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Sum Formulas of Generalized Pentanacci Numbers: Closed Forms of the Sum Formulas $\sum_{k=0}^{n} kW_k$ and $\sum_{k=1}^{n} kW_{-k}$

Research Article

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Abstract: In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} kW_k$ and $\sum_{k=1}^{n} kW_{-k}$ for generalized Pentanacci numbers are presented. As special cases, we give summation formulas of Pentanacci, Pentanacci-Lucas, and other fifth-order recurrence sequences.

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Keywords: Pentanacci numbers • Pentanacci-Lucas numbers • sum formulas, summing formulas.

1. Introduction

The generalized Pentanacci sequence $\{W_n(W_0, W_1, W_2, W_3, W_4; r, s, t, u, v)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$\begin{aligned} W_n &= rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}, \\ W_0 &= c_0, W_1 = c_1, W_2 = c_2,\ W_3 = c_3, W_4 = c_4,\ n \geq 5 \end{aligned} \tag{1}$$

where W_0, W_1, W_2, W_3, W_4 are arbitrary real or complex numbers and r, s, t, u, v are real numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{u}{v}W_{-n+1} - \frac{t}{v}W_{-n+2} - \frac{s}{v}W_{-n+3} - \frac{r}{v}W_{-n+4} + \frac{1}{v}W_{-n+5}$$

for $n = 1, 2, 3, ...$ when $v \neq 0$. Therefore, recurrence (1) holds for all integer n. Pentanacci sequence has been studied by many authors, see for example [8, 9, 11, 26].

Table 1. A few special case of generalized Pentanacci sequences.

No	Sequences (Numbers)	Notation	References
1	Generalized Pentanacci	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 1, 1, 1, 1, 1)\}$	[26]
2	Generalized Fifth order Pell	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 2, 1, 1, 1, 1)\}$	[27]
3	Generalized Fifth order Jacobsthal	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 1, 1, 1, 1, 2)\}$	[28]
4	Generalized 5-primes	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 2, 3, 5, 7, 11)\}$	[29]

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Table 2. A few members of generalized Pentanacci sequences.

Sequences (Numbers)	Notation	OEIS [12]	Ref
Pentanacci	$\{P_n\} = \{W_n(0,1,1,2,4;1,1,1,1,1)\}$	A001591	[26]
Pentanacci-Lucas	$\{Q_n\} = \{W_n(5,1,3,7,15;1,1,1,1,1)\}$	A074048	[26]
fifth order Pell	$\{P_n^{(5)}\} = \{W_n(0,1,2,5,13;2,1,1,1,1)\}$	A141448	[27]
fifth order Pell-Lucas	$\{Q_n^{(5)}\} = \{W_n(5,2,6,17,46;2,1,1,1,1)\}$		[27]
modified fifth-order Pell	$\{E_n^{(5)}\} = \{W_n(0,1,1,3,8;2,1,1,1,1)\}$		[27]
fifth order Jacobsthal	$\{J_n^{(5)}\} = \{W_n(0,1,1,1,1;1,1,1,1,2)\}$	A226310	[28],[2]
fifth order Jacobsthal-Lucas	$\{j_n^{(5)}\} = \{W_n(2,1,5,10,20;1,1,1,1,2)\}$	A226311	[28],[2]
modified fifth order Jacobsthal	$\{K_n^{(5)}\} = \{W_n(3,1,3,10,20;1,1,1,1,2)\}$		[28]
fifth-order Jacobsthal Perrin	$\{Q_n^{(5)}\} = \{W_n(3,0,2,8,16;1,1,1,1,2)\}$		[28]
adjusted fifth-order Jacobsthal	$\{S_n^{(5)}\} = \{W_n(0,1,1,2,4;1,1,1,1,2)\}$		[28]
modified fifth-order Jacobsthal-Lucas	$\{R_n^{(5)}\} = \{W_n(5,1,3,7,15;1,1,1,1,2)\}$		[28]
5-primes	$\{G_n\} = \{W_n(0,0,0,1,2;2,3,5,7,11)\}$		[29]
Lucas 5-primes	$\{H_n\} = \{W_n(5,2,10,41,150;2,3,5,7,11)\}$		[29]
modified 5-primes	$\{E_n\} = \{W_n(0,0,0,1,1;2,3,5,7,11)\}$		[29]

For some specific values of W_0, W_1, W_2, W_3, W_4 and r, s, t, u, v it is worth presenting these special Pentanacci numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 2) are used for the special cases of r, s, t, u, v and initial values.

For easy writing, from now on, we drop the superscripts from the sequences, for example we write P_n for $P_n^{(5)}$.

We present some works on summing formulas of the numbers in the following Table 3.

Table 3. A few special study of sum formulas.

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[1],[4],[30],[6],[7]
Generalized Fibonacci	[5],[13],[14],[15],[16],[17],[19]
Generalized Tribonacci	[3],[10],[18]
Generalized Tetranacci	[20],[25],[31]
Generalized Pentanacci	[[21],[22]]
Generalized Hexanacci	[23],[24]

The following Theorem presents some linear summing formulas of generalized Pentanacci numbers with positive subscripts.

Theorem 1.1.
For $n \geq 0$ we have the following formulas:

(a) *(Sum of the generalized Pentanacci numbers) If $r+s+t+u+v-1 \neq 0$ then*

$$\sum_{k=0}^{n} W_k = \frac{\Theta_1}{r+s+t+u+v-1}$$

where

$\Theta_1 = W_{n+5}+(1-r)W_{n+4}+(1-r-s)W_{n+3}+(1-r-s-t)W_{n+2}+(1-r-s-t-u)W_{n+1}-W_4+(r-1)W_3+(r+s-1)W_2+(r+s+t-1)W_1+(r+s+t+u-1)W_0.$

(b) *If $(r-s+t-u+v+1)(r+s+t+u+v-1) \neq 0$ then*

$$\sum_{k=0}^{n} W_{2k} = \frac{\Theta_2}{(r+s+t+u+v-1)(r-s+t-u+v+1)}$$

where

$\Theta_2 = (1-s-u)W_{2n+2}+(t+v+rs+ru)W_{2n+1}+(t^2-u^2+v^2+rt+rv-su+2tv+u)W_{2n}+(v+ru-sv+tu)W_{2n-1}+(v^2+rv+tv)W_{2n-2}+(s+u-1)W_4-(t+v+rs+ru)W_3+(r^2-s^2+rv-su+rt+2s+u-1)W_2+(sv-ru-tu-v)W_1+(r^2-s^2+t^2-u^2+2rt+rv-2su+tv+2s+2u-1)W_0.$

(c) *If* $(r-s+t-u+v+1)(r+s+t+u+v-1)\neq 0$ *then*

$$\sum_{k=0}^{n} W_{2k+1} = \frac{\Theta_3}{(r-s+t-u+v+1)(r+s+t+u+v-1)}$$

where

$\Theta_3 = (r+t+v)W_{2n+2} + sW_{2n+1} + (-s^2+t^2+v^2-u^2+rv+rt-2su+2tv+u)W_{2n+1} + (ru-st-sv+t+v)W_{2n} + (-u^2+v^2+rv-su+tv+u)W_{2n-1} + (-sv-uv+v)W_{2n-2} - (r+t+v)W_4 + (s+u+rt+rv+r^2-1)W_3 - (t+v+ru-st-sv)W_2 + (2s+u+2rt+rv-su+tv+r^2-s^2+t^2-1)W_1 + (sv+uv-v)W_0.$

Proof. It is given in Soykan [[21], Theorem 2.1].

The following Theorem presents some linear summing formulas of generalized Pentanacci numbers with negative subscripts.

Theorem 1.2.
For $n\geq 1$ *we have the following formulas:If* $r+s+t+u+v-1\neq 0$, *then*

$$\sum_{k=1}^{n} W_{-k} = \frac{\Theta_4}{r+s+t+u+v-1}$$

where

$\Theta_4 = -W_{-n+4} + (r-1)W_{-n+3} + (r+s-1)W_{-n+2} + (r+s+t-1)W_{-n+1} + (r+s+t+u-1)W_{-n} + W_4 + (1-r)W_3 + (1-r-s)W_2 + (1-r-s-t)W_1 + (1-r-s-t-u)W_0.$

Proof. It is given in Soykan [[21], Theorem 3.1].

In this work, we investigate linear summation formulas of generalized Pentanacci numbers.

2. Sum Formulas of Generalized Pentanacci Numbers with Positive Subscripts

The following Theorem presents some linear summing formulas of generalized Pentanacci numbers with positive subscripts.

Theorem 2.1.
For $n\geq 0$ *we have the following formulas:*

(a) *If* $r+s+t+u+v-1\neq 0$ *then*

$$\sum_{k=0}^{n} kW_k = \frac{\Psi_1}{(r+s+t+u+v-1)^2}$$

where

$\Psi_1 = (-n+3r+2s+t-v+nr+ns+nt+nu+nv-4)W_{n+4} - (n-6r-s+u+2v+nr^2-2nr-ns-nt-nu-nv+2rs+rt-rv+3r^2+nrs+nrt+nru+nrv+3)W_{n+3} - (n-4r-4s+t+2u+3v+nr^2+ns^2-2nr-2ns-nt-nu-nv+4rs-ru+st-2rv-sv+2r^2+2s^2+2nrs+nrt+nru+nst+nrv+nsu+nsv+2)W_{n+2} - (n-2r-2s-2t+3u+4v+nr^2+ns^2+nt^2-2nr-2ns-2nt-nu-nv+2rs+2rt-2ru+2st-3rv-su-2sv-tv+r^2+s^2+t^2+2nrs+2nrt+nru+2nst+nrv+nsu+nsv+ntu+ntv+1)W_{n+1} + v(-n+4r+3s+2t+u+nr+ns+nt+nu+nv-5)W_n - (3r+2s+t-v-4)W_4 + (-6r-s+u+2v+2rs+rt-rv+3r^2+3)W_3 + (-4r-4s+t+2u+3v+4rs-ru+st-2rv-sv+2r^2+2s^2+2)W_2 + (-2r-2s-2t+3u+4v+2rs+2rt-2ru+2st-3rv-su-2sv-tv+r^2+s^2+t^2+1)W_1 - v(4r+3s+2t+u-5)W_0.$

(b) *If* $(r-s+t-u+v+1)(r+s+t+u+v-1)\neq 0$ *then*

$$\sum_{k=0}^{n} kW_{2k} = \frac{\Psi_2}{(r-s+t-u+v+1)^2(r+s+t+u+v-1)^2}$$

where

$\Psi_2 = -(n(u+s-1)(r^2+2u-s^2+t^2-u^2+v^2+2s+2rt+2rv-2su+2tv-1)+1+s^2+2t^2-u^2+u^3+4v^2-2s+2rt+4rv+6tv+2r^2u-st^2+s^2u+2su^2-3sv^2-2uv^2-2rsv+2rtu-4stv-2tuv+r^2s-u)W_{2n+2} + (n(t+rs+v+ru)(r^2+2u-s^2+t^2-u^2+v^2+2s+2rt+2rv-2su+2tv-1)+2rs^2-t^3-2v^3-2rs-3v-2t+r^3s+r^2t+4sv+2uv+2ru^2+2r^3u+2r^2v+ru^3-s^2v+2tu^2-4t^2v-5tv^2+u^2v+2stu-rst^2+rs^2u+2rsu^2-2r^2sv+2r^2tu-3rsv^2-2ruv^2-3ru+2st+4rsu-4rstv-2rtuv)W_{2n+1} + (n(u+t^2-u^2+v^2+rt+2tv+rv-su)(r^2+2u-s^2+t^2-u^2+v^2+2s+$

$2rt+2rv-2su+2tv-1)+4u^2-3t^2-2u-2u^3-5v^2-2rt+2r^2t^2-3r^2u^2-s^2t^2+4r^2v^2+2s^2u^2-3s^2v^2+r^3t+r^2 u-8tv+rt^3+4st^2-4s^2u-6su^2+2r^3v+s^3u+t^2u+su^3+8sv^2+2rv^3+3uv^2+4rsv+12stv-2r^2su-rs^2v-2rtu^2 +6r^2tv+4rt^2v+5rtv^2-4s^2tv-ru^2v-2suv^2+2rst-2stuv-3rv+5su+4tuv)W_{2n}+(n(v+ru-sv+tu)(r^2+ 2u-s^2+t^2-u^2+v^2+2s+2rt+2rv-2su+2tv-1)+r^3u-3v^3-2ru-2v+r^2v+2ru^3-2rv^2-4s^2v+2tu^2+s^3v -t^2v+tu^3-4tv^2+2sv^3+2u^2v+4stu+2rsu^2-2r^2sv+2r^2tu+rt^2u-s^2tu-2r^2uv-3ruv^2+2stv^2-su^2v- 2t^2uv-2tuv^2-2rstv-4rtuv+5sv-3tu+2rsu)W_{2n-1}+v(n(r+v+t)(r^2+2u-s^2+t^2-u^2+v^2+2s+2rt+2rv- 2su+2tv-1)+r^3-2r-4v-v^3-3t+4st+6sv+2tu+4uv+2r^2t+rt^2-s^2t+2ru^2+r^2v-rv^2-2s^2v+tu^2-t^2v -2tv^2+2rs+2rsu-2suv)W_{2n-2}+(-r^2-4u+4s^2-s^3+t^2+2u^2+3v^2-5s+2rv+6su+4tv+2r^2s+3r^2u-2s^2u- su^2+t^2u-2sv^2-uv^2+2rst+4rtu+2ruv-2stv+2)W_4+(3t+v^3+3rs+4v-4rs^2-2r^3s-2r^2t-6sv-2tu- 4uv+rs^3-2rt^2+s^2t-4ru^2-3r^3u-3r^2v-2rv^2+2s^2v-tu^2+t^2v+2tv^2+4ru-4st-8rsu-4rtv+2suv- 2r^2st+2rs^2u+rsu^2-4r^2tu-rt^2u+2rsv^2-2r^2uv+ruv^2+2rstv)W_3+(r^4+2r^3t+r^3v-2r^2s^2-r^2su+4r^2s+ r^2t^2+2r^2u^2+2r^2u-r^2v^2-2r^2-3rs^2t-2rs^2v-4rstu+4rst-4rsuv+2rsv-rt^2v+rtu^2+4rtu-2rtv^2-rt+4 ruv-rv^3+s^4+2s^3u-4s^3+2s^2tv+s^2u^2-5s^2u+2s^2v^2+6s^2-st^2u-2st^2-8stv+suv^2+4su-6sv^2-4s+2t^2-2tuv+6 tv+u^3-u^2-2uv^2-u+4v^2+1)W_2+(3v+2v^3+3ru-2r^3u-2r^2v-2uv-2ru^2-ru^3+7s^2v-4tu^2-2s^3v-t^3u+ 2tv^2-sv^3-u^2v-8sv+4tu-2rtv-6stu+4suv+rs^2u+3r^2sv-5r^2tu-4rt^2u+2rsv^2+2s^2tu+2stu^2+st^2v+ 2ruv^2-2s^2uv+tuv^2-4rsu+4rstv)W_1+v(3r-2r^3+5v-t^3+4t-2ru-6st-8sv-4tu-6uv+rs^2-5r^2t- 4rt^2+2s^2t-ru^2-4r^2v-2rv^2+3s^2v-2t^2v-tv^2+u^2v-4rs-6rtv+2stu+4suv)W_0.$

(c) *If* $(r-s+t-u+v+1)(r+s+t+u+v-1)\neq 0$ *then*

$$\sum_{k=0}^{n} kW_{2k+1}=\frac{\Psi_3}{(r-s+t-u+v+1)^2(r+s+t+u+v-1)^2}$$

where

$\Psi_3=(n(r+v+t)(r^2+2u-s^2+t^2-u^2+v^2+2s+2rt+2rv-2su+2tv-1)-3v-t^3-2v^3-2t-r-2ru+2st+ 4sv+2uv+rs^2-r^2t-2rt^2+3ru^2-2r^2v-4rv^2-s^2v+2tu^2-4t^2v-5tv^2+u^2v+4rsu-6rtv+2stu)W_{2n+2} +(n(s-s^2+t^2-u^2+v^2+u+rv-2su+2tv+rt)(r^2+2u-s^2+t^2-u^2+v^2+2s+2rt+2rv-2su+2tv-1) +2s^2-s-s^3-3t^2+4u^2-2u^3-5v^2-r^2s^2+2r^2t^2-3r^2u^2+4r^2v^2-3rv+6su-8tv+r^3t+r^2u+rt^3+2st^2-4s^2u- 5su^2+2r^3v+t^2u+4sv^2+2rv^3+3uv^2+6stv+4tuv+rs^2v-2rtu^2+6r^2tv+4rt^2v+5rtv^2-ru^2v-4r^2su-2rstu- 2rt-2u)W_{2n+1}+(n(t+v-sv+ru-st)(r^2+2u-s^2+t^2-u^2+v^2+2s+2rt+2rv-2su+2tv-1)+5sv-2t^3-3v^3-2v- t-2tu-2rt^2-s^2t+r^3u+r^2v+st^3+2ru^3-2rv^2-4s^2v+3tu^2+s^3v-7t^2v-8tv^2+2sv^3+2u^2v-2ru+2st+2rsu- 4rtv+4stu-r^2st-2r^2sv-rt^2u-2s^2tu-2stu^2-2r^2uv+4st^2v-3ruv^2+5stv^2-su^2v+2rsu^2-4rtuv)W_{2n}+ (n(u-u^2+v^2+tv+rv-su)(r^2+2u-s^2+t^2-u^2+v^2+2s+2rt+2rv-2su+2tv-1)+u^2-u+u^3-4v^2-u^4-v^4- 2r^2u^2+r^2v^2-s^2u^2-2s^2v^2-t^2v^2+2u^2v^2-3tv-s^2u+r^3v-2t^2u-2su^3+6sv^2-rv^3-2tv^3-2rv+2su+2rsv-2rtu -4ruv+4stv-4tuv-r^2su-2rtu^2+2r^2tv+rt^2v+st^2u-s^2tv+2ru^2v+suv^2+3tu^2v+4rsuv+4stuv)W_{2n-1}- v(n(u+s-1)(r^2+2u-s^2+t^2-u^2+v^2+2s+2rt+2rv-2su+2tv-1)+1+s^2+2t^2-u-u^2+u^3+4v^2-2s+2rt+4rv+6 tv+r^2s+2r^2u-st^2+s^2u+2su^2-3sv^2-2uv^2-2rsv+2rtu-4stv-2tuv)W_{2n-2}+(2r-r^3+4v+v^3+3t-4st- 6sv-2tu-4uv-2r^2t-rt^2+s^2t-2ru^2-r^2v+rv^2+2s^2v-tu^2+t^2v+2tv^2-2rs-2rsu+2suv)W_4+(r^4+2r^3t+ r^3v+2r^2su+3r^2s+r^2t^2+2r^2u^2+2r^2u-r^2v^2-2r^2-rs^2t-2rs^2v+4rst-2rsuv+4rsv-rt^2v+rtu^2+4rtu- 2rtv^2-rt+4ruv-rv^3+s^2u+s^2-st^2-4stv+2su^2-3sv^2-2s+2t^2-2tuv+6tv+u^3-u^2-2uv^2-u+4v^2+1)W_3+ (-2r^3u+2r^2st+3r^2sv-2r^2tu-r^2t-2r^2v+rs^2u+2rst^2+4rstv-4rsu+2rsv^2+2rtuv-ru^3-2ru^2+2ruv^2+ 3ru-s^3t-2s^3v+4s^2t-2s^2uv+7s^2v-st^2v+stu^2-2stv^2-5st+4suv-sv^3-8sv+t^3+4t^2v-2tu^2+5tv^2+ 2t-u^2v-2uv+2v^3+3v)W_2+(-2r^3v+2r^2su-5r^2tv+3r^2u^2-r^2u-4r^2v^2+rs^2v+2rstu-4rsv-4rt^2v+ 4rtu^2-6rtv^2+ru^2v-2rv^3+3rv-s^3u+2s^2tv-2s^2u^2+4s^2u+3s^2v^2-6stv-su^3+6su^2+2suv^2-5su-8sv^2-t^3 v+t^2u^2+t^2u-2t^2v^2-tv^3+4tv+2u^3-4u^2-3uv^2+2u+5v^2)W_1+v(-r^2-4u+4s^2-s^3+t^2+2u^2+3v^2-5s+ 2rv+6su+4tv+2r^2s+3r^2u-2s^2u-su^2+t^2u-2sv^2-uv^2+2rst+4rtu+2ruv-2stv+2)W_0.$

Proof.

(a) Using the recurrence relation

$$W_n=rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}+vW_{n-5}$$

i.e.

$$vW_{n-5}=W_n-rW_{n-1}-sW_{n-2}-tW_{n-3}-uW_{n-4}$$

we obtain

$$
\begin{aligned}
v\times 0\times W_0 &= 0\times W_5 - r\times 0\times W_4 - s\times 0\times W_3 - t\times 0\times W_2 - u\times 0\times W_1\\
v\times 1\times W_1 &= 1\times W_6 - r\times 1\times W_5 - s\times 1\times W_4 - t\times 1\times W_3 - u\times 1\times W_2\\
v\times 2\times W_2 &= 2\times W_7 - r\times 2\times W_6 - s\times 2\times W_5 - t\times 2\times W_4 - u\times 2\times W_3\\
v\times 3\times W_3 &= 3\times W_8 - r\times 3\times W_7 - s\times 3\times W_6 - t\times 3\times W_5 - u\times 3\times W_4\\
v\times 4\times W_4 &= 4\times W_9 - r\times 4\times W_8 - s\times 4\times W_7 - t\times 4\times W_6 - u\times 4\times W_5\\
&\vdots\\
v(n-3)W_{n-3} &= (n-3)W_{n+2} - r(n-3)W_{n+1} - s(n-3)W_n - t(n-3)W_{n-1} - u(n-3)W_{n-2}\\
v(n-2)W_{n-2} &= (n-2)W_{n+3} - r(n-2)W_{n+2} - s(n-2)W_{n+1} - t(n-2)W_n - u(n-2)W_{n-1}\\
v(n-1)W_{n-1} &= (n-1)W_{n+4} - r(n-1)W_{n+3} - s(n-1)W_{n+2} - t(n-1)W_{n+1} - u(n-1)W_n\\
v\times n\times W_n &= n\times W_{n+5} - r\times n\times W_{n+4} - s\times n\times W_{n+3} - t\times n\times W_{n+2} - u\times n\times W_{n+1}
\end{aligned}
$$

If we add the equations side by side (and using Theorem 1.1 (a)), we get (a).

(b) and (c) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5}$$

we obtain

$$
\begin{aligned}
r\times 0\times W_1 &= 0\times W_2 - s\times 0\times W_0 - t\times 0\times W_{-1} - u\times 0\times W_{-2} - v\times 0\times W_{-3}\\
r\times 1\times W_3 &= 1\times W_4 - s\times 1\times W_2 - t\times 1\times W_1 - u\times 1\times W_0 - v\times 1\times W_{-1}\\
r\times 2\times W_5 &= 2\times W_6 - s\times 2\times W_4 - t\times 2\times W_3 - u\times 2\times W_2 - v\times 2\times W_1\\
&\vdots\\
r\times(n-1)\times W_{2n-1} &= (n-1)\times W_{2n} - s\times(n-1)\times W_{2n-2} - t\times(n-1)\times W_{2n-3}\\
&\quad -u\times(n-1)\times W_{2n-4} - v\times(n-1)\times W_{2n-5}\\
r\times n\times W_{2n+1} &= n\times W_{2n+2} - s\times n\times W_{2n} - t\times n\times W_{2n-1}\\
&\quad -u\times n\times W_{2n-2} - v\times n\times W_{2n-3}
\end{aligned}
$$

Now, if we add the above equations side by side, we get

$$
\begin{aligned}
r(-0\times W_1 + \sum_{k=0}^{n} kW_{2k+1}) &= (n\times W_{2n+2} - 0\times W_2 - (-1)\times W_0 + \sum_{k=0}^{n}(k-1)W_{2k})\\
&\quad -s(-0\times W_0 + \sum_{k=0}^{n} kW_{2k}) - t(-(n+1)W_{2n+1} + \sum_{k=0}^{n}(k+1)W_{2k+1})\\
&\quad -u(-(n+1)W_{2n} + \sum_{k=0}^{n}(k+1)W_{2k})\\
&\quad -v(-(n+2)W_{2n+1} - (n+1)W_{2n-1} + 1\times W_{-1} + \sum_{k=0}^{n}(k+2)W_{2k+1}).
\end{aligned}
$$

Since

$$W_{-1} = -\frac{u}{v}W_0 - \frac{t}{v}W_1 - \frac{s}{v}W_2 - \frac{r}{v}W_3 + \frac{1}{v}W_4$$

we obtain

$$
\begin{aligned}
r(-0\times W_1+\sum_{k=0}^{n}kW_{2k+1}) &= (n\times W_{2n+2}-0\times W_2-(-1)\times W_0+\sum_{k=0}^{n}kW_{2k}-\sum_{k=0}^{n}W_{2k}) \\
&\quad -s(-0\times W_0+\sum_{k=0}^{n}kW_{2k}) \\
&\quad -t(-(n+1)W_{2n+1}+\sum_{k=0}^{n}kW_{2k+1}+\sum_{k=0}^{n}W_{2k+1}) \\
&\quad -u(-(n+1)W_{2n}+\sum_{k=0}^{n}kW_{2k}+\sum_{k=0}^{n}W_{2k}) \\
&\quad -v(-(n+2)W_{2n+1}-(n+1)W_{2n-1} \\
&\quad +1\times(-\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3+\frac{1}{v}W_4) \\
&\quad +\sum_{k=0}^{n}kW_{2k+1}+2\sum_{k=0}^{n}W_{2k+1}).
\end{aligned} \tag{2}
$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}+vW_{n-5}$$

i.e.

$$rW_{n-1} = W_n-sW_{n-2}-tW_{n-3}-uW_{n-4}-vW_{n-5}$$

we write the following obvious equations;

$$
\begin{aligned}
r\times 1\times W_2 &= 1\times W_3-s\times 1\times W_1-t\times 1\times W_0-u\times 1\times W_{-1}-v\times 1\times W_{-2} \\
r\times 2\times W_4 &= 2\times W_5-s\times 2\times W_3-t\times 2\times W_2-u\times 2\times W_1-v\times 2\times W_0 \\
r\times 3\times W_6 &= 3\times W_7-s\times 3\times W_5-t\times 3\times W_4-u\times 3\times W_3-v\times 3\times W_2 \\
&\vdots \\
r\times(n-1)\times W_{2n-2} &= (n-1)\times W_{2n-1}-s\times(n-1)\times W_{2n-3}-t\times(n-1)\times W_{2n-4} \\
&\quad -u\times(n-1)\times W_{2n-5}-v\times(n-1)\times W_{2n-6} \\
r\times n\times W_{2n} &= n\times W_{2n+1}-s\times n\times W_{2n-1}-t\times n\times W_{2n-2} \\
&\quad -u\times n\times W_{2n-3}-v\times n\times W_{2n-4}
\end{aligned}
$$

Now, if we add the above equations side by side, we obtain

$$
\begin{aligned}
r(-0\times W_0+\sum_{k=0}^{n}kW_{2k}) &= (-0\times W_1+\sum_{k=0}^{n}kW_{2k+1})-s(-(n+1)W_{2n+1} \\
&\quad +\sum_{k=0}^{n}(k+1)W_{2k+1})-t(-(n+1)W_{2n}+\sum_{k=0}^{n}(k+1)W_{2k}) \\
&\quad -u(-(n+2)W_{2n+1}-(n+1)W_{2n-1}+1\times W_{-1}+\sum_{k=0}^{n}(k+2)W_{2k+1}) \\
&\quad -v(-(n+2)W_{2n}-(n+1)W_{2n-2}+1\times W_{-2}+\sum_{k=0}^{n}(k+2)W_{2k}).
\end{aligned}
$$

Since

$$
\begin{aligned}
W_{-1} &= -\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3+\frac{1}{v}W_4 \\
W_{-2} &= -\frac{u}{v}(-\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3+\frac{1}{v}W_4)-\frac{t}{v}W_0-\frac{s}{v}W_1-\frac{r}{v}W_2+\frac{1}{v}W_3
\end{aligned}
$$

we have

$$\begin{aligned} r(-0\times W_0+\sum_{k=0}^{n}kW_{2k}) &= (-0\times W_1+\sum_{k=0}^{n}kW_{2k+1}) \\ &-s(-(n+1)W_{2n+1}+\sum_{k=0}^{n}kW_{2k+1}+\sum_{k=0}^{n}W_{2k+1}) \\ &-t(-(n+1)W_{2n}+\sum_{k=0}^{n}kW_{2k}+\sum_{k=0}^{n}W_{2k}) \\ &-u(-(n+2)W_{2n+1}-(n+1)W_{2n-1} \\ &+1\times(-\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3+\frac{1}{v}W_4) \\ &+\sum_{k=0}^{n}kW_{2k+1}+2\sum_{k=0}^{n}W_{2k+1}) \\ &-v(-(n+2)W_{2n}-(n+1)W_{2n-2} \\ &+1\times(-\frac{u}{v}(-\frac{u}{v}W_0-\frac{t}{v}W_1-\frac{s}{v}W_2-\frac{r}{v}W_3+\frac{1}{v}W_4) \\ &-\frac{t}{v}W_0-\frac{s}{v}W_1-\frac{r}{v}W_2+\frac{1}{v}W_3)+\sum_{k=0}^{n}kW_{2k}+2\sum_{k=0}^{n}W_{2k}). \end{aligned} \tag{3}$$

Then, solving the system (2)-(3), the required result of (b) and (c) follow. □

Taking $r=s=t=u=v=1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.1.
If $r=s=t=u=v=1$ then for $n\geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n}kW_k=\frac{1}{16}((4n+1)W_{n+4}-4W_{n+3}-(4n+5)W_{n+2}-(8n+2)W_{n+1}+(4n+5)W_n-W_4+4W_3+5W_2+2W_1-5W_0)$.

(b) $\sum_{k=0}^{n}kW_{2k}=\frac{1}{64}(-(8n+11)W_{2n+2}+(32n-4)W_{2n+1}+(40n+23)W_{2n}+(16n-10)W_{2n-1}+(24n+9)W_{2n-2}+19W_4-28W_3+W_2-6W_1-33W_0)$.

(c) $\sum_{k=0}^{n}kW_{2k+1}=\frac{1}{64}((24n-15)W_{2n+2}+(32n+12)W_{2n+1}+(8n-21)W_{2n}+(16n-2)W_{2n-1}-(8n+11)W_{2n-2}-9W_4+20W_3+13W_2-14W_1+19W_0)$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Pentanacci numbers (take $W_n=P_n$ with $P_0=0,P_1=1,P_2=1,P_3=2,P_4=4$).

Corollary 2.1.
For $n\geq 0$, Pentanacci numbers have the following properties.

(a) $\sum_{k=0}^{n}kP_k=\frac{1}{16}((4n+1)P_{n+4}-4P_{n+3}-(4n+5)P_{n+2}-(8n+2)P_{n+1}+(4n+5)P_n+11)$.

(b) $\sum_{k=0}^{n}kP_{2k}=\frac{1}{64}(-(8n+11)P_{2n+2}+(32n-4)P_{2n+1}+(40n+23)P_{2n}+(16n-10)P_{2n-1}+(24n+9)P_{2n-2}+15)$.

(c) $\sum_{k=0}^{n}kP_{2k+1}=\frac{1}{64}((24n-15)P_{2n+2}+(32n+12)P_{2n+1}+(8n-21)P_{2n}+(16n-2)P_{2n-1}-(8n+11)P_{2n-2}+3)$.

Taking $W_n=Q_n$ with $Q_0=5,Q_1=1,Q_2=3,Q_3=7,Q_4=15$ in the above Proposition, we have the following Corollary which presents linear sum formulas of Pentanacci-Lucas numbers.

Corollary 2.2.
For $n\geq 0$, Pentanacci-Lucas numbers have the following properties.

(a) $\sum_{k=0}^{n}kQ_k=\frac{1}{16}((4n+1)Q_{n+4}-4Q_{n+3}-(4n+5)Q_{n+2}-(8n+2)Q_{n+1}+(4n+5)Q_n+5)$.

(b) $\sum_{k=0}^{n}kQ_{2k}=\frac{1}{64}(-(8n+11)Q_{2n+2}+(32n-4)Q_{2n+1}+(40n+23)Q_{2n}+(16n-10)Q_{2n-1}+(24n+9)Q_{2n-2}-79)$.

(c) $\sum_{k=0}^{n}kQ_{2k+1}=\frac{1}{64}((24n-15)Q_{2n+2}+(32n+12)Q_{2n+1}+(8n-21)Q_{2n}+(16n-2)Q_{2n-1}-(8n+11)Q_{2n-2}+125)$.

Taking $r=2,s=t=u=v=1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.2.
If $r=2, s=t=u=v=1$ *then for* $n\geq 0$ *we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k = \frac{1}{25}((5n+4)W_{n+4}-(5n+9)W_{n+3}-(10n+8)W_{n+2}-(15n+2)W_{n+1}+(5n+9)W_n-4W_4+9W_3+8W_2+2W_1-9W_0)$.

(b) $\sum_{k=0}^{n} kW_{2k} = \frac{1}{225}(-(15n+26)W_{2n+2}+(90n+21)W_{2n+1}+(105n+77)W_{2n}+(45n-12)W_{2n-1}+(60n+29)W_{2n-2}+41W_4-111W_3+43W_2-33W_1-89W_0)$.

(c) $\sum_{k=0}^{n} kW_{2k+1} = \frac{1}{225}((60n-31)W_{2n+2}+(90n+51)W_{2n+1}+(30n-38)W_{2n}-(15n+26)W_{2n-2}+(45n+3)W_{2n-1}-29W_4+84W_3+8W_2-48W_1+41W_0)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of fifth-order Pell numbers (take $W_n=P_n$ with $P_0=0, P_1=1, P_2=2, P_3=5, P_4=13$).

Corollary 2.3.
For $n\geq 0$, *fifth-order Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} kP_k = \frac{1}{25}((5n+4)P_{n+4}-(5n+9)P_{n+3}-(10n+8)P_{n+2}-(15n+2)P_{n+1}+(5n+9)P_n+11)$.

(b) $\sum_{k=0}^{n} kP_{2k} = \frac{1}{225}(-(15n+26)P_{2n+2}+(90n+21)P_{2n+1}+(105n+77)P_{2n}+(45n-12)P_{2n-1}+(60n+29)P_{2n-2}+31)$.

(c) $\sum_{k=0}^{n} kP_{2k+1} = \frac{1}{225}((60n-31)P_{2n+2}+(90n+51)P_{2n+1}+(30n-38)P_{2n}-(15n+26)P_{2n-2}+(45n+3)P_{2n-1}+11)$.

Taking $W_n=Q_n$ with $Q_0=5, Q_1=2, Q_2=6, Q_3=17, Q_4=46$ in the last Proposition, we have the following Corollary which presents linear sum formulas of fifth-order Pell-Lucas numbers.

Corollary 2.4.
For $n\geq 0$, *fifth-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} kQ_k = \frac{1}{25}((5n+4)Q_{n+4}-(5n+9)Q_{n+3}-(10n+8)Q_{n+2}-(15n+2)Q_{n+1}+(5n+9)Q_n-24)$.

(b) $\sum_{k=0}^{n} kQ_{2k} = \frac{1}{225}(-(15n+26)Q_{2n+2}+(90n+21)Q_{2n+1}+(105n+77)Q_{2n}+(45n-12)Q_{2n-1}+(60n+29)Q_{2n-2}-254)$.

(c) $\sum_{k=0}^{n} kQ_{2k+1} = \frac{1}{225}((60n-31)Q_{2n+2}+(90n+51)Q_{2n+1}+(30n-38)Q_{2n}-(15n+26)Q_{2n-2}+(45n+3)Q_{2n-1}+251)$.

Taking $r=1, s=1, t=1, u=1, v=2$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.3.
If $r=1, s=1, t=1, u=1, v=2$ *then for* $n\geq 0$ *we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k = \frac{1}{5}(nW_{n+4}-W_{n+3}-(n+1)W_{n+2}-2nW_{n+1}+2(n+1)W_n+W_3+W_2-2W_0)$.

(b) $\sum_{k=0}^{n} kW_{2k} = \frac{1}{45}(-(3n+2)W_{2n+2}+(15n-11)W_{2n+1}+(33n+16)W_{2n}+(6n-11)W_{2n-1}+2(12n-1)W_{2n-2}+5W_4-4W_3-4W_2+5W_1-22W_0)$.

(c) $\sum_{k=0}^{n} kW_{2k+1} = \frac{1}{45}((12n-13)W_{2n+2}+(30n+14)W_{2n+1}+(3n-13)W_{2n}+(21n-4)W_{2n-1}-2(3n+2)W_{2n-2}+W_4+W_3+10W_2-17W_1+10W_0)$.

Taking $W_n=J_n$ with $J_0=0, J_1=1, J_2=1, J_3=1, J_4=1$ in the last Proposition, we have the following Corollary which presents linear sum formulas of fifth-order Jacobsthal numbers.

Corollary 2.5.
For $n\geq 0$, *fifth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} kJ_k = \frac{1}{5}(nJ_{n+4}-J_{n+3}-(n+1)J_{n+2}-2nJ_{n+1}+2(n+1)J_n+2)$.

(b) $\sum_{k=0}^{n} kJ_{2k} = \frac{1}{45}(-(3n+2)J_{2n+2}+(15n-11)J_{2n+1}+(33n+16)J_{2n}+(6n-11)J_{2n-1}+2(12n-1)J_{2n-2}+2)$.

(c) $\sum_{k=0}^{n} kJ_{2k+1} = \frac{1}{45}((12n-13)J_{2n+2}+(30n+14)J_{2n+1}+(3n-13)J_{2n}+(21n-4)J_{2n-1}-2(3n+2)J_{2n-2}-5)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of fifth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20$).

Corollary 2.6.
For $n \geq 0$, fifth order Jacobsthal-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} k j_k = \frac{1}{5}(n j_{n+4} - j_{n+3} - (n+1) j_{n+2} - 2n j_{n+1} + 2(n+1) j_n + 11)$.

(b) $\sum_{k=0}^{n} k j_{2k} = \frac{1}{45}(-(3n+2) j_{2n+2} + (15n-11) j_{2n+1} + (33n+16) j_{2n} + (6n-11) j_{2n-1} + 2(12n-1) j_{2n-2} + 1)$.

(c) $\sum_{k=0}^{n} k j_{2k+1} = \frac{1}{45}((12n-13) j_{2n+2} + (30n+14) j_{2n+1} + (3n-13) j_{2n} + (21n-4) j_{2n-1} - 2(3n+2) j_{2n-2} + 83)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20$ in the last proposition, we have the following corollary which presents linear sum formula of modified fifth order Jacobsthal numbers.

Corollary 2.7.
For $n \geq 0$, modified fifth order Jacobsthal numbers have the following property:

(a) $\sum_{k=0}^{n} k K_k = \frac{1}{5}(n K_{n+4} - K_{n+3} - (n+1) K_{n+2} - 2n K_{n+1} + 2(n+1) K_n + 7)$.

(b) $\sum_{k=0}^{n} k K_{2k} = \frac{1}{45}(-(3n+2) K_{2n+2} + (15n-11) K_{2n+1} + (33n+16) K_{2n} + (6n-11) K_{2n-1} + 2(12n-1) K_{2n-2} - 13)$.

(c) $\sum_{k=0}^{n} k K_{2k+1} = \frac{1}{45}((12n-13) K_{2n+2} + (30n+14) K_{2n+1} + (3n-13) K_{2n} + (21n-4) K_{2n-1} - 2(3n+2) K_{2n-2} + 73)$.

From the last proposition, we have the following corollary which gives linear sum formula of fifth-order Jacobsthal Perrin numbers (take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16$).

Corollary 2.8.
For $n \geq 0$, fifth-order Jacobsthal Perrin numbers have the following property:

(a) $\sum_{k=0}^{n} k Q_k = \frac{1}{5}(n Q_{n+4} - Q_{n+3} - (n+1) Q_{n+2} - 2n Q_{n+1} + 2(n+1) Q_n + 4)$.

(b) $\sum_{k=0}^{n} k Q_{2k} = \frac{1}{45}(-(3n+2) Q_{2n+2} + (15n-11) Q_{2n+1} + (33n+16) Q_{2n} + (6n-11) Q_{2n-1} + 2(12n-1) Q_{2n-2} - 26)$.

(c) $\sum_{k=0}^{n} k Q_{2k+1} = \frac{1}{45}((12n-13) Q_{2n+2} + (30n+14) Q_{2n+1} + (3n-13) Q_{2n} + (21n-4) Q_{2n-1} - 2(3n+2) Q_{2n-2} + 74)$.

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4$ in the proposition, we have the following corollary which presents linear sum formula of adjusted fifth-order Jacobsthal numbers.

Corollary 2.9.
For $n \geq 0$, adjusted fifth-order Jacobsthal numbers have the following property:

(a) $\sum_{k=0}^{n} k S_k = \frac{1}{5}(n S_{n+4} - S_{n+3} - (n+1) S_{n+2} - 2n S_{n+1} + 2(n+1) S_n + 3)$.

(b) $\sum_{k=0}^{n} k S_{2k} = \frac{1}{45}(-(3n+2) S_{2n+2} + (15n-11) S_{2n+1} + (33n+16) S_{2n} + (6n-11) S_{2n-1} + 2(12n-1) S_{2n-2} + 13)$.

(c) $\sum_{k=0}^{n} k S_{2k+1} = \frac{1}{45}((12n-13) S_{2n+2} + (30n+14) S_{2n+1} + (3n-13) S_{2n} + (21n-4) S_{2n-1} - 2(3n+2) S_{2n-2} - 1)$.

From the last proposition, we have the following corollary which gives linear sum formula of modified fifth-order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 5, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15$).

Corollary 2.10.
For $n \geq 0$, modified fifth-order Jacobsthal-Lucas numbers have the following property:

(a) $\sum_{k=0}^{n} k R_k = \frac{1}{5}(n R_{n+4} - R_{n+3} - (n+1) R_{n+2} - 2n R_{n+1} + 2(n+1) R_n)$.

(b) $\sum_{k=0}^{n} k R_{2k} = \frac{1}{45}(-(3n+2) R_{2n+2} + (15n-11) R_{2n+1} + (33n+16) R_{2n} + (6n-11) R_{2n-1} + 2(12n-1) R_{2n-2} - 70)$.

(c) $\sum_{k=0}^{n} k R_{2k+1} = \frac{1}{45}((12n-13) R_{2n+2} + (30n+14) R_{2n+1} + (3n-13) R_{2n} + (21n-4) R_{2n-1} - 2(3n+2) R_{2n-2} + 85)$.

Taking $r = 2, s = 3, t = 5, u = 7, v = 11$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 2.4.
If $r=2, s=3, t=5, u=7, v=11$ then for $n\geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} kW_k = \frac{1}{729}((27n+2)W_{n+4}-(27n+29)W_{n+3}-(108n+8)W_{n+2}-(243n-90)W_{n+1}+11(27n+29)W_n-2W_4+29W_3-90W_1+8W_2-319W_0)$.

(b) $\sum_{k=0}^{n} kW_{2k} = \frac{1}{729}(-(27n-32)W_{2n+2}+(108n-149)W_{2n+1}+(675n+133)W_{2n}+81(n-2)W_{2n-1}+11(54n-31)W_{2n-2}-5W_4+41W_3-79W_2+81W_1-253W_0)$.

(c) $\sum_{k=0}^{n} kW_{2k+1} = \frac{1}{729}((54n-85)W_{2n+2}+(594n+229)W_{2n+1}-2(27n+1)W_{2n}+9(45n-13)W_{2n-1}-11(27n-32)W_{2n-2}+31W_4-94W_3+56W_2-288W_1-55W_0)$.

From the last proposition, we have the following corollary which gives linear sum formulas of 5-primes numbers (take $W_n=G_n$ with $G_0=0, G_1=0, G_2=0, G_3=1, G_4=2$).

Corollary 2.11.
For $n\geq 0$, 5-primes numbers have the following properties:

(a) $\sum_{k=0}^{n} kG_k = \frac{1}{729}((27n+2)G_{n+4}-(27n+29)G_{n+3}-(108n+8)G_{n+2}-(243n-90)G_{n+1}+11(27n+29)G_n+25)$.

(b) $\sum_{k=0}^{n} kG_{2k} = \frac{1}{729}(-(27n-32)G_{2n+2}+(108n-149)G_{2n+1}+(675n+133)G_{2n}+81(n-2)G_{2n-1}+11(54n-31)G_{2n-2}+31)$.

(c) $\sum_{k=0}^{n} kG_{2k+1} = \frac{1}{729}((54n-85)G_{2n+2}+(594n+229)G_{2n+1}-2(27n+1)G_{2n}+9(45n-13)G_{2n-1}-11(27n-32)G_{2n-2}-32)$.

Taking $W_n=H_n$ with $H_0=5, H_1=2, H_2=10, H_3=41, H_4=150$ in the last proposition, we have the following corollary which presents linear sum formulas of Lucas 5-primes numbers.

Corollary 2.12.
For $n\geq 0$, Lucas 5-primes numbers have the following properties:

(a) $\sum_{k=0}^{n} kH_k = \frac{1}{729}((27n+2)H_{n+4}-(27n+29)H_{n+3}-(108n+8)H_{n+2}-(243n-90)H_{n+1}+11(27n+29)H_n-806)$.

(b) $\sum_{k=0}^{n} kH_{2k} = \frac{1}{729}(-(27n-32)H_{2n+2}+(108n-149)H_{2n+1}+(675n+133)H_{2n}+81(n-2)H_{2n-1}+11(54n-31)H_{2n-2}-962)$.

(c) $\sum_{k=0}^{n} kH_{2k+1} = \frac{1}{729}((54n-85)H_{2n+2}+(594n+229)H_{2n+1}-2(27n+1)H_{2n}+9(45n-13)H_{2n-1}-11(27n-32)H_{2n-2}+505)$.

From the last proposition, we have the following corollary which gives linear sum formulas of modified 5-primes numbers (take $W_n=E_n$ with $E_0=0, E_1=0, E_2=0, E_3=1, E_4=1$).

Corollary 2.13.
For $n\geq 0$, modified 5-primes numbers have the following properties:

(a) $\sum_{k=0}^{n} kE_k = \frac{1}{729}((27n+2)E_{n+4}-(27n+29)E_{n+3}-(108n+8)E_{n+2}-(243n-90)E_{n+1}+11(27n+29)E_n+27)$.

(b) $\sum_{k=0}^{n} kE_{2k} = \frac{1}{729}(-(27n-32)E_{2n+2}+(108n-149)E_{2n+1}+(675n+133)E_{2n}+81(n-2)E_{2n-1}+11(54n-31)E_{2n-2}+36)$.

(c) $\sum_{k=0}^{n} kE_{2k+1} = \frac{1}{729}((54n-85)E_{2n+2}+(594n+229)E_{2n+1}-2(27n+1)E_{2n}+9(45n-13)E_{2n-1}-11(27n-32)E_{2n-2}-63)$.

3. Sum Formulas of Generalized Pentanacci Numbers with Negative Subscripts

The following Theorem presents some linear summing formulas of generalized Pentanacci numbers with negative subscripts.

Theorem 3.1.
For $n\geq 1$ we have the following formula:If $r+s+t+u+v-1\neq 0$, then

$$\sum_{k=1}^{n} kW_{-k} = \frac{\Psi_4}{(r+s+t+u+v-1)^2}$$

where

$\Psi_4 = (n+3r+2s+t-v-nr-ns-nt-nu-nv-4)W_{-n+4}+(n+6r+s-u-2v+nr^2-2nr-ns-nt-nu-nv-2rs-rt+rv-3r^2+nrs+nrt+nru+nrv-3)W_{-n+3}+(n+4r+4s-t-2u-3v+nr^2+ns^2-2nr-2ns-nt-nu-nv-4rs+ru-st+2rv+sv-2r^2-2s^2+2nrs+nrt+nru+nst+nrv+nsu+nsv-2)W_{-n+2}+(n+2r+2s+2t-3u-4v+nr^2+ns^2+nt^2-2nr-2ns-2nt-nu-nv-2rs-2rt+2ru-2st+3rv+su+2sv+tv-r^2-s^2-t^2+2nrs+2nrt+nru+2nst+nrv+nsu+nsv+ntu+ntv-1)W_{-n+1}+(n-5v+nr^2+ns^2+nt^2+nu^2-2nr-2ns-2nt-2nu-nv+4rv+3sv+2tv+uv+2nrs+2nrt+2nru+2nst+nrv+2nsu+nsv+2ntu+ntv+nuv)W_{-n}-(3r+2s+t-v-4)W_4+(-6r-s+u+2v+2rs+rt-rv+3r^2+3)W_3+(-4r-4s+t+2u+3v+4rs-ru+st-2rv-sv+2r^2+2s^2+2)W_2+(-2r-2s-2t+3u+4v+2rs+2rt-2ru+2st-3rv-su-tv+r^2+s^2+t^2-2sv+1)W_1-v(4r+3s+2t+u-5)W_0.$

Proof. Using the recurrence relation

$$\begin{aligned} W_{n+5} &= rW_{n+4}+sW_{n+3}+tW_{n+2}+uW_{n+1}+vW_n \\ \Rightarrow\ W_{-n} &= \frac{1}{v}W_{-n+5}-\frac{u}{v}W_{-n+1}-\frac{t}{v}W_{-n+2}-\frac{s}{v}W_{-n+3}-\frac{r}{v}W_{-n+4} \end{aligned}$$

i.e.

$$vW_{-n} = W_{-n+5}-rW_{-n+4}-sW_{-n+3}-tW_{-n+2}-uW_{-n+1}$$

we obtain

$$\begin{aligned} v\times n\times W_{-n} &= n\times W_{-n+5}-r\times n\times W_{-n+4}-s\times n\times W_{-n+3} \\ &\quad -t\times n\times W_{-n+2}-u\times n\times W_{-n+1} \\ v(n-1)W_{-n+1} &= (n-1)W_{-n+6}-r(n-1)W_{-n+5}-s(n-1)W_{-n+4} \\ &\quad -t(n-1)W_{-n+3}-u(n-1)W_{-n+2} \\ v(n-2)W_{-n+2} &= (n-2)W_{-n+7}-r(n-2)W_{-n+6}-s(n-2)W_{-n+5} \\ &\quad -t(n-2)W_{-n+4}-u(n-2)W_{-n+3} \\ &\ \ \vdots \\ v\times 4\times W_{-4} &= 4\times W_1-r\times 4\times W_0-s\times 4\times W_{-1}-t\times 4\times W_{-2}-u\times 4\times W_{-3} \\ v\times 3\times W_{-3} &= 3\times W_2-r\times 3\times W_1-s\times 3\times W_0-t\times 3\times W_{-1}-u\times 3\times W_{-2} \\ v\times 2\times W_{-2} &= 2\times W_3-r\times 2\times W_2-s\times 2\times W_1-t\times 2\times W_0-u\times 2\times W_{-1} \\ v\times 1\times W_{-1} &= 1\times W_4-r\times 1\times W_3-s\times 1\times W_2-t\times 1\times W_1-u\times 1\times W_0. \end{aligned}$$

If we add the equations side by side (and using Theorem 1.2), we get the result. □

Taking $r=s=t=u=v=1$ in Theorem 3.1, we obtain the following Proposition.

Proposition 3.1.
If $r=s=t=u=v=1$ then for $n\geq 1$ we have the following formulas:

$\sum_{k=1}^{n} kW_{-k} = \frac{1}{16}(-(4n-1)W_{-n+4}-4W_{-n+3}+(4n-5)W_{-n+2}+(8n-2)W_{-n+1}+(12n+5)W_{-n}-W_4+4W_3+5W_2+2W_1-5W_0).$

From the above Proposition, we have the following Corollary which gives linear sum formulas of Pentanacci numbers (take $W_n=P_n$ with $P_0=0, P_1=1, P_2=1, P_3=2, P_4=4$).

Corollary 3.1.
For $n\geq 1$, Pentanacci numbers have the following properties.

$\sum_{k=1}^{n} kP_{-k} = \frac{1}{16}(-(4n-1)P_{-n+4}-4P_{-n+3}+(4n-5)P_{-n+2}+(8n-2)P_{-n+1}+(12n+5)P_{-n}+11).$

Taking $W_n=Q_n$ with $Q_0=5, Q_1=1, Q_2=3, Q_3=7, Q_4=15$ in the above Proposition, we have the following Corollary which presents linear sum formulas of Pentanacci-Lucas numbers.

Corollary 3.2.
For $n\geq 1$, Pentanacci-Lucas numbers have the following properties.

$\sum_{k=1}^{n} kQ_{-k} = \frac{1}{16}(-(4n-1)Q_{-n+4}-4Q_{-n+3}+(4n-5)Q_{-n+2}+(8n-2)Q_{-n+1}+(12n+5)Q_{-n}+5).$

Taking $r=2, s=t=u=v=1$ in Theorem 3.1, we obtain the following Proposition.

Proposition 3.2.
If $r=2, s=t=u=v=1$ then for $n\geq 1$ we have the following formulas:
$\sum_{k=1}^{n} kW_{-k} = \frac{1}{25}(-(5n-4)W_{-n+4}+(5n-9)W_{-n+3}+(10n-8)W_{-n+2}+(15n-2)W_{-n+1}+(20n+9)W_{-n}-4W_4+9W_3+8W_2+2W_1-9W_0)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of fifth-order Pell numbers (take $W_n=P_n$ with $P_0=0, P_1=1, P_2=2, P_3=5, P_4=13$).

Corollary 3.3.
For $n\geq 1$, fifth-order Pell numbers have the following properties:
$\sum_{k=1}^{n} kP_{-k} = \frac{1}{25}(-(5n-4)P_{-n+4}+(5n-9)P_{-n+3}+(10n-8)P_{-n+2}+(15n-2)P_{-n+1}+(20n+9)P_{-n}+11)$.

Taking $W_n=Q_n$ with $Q_0=5, Q_1=2, Q_2=6, Q_3=17, Q_4=46$ in the last Proposition, we have the following Corollary which presents linear sum formulas of fifth-order Pell-Lucas numbers.

Corollary 3.4.
For $n\geq 1$, fifth-order Pell-Lucas numbers have the following properties:
$\sum_{k=1}^{n} kQ_{-k} = \frac{1}{25}(-(5n-4)Q_{-n+4}+(5n-9)Q_{-n+3}+(10n-8)Q_{-n+2}+(15n-2)Q_{-n+1}+(20n+9)Q_{-n}-24)$.

Taking $r=s=t=1, u=1, v=2$ in Theorem 3.1, we obtain the following Proposition.

Proposition 3.3.
If $r=s=t=1, u=1, v=2$ then for $n\geq 1$ we have the following formulas:
$\sum_{k=1}^{n} kW_{-k} = \frac{1}{5}(-nW_{-n+4}-W_{-n+3}+(n-1)W_{-n+2}+2nW_{-n+1}+(3n+2)W_{-n}+W_3+W_2-2W_0)$.

Taking $W_n=J_n$ with $J_0=0, J_1=1, J_2=1, J_3=1, J_4=1$ in the last Proposition, we have the following Corollary which presents linear sum formulas of fifth-order Jacobsthal numbers.

Corollary 3.5.
For $n\geq 1$, fifth order Jacobsthal numbers have the following properties:
$\sum_{k=1}^{n} kJ_{-k} = \frac{1}{5}(-nJ_{-n+4}-J_{-n+3}+(n-1)J_{-n+2}+2nJ_{-n+1}+(3n+2)J_{-n}+2)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of fifth order Jacobsthal-Lucas numbers (take $W_n=j_n$ with $j_0=2, j_1=1, j_2=5, j_3=10, j_4=20$).

Corollary 3.6.
For $n\geq 1$, fifth order Jacobsthal-Lucas numbers have the following properties:
$\sum_{k=1}^{n} kj_{-k} = \frac{1}{5}(-nj_{-n+4}-j_{-n+3}+(n-1)j_{-n+2}+2nj_{-n+1}+(3n+2)j_{-n}+11)$.

Taking $W_n=K_n$ with $K_0=3, K_1=1, K_2=3, K_3=10, K_4=20$ in the last proposition, we have the following corollary which presents linear sum formula of modified fifth order Jacobsthal numbers.

Corollary 3.7.
For $n\geq 1$, modified fifth order Jacobsthal numbers have the following property:
$\sum_{k=1}^{n} kK_{-k} = \frac{1}{5}(-nK_{-n+4}-K_{-n+3}+(n-1)K_{-n+2}+2nK_{-n+1}+(3n+2)K_{-n}+7)$.

From the last proposition, we have the following corollary which gives linear sum formula of fifth-order Jacobsthal Perrin numbers (take $W_n=Q_n$ with $Q_0=3, Q_1=0, Q_2=2, Q_3=8, Q_4=16$).

Corollary 3.8.
For $n\geq 1$, fifth-order Jacobsthal Perrin numbers have the following property:
$\sum_{k=1}^{n} kQ_{-k} = \frac{1}{5}(-nQ_{-n+4}-Q_{-n+3}+(n-1)Q_{-n+2}+2nQ_{-n+1}+(3n+2)Q_{-n}+4)$.

Taking $W_n=S_n$ with $S_0=0, S_1=1, S_2=1, S_3=2, S_4=4$ in the proposition, we have the following corollary which presents linear sum formula of adjusted fifth-order Jacobsthal numbers.

Corollary 3.9.
For $n \geq 1$, adjusted fifth-order Jacobsthal numbers have the following property:
$\sum_{k=1}^{n} kS_{-k} = \frac{1}{5}(-nS_{-n+4} - S_{-n+3} + (n-1)S_{-n+2} + 2nS_{-n+1} + (3n+2)S_{-n} + 3)$.

From the last proposition, we have the following corollary which gives linear sum formula of modified fifth-order Jacobsthal-Lucas numbers (take $W_n = R_n$ with $R_0 = 5, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15$).

Corollary 3.10.
For $n \geq 1$, modified fifth-order Jacobsthal-Lucas numbers have the following property:
$\sum_{k=1}^{n} kR_{-k} = \frac{1}{5}(-nR_{-n+4} - R_{-n+3} + (n-1)R_{-n+2} + 2nR_{-n+1} + (3n+2)R_{-n})$.

Taking $r = 2, s = 3, t = 5, u = 7, v = 11$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.4.
If $r = 2, s = 3, t = 5, u = 7, v = 11$ then for $n \geq 1$ we have the following formulas:
$\sum_{k=1}^{n} kW_{-k} = \frac{1}{729}(-(27n-2)W_{-n+4} + (27n-29)W_{-n+3} + (108n-8)W_{-n+2} + (243n+90)W_{-n+1} + (432n+319)W_{-n} - 2W_4 + 29W_3 + 8W_2 - 90W_1 - 319W_0)$.

From the last proposition, we have the following corollary which gives linear sum formulas of 5-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 1, G_4 = 2$).

Corollary 3.11.
For $n \geq 1$, 5-primes numbers have the following properties:
$\sum_{k=1}^{n} kG_{-k} = \frac{1}{729}(-(27n-2)G_{-n+4} + (27n-29)G_{-n+3} + (108n-8)G_{-n+2} + (243n+90)G_{-n+1} + (432n+319)G_{-n} + 25)$.

Taking $W_n = H_n$ with $H_0 = 5, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150$ in the last proposition, we have the following corollary which presents linear sum formulas of Lucas 5-primes numbers.

Corollary 3.12.
For $n \geq 1$, Lucas 5-primes numbers have the following properties:
$\sum_{k=1}^{n} kH_{-k} = \frac{1}{729}(-(27n-2)H_{-n+4} + (27n-29)H_{-n+3} + (108n-8)H_{-n+2} + (243n+90)H_{-n+1} + (432n+319)H_{-n} - 806)$.

From the last proposition, we have the following corollary which gives linear sum formulas of modified 5-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 1, E_4 = 1$).

Corollary 3.13.
For $n \geq 1$, modified 5-primes numbers have the following properties:
$\sum_{k=1}^{n} kE_{-k} = \frac{1}{729}(-(27n-2)E_{-n+4} + (27n-29)E_{-n+3} + (108n-8)E_{-n+2} + (243n+90)E_{-n+1} + (432n+319)E_{-n} + 27)$.

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