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On $\mathscr{I}^{\omega}-$ openness property in ideal topological spaces

Research Article

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Abstract: In this paper, we introduec and investigate the new class of $\omega-$ open sets in ideal toplogical spaces by giving the concept of $\mathscr{I}^{\omega}-$ open sets then we study the interior operator and clouser operator via this class.

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1. Introduction

The notion of $\omega-$open sets in topological spaces is introduced by Hdeib in 1982, [7], several mathematical researcher introduced the new forms of $\omega-$ open sets such as in 2009, Al-omari and Noorani, [10] introduced the weak form of $\omega-$ open sets, called $pre-\omega-$open sets.

The study of ideal topological spaces is introduced by Kuratowski, [8]. Many researcher studid abawt the ideal topological spaces. An idea $\mathscr{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies the following conditions:
1- if $A\in\mathscr{I}$ and $B\in A$ then $B\in\mathscr{I}$,
2- if $A\in\mathscr{I}$ and $B\in\mathscr{I}$ then $A\cup B\in\mathscr{I}$.
Applications to various fields were further investigated by Jankovic and Hamlett [1], Dontchev [6] and Arenas et al [4]. An ideal topological space is a topological space (X,τ) with an ideal $\mathscr{I}$ on X, and is denoted by $(X,\tau,\mathscr{I})$. Ekici and Noiri, [2] introduced subsets and decompositions of continuity in ideal topological spaces.

In this paper, we introduce the notion of $\mathscr{I}^{\omega}-$open sets as a form stronger than $pre-\omega-$open sets and weaker than $\omega-$open sets and weaker than pre$-\mathscr{I}-$ open sets. This paper is organized as follows. In Section 3, we introduce the concept of $\mathscr{I}^{\omega}-$open sets and we give its relationship with the other known sets. In Section 4, we study the interior operator and closure operator via the class of $\mathscr{I}^{\omega}-$open sets in ideal topological spaces.

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2. Preliminaries

By $Cl(A)$ and $Int(A)$ we mean the closure set and the interior set of A in topological space (X,τ), respectively.

Theorem 2.1.
[5] For a topological space (X,τ) and $A,B \subseteq X$, if B is an open set in X then $Cl(A)\cap B \subseteq Cl(A\cap B)$.

Theorem 2.2.
[5] For a topological space (X,τ),

1. $Cl(X-A) = X - Int(A)$ *for all* $A \subseteq X$.
2. $Int(X-A) = X - Cl(A)$ *for all* $A \subseteq X$.

Definition 2.3.
[7] A subset A of a space X is called $\omega-$open set if for each $x\in A$, there is an open set U_x containing x such that $U_x - A$ is a countable set. The complement of a $\omega-$open set is called a $\omega-$closed set.

In this work, $Int_{\omega}(A)$ denotes the $\omega-$interior operator of A defined as the union of all $\omega-$open sets which contained in A and $Cl_{\omega}(A)$ denotes the ω?closure operator of A defined as the intersection of all $\omega-$closed sets which contain A .

Definition 2.4.
[10] A subset A of a space X is called a $pre-\omega-$open set if $A \subseteq Int_{\omega}(Cl(A))$. The complement of a $pre-\omega-$open set is called $pre-\omega-$closed set.

In The idea topological space $(X,\tau,\mathscr{I})$, $A^*(\mathscr{I})$ is defined by:

$$A^*(\mathscr{I}) = \{x\in X : U\cap A \notin \mathscr{I} \text{ for each open neighborhood } U of x\}$$

is called the local function of asubset A of X with respect to $\mathscr{I}$ and τ , [8]. When there is no chance for confusion $A^*(\mathscr{I})$ is denoted by A^*. For every ideal topological space $(X,\tau,\mathscr{I})$, there exists a topology τ^* finer than τ, generated by the base

$$\beta(\mathscr{I},\tau) = \{U - I : U\in\tau \text{ and } I\in\mathscr{I}\}$$

Observe additionally that $Cl^*(A) = A\cup A^*$, [9], defines a Kuratowski closure operator for τ^*. $Int^*(A)$ will denote the interior of A in (X,τ^*).

Theorem 2.5.
[3] Let $(X,\tau,\mathscr{I})$ be an ideal topological space. Then for $A,B\subseteq X$, the following properties hold:

1. $A\subseteq B$ *implies that* $A^*\subseteq B^*$.
2. $(A\cup B)^* = A^*\cup B^*$.
3. $A^* = Cl(A^*)\subseteq Cl(A)$.
4. *If* $U\in\tau$ *then* $U\cap A^*\subseteq (U\cap A)^*$.
5. $(A^*)^*\subseteq A^*$.

Definition 2.6.
[2] A subset A of an ideal topological space $(X,\tau,\mathscr{I})$ is called pre–$\mathscr{I}-$open set, if $A\subseteq Int(Cl^*(A))$. The complement of pre–$\mathscr{I}-$open set is called pre–$\mathscr{I}-$closed set.

3. $\mathscr{I}^{\omega}-$Open sets

For a topological space (X,τ) and $A \subseteq X$, the $\omega-$*closure operator* of A is a set defined as the intersection of all $\omega-$closed subsets of X containing A and denoted by $Cl_{\omega}(A)$. The $\omega-$*interior operator* of A is a set defined as the union of all $\omega-$open subsets of X contained in A and denoted by $Int_{\omega}(A)$.

Definition 3.1.
*A subset A of ideal topological space $(X,\tau,\mathscr{I})$ is called $\mathscr{I}^{\omega}-$*open set *if $A \subseteq Int_{\omega}(Cl^{*}(A))$. The complement of $\mathscr{I}^{\omega}-$open set is called $\mathscr{I}^{\omega}-$*closed set. *The set of all $\mathscr{I}^{\omega}-$open sets in X denoted by $\mathscr{I}^{\omega}_{O}(X,\tau)$ and the set of all $\mathscr{I}^{\omega}-$closed sets in X denoted by $\mathscr{I}^{\omega}_{C}(X,\tau)$.*

Example 3.2.
Let $(X,\tau,\mathscr{I})$ be an ideal topological space, where $X = \mathbb{R}$, $\tau = \{\emptyset,\mathbb{R},\{1,2\}\}$ and $\mathscr{I} = \{\emptyset,\{1\}\}$. The set $A = \mathbb{R}-\{1,3\}$ is $\mathscr{I}^{\omega}-$open set.

Example 3.3.
For any ideal topological space $(X,\tau,\mathscr{I})$ with countable set X,
$\mathscr{I}^{\omega}_{O}(X,\tau) = P(X) = \mathscr{I}^{\omega}_{C}(X,\tau)$

Theorem 3.4.
Every $\mathscr{I}^{\omega}-$open set in an ideal topological space $(X,\tau,\mathscr{I})$ is $pre-\omega-$open set in a space (X,τ).

Proof. Let A be $\mathscr{I}^{\omega}-$open subset of an ideal topological space $(X,\tau,\mathscr{I})$. Then

$$A \subseteq Int_{\omega}(Cl^{*}(A)) \subseteq Int_{\omega}(Cl(A)).$$

That is, A is $pre-\omega-$open set in a space (X,τ). □

The converse of above theorem no need to be true.

Example 3.5.
Let $(\mathbb{R},\tau,\mathscr{I})$ be an ideal topological space, where $\tau = \{\emptyset,\mathbb{R}\}$ and $\mathscr{I} = \{\emptyset,\{1\}\}$. The set {1} is $pre-\omega-$open set but it is not $\mathscr{I}^{\omega}-$open set.

Theorem 3.6.
Every $\omega-$open set is $\mathscr{I}^{\omega}-$open set.

Proof. Similar for the proof of Theorem(3.4). □

The converse of above theorem no need to be true.

Example 3.7.
*In Example **3.5** The set {2} is $\mathscr{I}^{\omega}-$open set but it is not $\omega-$ open set.*

Theorem 3.8.
Every $pre-\mathscr{I}-$open set is $\mathscr{I}^{\omega}-$open set.

Proof. Let A be pre$-\mathscr{I}-$open subset of an ideal topological space $(X,\tau,\mathscr{I})$. Then

$$A \subseteq Int(Cl^{*}(A)) \subseteq Int_{\omega}(Cl^{*}(A)).$$

That is, A is $\mathscr{I}^{\omega}-$open set in a space $(X,\tau,\mathscr{I})$. □

The converse of above theorem no need to be true.

Example 3.9.
*In Example **3.2** The set $\mathbb{R}-\{1,4\}$ is $\mathscr{I}^{\omega}-$open set but it is not $pre-\mathscr{I}-open$ set.*

From Theorems (3.4), (3.6),(3.8) , We have the following relation for $\mathscr{I}^{\omega}$-open set with the other known sets.

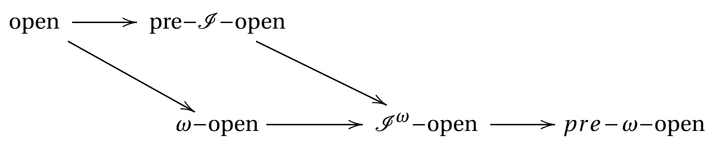

Fig. 1. Relation for sets 1

Theorem 3.10.
A subset A of an ideal topological space $(X,\tau,\mathscr{I})$ is $\mathscr{I}^{\omega}-$closed set if and only if $CL_{\omega}(Int^{*}(A)) \subseteq A$.

Proof. Let A be any $\mathscr{I}^{\omega}-$closed set in ideal topological space $(X,\tau,\mathscr{I})$. That is, $X-A$ is $\mathscr{I}^{\omega}-$open set in ideal topological space $(X,\tau,\mathscr{I})$. Then we have

$$(X-A) \subseteq Int_{\omega}(CL^{*}(X-A)).$$

By using Theorems (2.2), this implies

$$\begin{aligned}(X-A) &\subseteq Int_{\omega}(CL^{*}(X-A)) = Int_{\omega}(X-Int^{*}(A))\\ &= X-Cl_{\omega}(Int^{*}(A)).\end{aligned}$$

Hence $Cl_{\omega}(Int^{*}(A)) \subseteq A$. The convers is similar. □

Theorem 3.11.
Let $(X,\tau,\mathscr{I})$ be an ideal topological space. If G_{λ} is $\mathscr{I}^{\omega}-$open set for each $\lambda \in \Delta$ then $\cup_{\lambda\in\Delta}G_{\lambda}$ is $\mathscr{I}^{\omega}-$open set, where Δ is an index set.

Proof. Since G_{λ} is $\mathscr{I}^{\omega}-$open set for each $\lambda \in \Delta$ then $G_{\lambda} \subseteq Int_{\omega}(CL^{*}(G_{\lambda}))$ for each $\lambda \in \Delta$. Then by Theorem (2.5),

$$\begin{aligned}\cup_{\lambda\in\Delta}G_{\lambda} &\subseteq \cup_{\lambda\in\Delta}Int_{\omega}(Cl^{*}(G_{\lambda}))\\ &\subseteq Int_{\omega}(\cup_{\lambda\in\Delta}Cl^{*}(G_{\lambda}))\\ &\subseteq Int_{\omega}(\cup_{\lambda\in\Delta}(G_{\lambda}\cup(G_{\lambda})^{*})\\ &\subseteq Int_{\omega}((\cup_{\lambda\in\Delta}G_{\lambda})\cup(\cup_{\lambda\in\Delta}(G_{\lambda})^{*})\\ &\subseteq Int_{\omega}(\cup_{\lambda\in\Delta}G_{\lambda}\cup(\cup_{\lambda\in\Delta}G_{\lambda})^{*})\\ &= Int_{\omega}(Cl^{*}(\cup_{\lambda\in\Delta}G_{\lambda})).\end{aligned}$$

Hence $\cup_{\lambda\in\Delta}G_{\lambda}$ is $\mathscr{I}^{\omega}-$open set. □

The intersection of two $\mathscr{I}^{\omega}-$open sets no need to be $\mathscr{I}^{\omega}-$open set.

Example 3.12.
Let $(\mathbb{R},\tau,\mathscr{I})$ be an ideal topological space, where $\tau=\{\emptyset,\mathbb{R}\}$ and $\mathscr{I}=\{\emptyset,\{1\}\}$. The sets $A=\{0,1\}$ and $B=\{1,4\}$ are $\mathscr{I}^{\omega}-$open sets but $A\cap B=\{1\}$ is not $\mathscr{I}^{\omega}-$open set.

Theorem 3.13.
Let $(X,\tau,\mathscr{I})$ be an ideal topological space. If A is an open set in (X,τ) and B is $\mathscr{I}^{\omega}-$open set then $A\cap B$ is $\mathscr{I}^{\omega}-$open set.

Proof. Since B is $\mathscr{I}^{\omega}-$open set then $B \subseteq Int_{\omega}(Cl^{*}(B))$. Then by Theorems (2.5) and (2.1),

$$\begin{aligned}A\cap B &\subseteq A\cap Int_{\omega}(Cl^{*}(B))\\ &= Int_{\omega}(A)\cap Int_{\omega}(Cl^{*}(B))\\ &= Int_{\omega}(A\cap Cl^{*}(B))\\ &\subseteq Int_{\omega}(Cl^{*}(A\cap B))\end{aligned}$$

Hence $A\cap B$ is $\mathscr{I}^{\omega}-$open set. □

We mean by *bitopological space* is a triple (X,τ,ρ) consists two topologies τ and ρ on a set X. A subset $A\subseteq X$ is said to be $\tau\rho-$*open set* in a bitopological space (X,τ,ρ) if $A\subseteq {}_{\tau}Int({}_{\rho}Cl(A))$. The complement of $\tau\rho$-open set is said to be $\tau\rho$*-closed set.*

Theorem 3.14.
A subset $A\subseteq X$ is $\mathscr{I}^{\omega}-$open set in ideal topological space $(X,\tau,\mathscr{I})$ if and only if it is $\tau\tau^{*}-$open set in bitopological space (X,τ,τ^{*}).

Proof. It is clear from the definitions and $Cl^{*}(A)\subseteq Cl(A)$. □

Theorem 3.15.
A subset A of a bitopological space (X,τ,ρ) is $\tau\rho-$closed set if and only if ${}_{\tau}Cl_{\omega}({}_{\rho}Int^{*}(A))\subseteq A$.

Proof. Let A be any $\tau\rho-$closed set in bitopological space (X,τ,ρ). That is, $X-A$ is $\tau\rho-$open set in bitopological space (X,τ,ρ). Hence

$$(X-A)\subseteq {}_{\tau}Int({}_{\rho}Cl(X-A)).$$

By using Theorems (2.2), we get that

$$\begin{aligned}(X-A) &\subseteq {}_{\tau}Int({}_{\rho}Cl(X-A))={}_{\tau}Int(X-{}_{\rho}Int(A))\\ &= X-{}_{\tau}Cl({}_{\rho}Int(A))=X-{}_{\tau}Cl({}_{\rho}Int(A)).\end{aligned}$$

Hence ${}_{\tau}Cl({}_{\rho}Int(A))\subseteq A$. The convers is similar. □

Theorem 3.16.
Let Y be an open subset of an ideal topological space $(X,\tau,\mathscr{I})$. If A is $\mathscr{I}^{\omega}-$open set in $(X,\tau,\mathscr{I})$ then $A\cap Y$ is $\tau|_{Y}\tau^{*}|_{Y}-$open set in bitopological space $(Y,\tau|_{Y},\tau^{*}|_{Y})$.

Proof. Since A is $\mathscr{I}^{\omega}-$open set in $(X,\tau,\mathscr{I})$ then $A\subseteq Int_{\omega}(Cl^{*}(A))$. Then by Theorems (2.5), and (2.1),

$$\begin{aligned}A\cap Y &\subseteq Int_{\omega}(Cl^{*}(A))\cap Y=Int_{\omega}(CL^{*}(A))\cap Y\cap Y\\ &= Int_{\omega}|_{Y}(Cl^{*}(A))\cap Y=Int_{\omega}|_{Y}[(Cl^{*}(A))\cap Int_{\omega}(Y)]\\ &= Int_{\omega}|_{Y}(Cl^{*}(A)\cap Y)=Int_{\omega}|_{Y}(Cl^{*}(A)\cap Y\cap Y)\\ &\subseteq Int_{\omega}|_{Y}(Cl^{*}(A\cap Y)\cap Y)=Int_{\omega}|_{Y}(Cl^{*}|_{Y}(A\cap Y)).\end{aligned}$$

Hence $A\cap Y$ is $\tau|_{Y}\tau^{*}|_{Y}-$open set in $(Y,\tau|_{Y},\tau^{*}|_{Y})$. □

Corollary 3.17.
Let Y be an open subset of an ideal topological space $(X,\tau,\mathscr{I})$. If A is $\mathscr{I}^{\omega}-$closed set in $(X,\tau,\mathscr{I})$ then $A\cap Y$ is $\tau|_{Y}\tau^{*}|_{Y}-$closed set in bitopological space $(Y,\tau|_{Y},\tau^{*}|_{Y})$.

Proof. Let A be $\mathscr{I}^{\omega}-$closed set in $(X,\tau,\mathscr{I})$. Then $X-A$ is $\mathscr{I}^{\omega}-$open set in $(X,\tau,\mathscr{I})$. By the above theorem, $Y-A=(X-A)\cap Y$ is $\tau|_{Y}\tau^{*}|_{Y}-$open set in $(Y,\tau|_{Y},\tau^{*}|_{Y})$. Hence

$$Y-(Y-A)=Y-(Y\cap(X-A))=Y\cap[(X-Y)\cup A]=A\cap Y$$

is $\tau|_{Y}\tau^{*}|_{Y}-$closed set in $(Y,\tau|_{Y},\tau^{*}|_{Y})$. □

Theorem 3.18.
Let Y be an open subset of an ideal topological space $(X,\tau,\mathscr{I})$. If A is $\tau|_{Y}\tau^{*}|_{Y}-$open set in bitopological space $(Y,\tau|_{Y},\tau^{*}|_{Y})$ then A is $\mathscr{I}^{\omega}-$open set in $(X,\tau,\mathscr{I})$.

Proof. Since A is $\tau|_{Y}\tau^{*}|_{Y}-$open set in $(Y,\tau|_{Y},\tau^{*}|_{Y})$ then

$$A\subseteq Int_{\omega}|_{Y}(Cl^{*}|_{Y}(A)).$$

Then by Theorems (2.5) and (2.1),

$$\begin{aligned}A &\subseteq Int_{\omega}|_{Y}(Cl^{*}|_{Y}(A))=Int_{\omega}(Cl^{*}|_{Y}(A))\cap Y\\ &= Int_{\omega}(Cl^{*}|_{Y}(A)\cap Y)\subseteq Int_{\omega}(Cl^{*}|_{Y}(A\cap Y))\\ &= Int_{\omega}(Cl^{*}(A)).\end{aligned}$$

Hence A is $\mathscr{I}^{\omega}-$open set in $(X,\tau,\mathscr{I})$. □

Corollary 3.19.
Let Y be an open subset of a ideal topological space $(X,\tau,\mathscr{I})$. If A is $\tau|_Y\tau^*|_Y-$closed set in bitopological space $(Y,\tau|_Y,\tau^*|_Y)$ then A is $\mathscr{I}^{\omega}-$closed set in $(X,\tau,\mathscr{I})$.

4. $\mathscr{I}^{\omega}-$Operators

Definition 4.1.
Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A\subseteq X$.

1. *The $\mathscr{I}^{\omega}-$*closure operator *of A is denoted by ${}_{\mathscr{I}^{\omega}}Cl(A)$ and defined by*

$$_{\mathscr{I}^{\omega}}Cl(A)=\cap\{B\subseteq X: A\subseteq B \text{ and } B\in \mathscr{I}^{\omega}_C(X,\tau)\}.$$

   That is, ${}_{\mathscr{I}^{\omega}}Cl(A)$ is the intersection of all $\mathscr{I}^{\omega}-$closed sets containing A.

2. *The $\mathscr{I}^{\omega}-$*interior operator *of A is denoted by ${}_{\mathscr{I}^{\omega}}Int(A)$ and defined by*

$$_{\mathscr{I}^{\omega}}Int(A)=\cup\{B\subseteq X: B\subseteq A \text{ and } B\in \mathscr{I}^{\omega}_O(X,\tau)\}.$$

   That is, ${}_{\mathscr{I}^{\omega}}Int(A)$ is the union of all $\mathscr{I}^{\omega}-$open sets contained in A.

Theorem 4.2.
Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A\subseteq X$. Then ${}_{\mathscr{I}^{\omega}}Int(A)=A$ if and only if A is a $\mathscr{I}^{\omega}-$open set.

Proof. Let ${}_{\mathscr{I}^{\omega}}Int(A)=A$. Then from definition of ${}_{\mathscr{I}^{\omega}}Int(A)$ and Theorem (3.11), ${}_{\mathscr{I}^{\omega}}Int(A)$ is $\mathscr{I}^{\omega}-$open set and so A is $\mathscr{I}^{\omega}-$open set.

Conversely, we have ${}_{\mathscr{I}^{\omega}}Int(A)\subseteq A$ by the definition. Since A is a $\mathscr{I}^{\omega}-$open set, then it is clear from the definition of ${}_{\mathscr{I}^{\omega}}Int(A)$, $A\subseteq {}_{\mathscr{I}^{\omega}}Int(A)$. Hence $A={}_{\mathscr{I}^{\omega}}Int(A)$. □

Theorem 4.3.
Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A\subseteq X$. Then ${}_{\mathscr{I}^{\omega}}Cl(A)=A$ if and only if A is a $\mathscr{I}^{\omega}-$closed set.

Proof. Similar for proof of Theorem (4.2). □

Theorem 4.4.
Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A\subseteq X$. Then $x\in {}_{\mathscr{I}^{\omega}}Int(A)$ if and only if there is $\mathscr{I}^{\omega}-$open set U such that $x\in U\subseteq A$.

Proof. Let $x\in {}_{\mathscr{I}^{\omega}}Int(A)$ and take $U={}_{\mathscr{I}^{\omega}}Int(A)$. Then by Theorem (3.11) and definition of ${}_{\mathscr{I}^{\omega}}Int(A)$ we get that U is a $\mathscr{I}^{\omega}-$open set and $x\in U\subseteq A$.

Conversely, Let there is $\mathscr{I}^{\omega}-$open set U such that $x\in U\subseteq A$. Then by definition of ${}_{\mathscr{I}^{\omega}}Int(A)$, $x\in U\subseteq {}_{\mathscr{I}^{\omega}}Int(A)$. □

Theorem 4.5.
Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A\subseteq X$. Then $x\in {}_{\mathscr{I}^{\omega}}Cl(A)$ if and only if for all $\mathscr{I}^{\omega}-$open set U containing x, $U\cap A\neq\emptyset$.

Proof. Let $x\in {}_{\mathscr{I}^{\omega}}Cl(A)$ and U be $\mathscr{I}^{\omega}-$open set containing x. If $U\cap A=\emptyset$ then $A\subseteq X-U$. Since $X-U$ is a $\mathscr{I}^{\omega}-$closed set containing A, then ${}_{\mathscr{I}^{\omega}}Cl(A)\subseteq X-U$ and so $x\in {}_{\mathscr{I}^{\omega}}Cl(A)\subseteq X-U$. This is contradiction, because $x\in U$. Therefore $U\cap A\neq\emptyset$.

Conversely, Let $x\notin {}_{\mathscr{I}^{\omega}}Cl(A)$. Then $X-{}_{\mathscr{I}^{\omega}}Cl(A)$ is $\mathscr{I}^{\omega}-$open set containing x. Hence by hypothesis, $[X-{}_{\mathscr{I}^{\omega}}Cl(A)]\cap A\neq\emptyset$. This is contradiction, because $X-{}_{\mathscr{I}^{\omega}}Cl(A)\subseteq X-A$. □

Theorem 4.6.
Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A,B\subseteq X$. Then the following hold:

1. *If $A\subseteq B$ then ${}_{\mathscr{I}^{\omega}}Int(A)\subseteq {}_{\mathscr{I}^{\omega}}Int(B)$.*
2. ${}_{\mathscr{I}^{\omega}}Int(A)\cup {}_{\mathscr{I}^{\omega}}Int(B)\subseteq {}_{\mathscr{I}^{\omega}}Int(A\cup B)$.
3. ${}_{\mathscr{I}^{\omega}}Int(A\cap B)\subseteq {}_{\mathscr{I}^{\omega}}Int(A)\cap {}_{\mathscr{I}^{\omega}}Int(B)$.

4. $Int(A) \subseteq {}_{\mathscr{I}^{\omega}}Int(A)$.

In the last theorem ${}_{\mathscr{I}^{\omega}}Int(A\cap B) \neq {}_{\mathscr{I}^{\omega}}Int(A) \cap {}_{\mathscr{I}^{\omega}}Int(B)$.

Example 4.7.
In Example (3.5), the sets $A = \{0,1\}$ and $B = \{1,4\}$ are $\mathscr{I}^{\omega}-$open sets in $(X,\tau,\mathscr{I})$. Then ${}_{\mathscr{I}^{\omega}}Int(A)\cap{}_{\mathscr{I}^{\omega}}Int(B) = A\cap B = \{1\}$ and

$${}_{\mathscr{I}^{\omega}}Int(A\cap B) = {}_{\mathscr{I}^{\omega}}Int(\{1\}) = \emptyset.$$

Theorem 4.8.
Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A,B \subseteq X$. Then the following hold:

1. *If $A \subseteq B$ then ${}_{\mathscr{I}^{\omega}}Cl(A) \subseteq {}_{\mathscr{I}^{\omega}}Cl(B)$.*
2. *${}_{\mathscr{I}^{\omega}}Cl(A) \cup {}_{\mathscr{I}^{\omega}}Cl(B) \subseteq {}_{\mathscr{I}^{\omega}}Cl(A\cup B)$.*
3. *${}_{\mathscr{I}^{\omega}}Cl(A\cap B) \subseteq {}_{\mathscr{I}^{\omega}}Cl(A) \cap {}_{\mathscr{I}^{\omega}}Cl(B)$.*
4. *${}_{\mathscr{I}^{\omega}}Cl(A) \subseteq Cl(B)$.*

In the last theorem ${}_{\mathscr{I}^{\omega}}Cl(A\cup B) \neq {}_{\mathscr{I}^{\omega}}Cl(A) \cup {}_{\mathscr{I}^{\omega}}Cl(B)$.

Example 4.9.
In Example (3.5), the sets $A = \mathbb{R} - \{0,1\}$ and $B = \mathbb{R} - \{1,4\}$ are $\mathscr{I}^{\omega}-$closed sets in $(X,\tau,\mathscr{I})$. Then ${}_{\mathscr{I}^{\omega}}Cl(A) \cup {}_{\mathscr{I}^{\omega}}Cl(B) = A\cup B = \mathbb{R} - \{1\}$ and

$${}_{\mathscr{I}^{\omega}}Cl(A\cup B) = {}_{\mathscr{I}^{\omega}}Cl(\mathbb{R} - \{1\}) = \mathbb{R}.$$

Theorem 4.10.
Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A \subseteq X$. Then the following hold:

1. *${}_{\mathscr{I}^{\omega}}Int(X - A) = X - {}_{\mathscr{I}^{\omega}}Cl(A)$.*
2. *${}_{\mathscr{I}^{\omega}}Cl(X - A) = X - {}_{\mathscr{I}^{\omega}}Int(A)$.*

Proof. 1. Since $A \subseteq {}_{\mathscr{I}^{\omega}}Cl(A)$ then $X - {}_{\mathscr{I}^{\omega}}Cl(A) \subseteq X - A$. Since $X - {}_{\mathscr{I}^{\omega}}Cl(A)$ is $\mathscr{I}^{\omega}-$open set in $(X,\tau,\mathscr{I})$ then

$$X - {}_{\mathscr{I}^{\omega}}Cl(A) = {}_{\mathscr{I}^{\omega}}Int[X - {}_{\mathscr{I}^{\omega}}Cl(A)] \subseteq {}_{\mathscr{I}^{\omega}}Int(X - A).$$

For the other side, let $x \in {}_{\mathscr{I}^{\omega}}Int(X - A)$. Then there is $\mathscr{I}^{\omega}-$open set U such that $x \in U \subseteq X - A$. Then $X - U$ is $\mathscr{I}^{\omega}-$closed set containing A and $x \notin X - U$. Hence $x \notin {}_{\mathscr{I}^{\omega}}Cl(A)$, that is, $x \in X - {}_{\mathscr{I}^{\omega}}Cl(A)$.
2. Since ${}_{\mathscr{I}^{\omega}}Int(A) \subseteq A$ then $X - A \subseteq X - {}_{\mathscr{I}^{\omega}}Int(A)$. Since $X - {}_{\mathscr{I}^{\omega}}Int(A)$ is $\mathscr{I}^{\omega}-$closed set in $(X,\tau,\mathscr{I})$ then

$${}_{\mathscr{I}^{\omega}}Cl(X - A) \subseteq {}_{\mathscr{I}^{\omega}}Cl[X - {}_{\mathscr{I}^{\omega}}Int(A)] = X - {}_{\mathscr{I}^{\omega}}Int(A).$$

For the other side, let $x \in {}_{\mathscr{I}^{\omega}}Int(X - A)$. Then there is $\mathscr{I}^{\omega}-$open set U such that $x \in U \subseteq X - A$. Then $X - U$ is $\mathscr{I}^{\omega}-$closed set containing A and $x \notin X - U$. Hence $x \notin {}_{\mathscr{I}^{\omega}}Cl(A)$, that is, $x \in X - {}_{\mathscr{I}^{\omega}}Cl(A)$. □

Theorem 4.11.
Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A \subseteq X$. Then the following hold:

1. *If B is an open set X then ${}_{\mathscr{I}^{\omega}}Cl(A) \cap B \subseteq {}_{\mathscr{I}^{\omega}}Cl(A\cap B)$.*
2. *If B is a closed set X then ${}_{\mathscr{I}^{\omega}}Int(A\cup B) \subseteq {}_{\mathscr{I}^{\omega}}Int(A) \cup B$.*

Proof. (1) Let $x \in {}_{\mathscr{I}^\omega}Cl(A) \cap B$. Then $x \in {}_{\mathscr{I}^\omega}Cl(A)$ and $x \in B$. Let V be any $\mathscr{I}^\omega-$open set in $(X,\tau,\mathscr{I})$ containing x. By Theorem (3.13), $V \cap B$ is $\mathscr{I}^\omega-$open set containing x. Since $x \in {}_{\mathscr{I}^\omega}Cl(A)$ then by Theorem (4.5), $(V \cap B) \cap A \neq \emptyset$. This implies, $V \cap (B \cap A) \neq \emptyset$. Hence by Theorem (4.5), $x \in {}_{\mathscr{I}^\omega}Cl(A \cap B)$. That is, ${}_{\mathscr{I}^\omega}Cl(A) \cap B \subseteq {}_{\mathscr{I}^\omega}Cl(A \cap B)$. (2) Since B is a closed set X then by the part (1) and Theorem (4.10),

$$\begin{aligned} X - [{}_{\mathscr{I}^\omega}Int(A) \cup B] &= [X - {}_{\mathscr{I}^\omega}Int(A)] \cap [X - B] \\ &= [{}_{\mathscr{I}^\omega}Cl(X - A)] \cap [X - B] \\ &\subseteq {}_{\mathscr{I}^\omega}Cl[(X - A) \cap (X - B)] \\ &\subseteq {}_{\mathscr{I}^\omega}Cl(X - A) \cap {}_{\mathscr{I}^\omega}Cl(X - B) \\ &= {}_{\mathscr{I}^\omega}Cl(X - A) \cap (X - B) \\ &= (X - {}_{\mathscr{I}^\omega}Int(A)) \cap (X - B) \\ &= X - ({}_{\mathscr{I}^\omega}Int(A)) \cup B). \end{aligned}$$

Hence ${}_{\mathscr{I}^\omega}Int(A \cup B) \subseteq {}_{\mathscr{I}^\omega}Int(A) \cup B$. □

Theorem 4.12.
Let Y be an open subset of a ideal topological space $(X,\tau,\mathscr{I})$ and A be a subset of Y. Then ${}_{\mathscr{I}^\omega}Cl|_Y(A) = {}_{\mathscr{I}^\omega}Cl(A) \cap Y$.

Proof. Let $x \in {}_{\mathscr{I}^\omega}Cl|_Y(A)$ and B be $\mathscr{I}^\omega-$open set in X containing x. By Theorem (3.16), $B \cap Y$ is $\tau|_Y\tau^*-$open set in bitopological space $(Y, \tau|_Y, \tau^*|_Y)$ containing x and since $x \in {}_{\mathscr{I}^\omega}Cl|_Y(A)$, then $A \cap B = (A \cap Y) \cap B \neq \emptyset$. Hence by Theorem (4.5), $x \in {}_{\mathscr{I}^\omega}Cl(A)$, and since $x \in Y$, this implies $x \in {}_{\mathscr{I}^\omega}Cl(A) \cap Y$. That is, ${}_{\mathscr{I}^\omega}Cl|_Y(A) \subseteq {}_{\mathscr{I}^\omega}Cl(A) \cap Y$.

On the other side, let $x \in {}_{\mathscr{I}^\omega}Cl(A) \cap Y$ and O be $\tau|_Y\tau^*|_Y-$open set in bitopological space $(Y, \tau|_Y, \tau^*|_Y)$ containing x. By Corollary (3.19), O is $\mathscr{I}^\omega-$open set in $(X,\tau,\mathscr{I})$. Since $x \in {}_{\mathscr{I}^\omega}Cl(A)$, then $O \cap A \neq \emptyset$. That is, $x \in {}_{\mathscr{I}^\omega}Cl|_Y(A)$. Hence ${}_{\mathscr{I}^\omega}Cl(A) \cap Y \subseteq {}_{\mathscr{I}^\omega}Cl|_Y(A)$. □

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