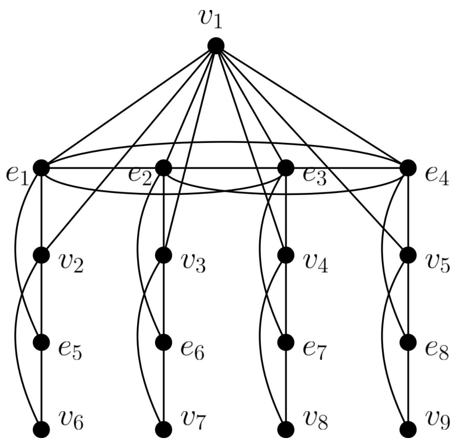

Fig. 9.

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References

[1] M. Behzad, *Graphs and their chromatic numbers*, Doctoral Thesis, Michigan State University (1965).
[2] B. Bresar, S. Klavzar and D.F. Rall, On the packing chromatic number of cartesian products, hexagonal lattice and trees, Discrete Appl. Math. 155, 2303-2311, 2007.
[3] Gallian, A dynamic survey of graph labeling, The electronics journal of Combinatorics, 17, 1–384, 2014.
[4] Gary chartrand and Fing zhang, Chromatic graph theory, Chapman and Hall/CRC, 2009.
[5] W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, J. M. Harris and D. F. Rall, *Broadcast chromatic numbers of graphs*, Ars Combin. 86, 33-49, 2008.
[6] F. Harary, Graph theory, Addison-Wesley, Reading Mass, 1969.
[7] T. R. Jensen and B. Toft, Graph Coloring Problem, John Wiley and Sons, Inc, New York, 1995.
[8] E. Sampathkumar and C. V. Venkatachalam, Chromatic partition of a graph and graph colring and variation, Discrete Maths., 74(1-2): 227-239, 1989.
[9] O. Tongi, On packing colorings of distance graphs, Discrete Appl. Math., 167, 280 - 289, 2014.
[10] J. Thomas and J. varghese, On the decomposition of total graphs, Advanced Modelling and Optimization, 15, 81-84, 2013.
[11] K. Rajalakshmi and M. Venkatachalam, On packing coloring of double wheel graph families, International Journal of Pure and Apllied Mathematics, 119(12), 2389-2396, 2018.