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FIBONACCI LACUNARY STATISTICAL CONVERGENCE OF ORDER γ IN IFNLS

Research Article

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Abstract: This article aims to present Fibonacci lacunary statistical convergence of order gamma in intuitionistic fuzzy normed linear spaces (shortly IFNLS). Certain features of Fibonacci lacunary statistical convergence of order gamma established then we show some important results on them.

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Keywords: Intuitionistic Fuzzy Normed Spaces • Fibonacci Sequence • Lacunary Statistical Convergence

1. Introduction

Zadeh established fuzzy set theory in [1]. After then, this concept has been very popular with researchers in mathematics and other fields.

About 20 years after Zadeh, Atanasov introduced intuitionistic fuzzy set theory in [2]. Also, statistical convergence was defined by Steinhaus, Fast independently of each other [3],[4]. After then, different generalizations of it have been made by many authors. Fridy and Orhan [5] established and worked on subject of lacunary statistical convergence. Then in [6], statistical convergence was defined on IFNS. After that, in this space Mursaleen and Mohiuddine were studied lacunary statistical convergence in [7].

Sen and Debnath [8] examined lacunary statistically convergent on IFnNLS. They were given some properties for lacunary statistically convergent. In [9], Savas and Ozturk have studied λ statistical convergence of order α in IFnNLS.

Apart from those mentioned above, many more generalizations of statistical convergence have been examined in IFNLS and important properties of these generalizations have been worked. In this article, we will consider a generalization of Fibonacci lacunary statistical convergence from such types of convergence. We will construct currently the concept of Fibonacci lacunary statistical convergence of order γ on IFNLS. The purpose of our study is to define and research the Fibonacci lacunary statistical convergence order γ also γ Fibonacci lacunary statistical Cauchy sequence on IFNLS then give very important properties.

Firstly, let's include some basic definitions used in the article.

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2. Preliminaries

Definition 2.1.
[5] Let $\theta = \{k_r\}$ be a sequence of increasing integers, where, $k_0 = 0$ with $\lim\limits_{r\to\infty} h_r := \lim\limits_{r\to\infty} k_r - k_{r-1} = \infty$. In this case, θ is said to be a lacunary sequences. The intervals defined by θ be indicated as $I_r := (k_{r-1}, k_r]$, k_r / k_{r-1} is shortened with q_r in this paper.

Definition 2.2.
[4] (x_k) is named to be statistically convergent to L where

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |x_k - L| \ge \varepsilon\}| = 0,$$

for all $\varepsilon > 0$. Then, $st - \lim x_k = L$ will be used. S represent, set of all statistical convergence sequences.

Definition 2.3.
[5] For lacunary sequence θ and every $\varepsilon > 0$, if,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \ge \varepsilon\}| = 0,$$

thus $x = (x_k)$ called to be lacunary statistical convergence sequence to x_0 (shortly $st(\theta)$-convergence). We write $x_k \to x_0\,(st(\theta))$. S_θ show, the set of all lacunary statistically convergent sequences.

Firstly, Fibonacci sequence were demonstrated in Fibonacci's book, written in 1202. Later, many researchers worked on the important properties of these numbers and their application to different subjects [10]-[19].

Definition 2.4.
The sequence (f_n) of Fibonacci numbers were given with

$$f_0 = f_1 = 1 \text{ also } f_n = f_{n-1} + f_{n-2} \text{ for } n \ge 2.$$

The numbers sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

is named the Fibonacci sequence.

Basic properties of sequences of these numbers [11] have the following features:

$$\begin{aligned} \sum_{k=0}^{n} f_k &= f_{n+2} - 1 \text{ for each } n \in \mathbb{N}, \\ &\sum_k \frac{1}{f_k} \text{ converges}, \\ f_{n-1} f_{n+1} - f_n^2 &= (-1)^{n+1} \text{ for all } n \ge 1. \end{aligned}$$

Definition 2.5.
[20] $* : I \times I \to I$ was named a continuous t-norm for $I = [0,1]$, where

(i) $*$ is commutative, associative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for each $a \in [0,1]$,
(iv) $a * b \le c * d$ if $a \le c$, $b \le d$ and $a, b, c, d \in [0,1]$.

Definition 2.6.
[20] $\triangle : [0,1] \times [0,1] \to [0,1]$ is called continuous t-conorm where it provides next four properties:

(i) $\triangle$ is associative, commutative,
(ii) $\triangle$ is continuous,
(iii) $a \triangle 0 = a$ for each $a \in [0,1]$,
(iv) for every $a, b, c, d \in [0,1]$, $a \triangle b \le c \triangle d$ if $a \le c$ and $b \le d$.

Definition 2.7.
[21] $(X,\mu,\nu,*,\triangle)$ were named an intuitionistic fuzzy normed linear space where μ,ν are fuzzy sets satisfying the next properties for each $w,z>0$ and x,y in X. Let $*$ and $\triangle$ be binary operations defined in Definitions 2.5 and 2.6.

(a) $\mu(x,z)>0$,
(b) $\nu(x,z)+\mu(x,z)\le 1$,
(c) $x=0 \Leftrightarrow \mu(x,z)=1$,
(d) For all $a\neq 0$, $\mu(ax,z)=\mu(x,\frac{z}{|a|})$,
(e) $\mu(x,z)*\mu(y,w)\le\mu(x+y,z+w)$,
(f) $\mu(x,z):(0,\infty)\to[0,1]$ is continuous in z,
(g) $\lim_{t\to\infty}\mu(x,z)=1$, $\lim_{t\to 0}\mu(x,z)=0$.
(h) $\nu(x,z)<1$,
(i) $\nu(x,z)=0\Leftrightarrow x=0$,
(j) $\nu(ax,z)=\nu(x,\frac{z}{|a|})$ for all $a\neq 0$,
(k) $\nu(x,z)\triangle\nu(y,w)\ge\nu(x+y,z+w)$,
(l) $\nu(x,z):(0,\infty)\to[0,1]$ is continuous in z,
(m) $\lim_{t\to\infty}\nu(x,z)=0$, $\lim_{t\to 0}\nu(x,z)=1$.

Then (μ,ν) is called to be an intuitionistic fuzzy norm.

Definition 2.8.
[22] Let for all $n\in\mathbb{N}$, f_n be the nth Fibonacci number. In this case, the infinite matrix $\hat{F}=\left(\hat{f}_{nk}\right)$ was defined with

$$\hat{f}_{nk}=\begin{cases}-\dfrac{f_{n+1}}{f_n}, & (k=n-1)\\ \dfrac{f_n}{f_{n+1}}, & (k=n)\\ 0, & (0\le k<n-1 \text{ or } k>n).\end{cases}$$

Kirişçi and Karaisa [13] defined Fibonacci statistical convergence. Afterward, Kirişçi [12] examined these concept on IFNS. Kisi and Tuzcuoglu introduced Fibonacci lacunary statistical convergence on this space in [14] and Kisi worked this concept for double sequence in [16].

Definition 2.9.
[12] (x_k) is called Fibonacci statistical convergence according to (μ,ν) in an IFNLS $(X,\mu,\nu,*,\triangle)$. For every $\varepsilon>0$, $u>0$, if there is a $\beta\in X$ whenever

$$K_\varepsilon\left(\hat{F}\right):=\left\{k\le n:\mu\left(\hat{F}(x_k)-\beta,u\right)\le 1-\varepsilon \text{ or } \nu\left(\hat{F}(x_k)-\beta,u\right)\ge\varepsilon\right\}$$

has natural density zero, so

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n:\mu\left(\hat{F}(x_k)-\beta,u\right)\le 1-\varepsilon \text{ or } \nu\left(\hat{F}(x_k)-\beta,u\right)\ge\varepsilon\right\}\right|=0.$$

Thus situated, we demonstrate $x_k\to\beta\left(st\left(\hat{F}\right)\right)$.

Definition 2.10.
[14] Let $(X,\mu,\nu,*,\triangle)$ be an IFNLS, (x_{nk}) is called Fibonacci lacunary statistically convergent to ξ according to (μ,ν), in this case, for every $\varepsilon>0$, $z>0$,

$$\lim_r\frac{1}{h_r}\left|\left\{k\in I_r:\mu\left(\hat{F}(x_{nk})-\xi;z\right)\le 1-\varepsilon \text{ and } \nu\left(\hat{F}(x_{nk})-\xi;z\right)\ge\varepsilon\right\}\right|=0.$$

It denoted with $x_{nk}\to\xi\left(st\left(\hat{F}(\theta)(\mu,\nu)\right)\right)$.

Here, set of all Fibonacci lacunary statistical convergent sequences was represented with $S\left(\hat{F}(\theta)(\mu,\nu)\right)$ in IFNLS.

Definition 2.11.
[23] Let $B\subseteq\mathbb{N}$, $\theta=\{k_r\}$ be a lacunary sequence and $\gamma\in(0,1]$ be given. h_r^γ demonstrate the γth power $(h_r)^\gamma$ of h_r, such that, $h^\gamma=(h_r^\gamma)=(h_1^\gamma,h_2^\gamma,\cdots,h_r^\gamma,\cdots)$. So we show $x_k\to\hat{L}(st_\theta^\gamma)$. The number

$$\delta_\theta^\gamma(B)=\lim_{r\to\infty}\frac{1}{h_r^\gamma}\left|\{k\in I_r:k\in B\}\right|$$

is called to be order γ of θ density of B, if the limit exists.

3. Fibonacci Lacunary Statistical Convergence Order γ in IFNLS

Definition 3.1.
Let $(X,\mu,\nu,*,\triangle)$ be an IFNLS, θ be a lacunary sequence. $x=(x_k)$ is called to be Fibonacci lacunary statistical convergent of order γ to $L_{\hat{F}}$ according to (μ,ν), where $0<\gamma\leq 1$, if for every $\varepsilon>0$, $z>0$;

$$\lim_{r\to\infty}\frac{1}{h_r^{\gamma}}\left|\left\{k\in I_r:\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\leq 1-\varepsilon \text{ or } \nu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\geq\varepsilon\right\}\right|=0$$

which denotes as

$$st_{\theta}^{\gamma}\left(\hat{F}\right)-\lim x=L_{\hat{F}}(\mu,\nu).$$

The set of all Fibonacci lacunary statistical convergence of γ (briefly $S_{\theta}^{\gamma}\left(\hat{F}\right)$-convergence) was demonstrated in IFNLS by $S_{\theta}^{\gamma}\left(\hat{F}\right)(\mu,\nu)$ and it will be represented by $S^{\gamma}\left(\hat{F}\right)(\mu,\nu)$ only if it is Fibonacci statistically convergent of γ.

Here, $S^{\gamma}\left(\hat{F}\right)(\mu,\nu)$ will be used for sequences that satisfy the following situation.

$$\lim_{n\to\infty}\frac{1}{n^{\gamma}}\left|\left\{k\leq n:\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\leq 1-\varepsilon \text{ or } \nu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\geq\varepsilon\right\}\right|=0.$$

Corollary 3.1.
If (x_k) is $S_{\theta}^{\gamma}\left(\hat{F}\right)$-convergence to $L_{\hat{F}}$ in IFNLS then

$$\delta_{\theta}^{\gamma}\left(\left\{k\in I_r:\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\leq 1-\varepsilon \text{ or } \nu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\geq\varepsilon\right\}\right)=0.$$

Theorem 3.1.
Let $\gamma\in(0,1]$ and θ be a lacunary sequence.

$$\begin{aligned} i)\ S_{\theta}^{\gamma}\left(\hat{F}\right)\text{-}\lim(cx_k) &= cx_0 \text{ if } S_{\theta}^{\gamma}\left(\hat{F}\right)\text{-}\lim x_k=x_0 \text{ and } c\in\mathbb{R},\\ ii)\ S_{\theta}^{\gamma}\left(\hat{F}\right)\text{-}\lim(x_k+y_k) &= x_0+y_0 \text{ if } S_{\theta}^{\gamma}\left(\hat{F}\right)\text{-}\lim x_k=x_0 \text{ and } S_{\theta}^{\gamma}\left(\hat{F}\right)\text{-}\lim y_k=y_0.\end{aligned}$$

So $S_{\theta}^{\gamma}\left(\hat{F}\right)(\mu,\nu)$ is a linear space.

Lemma 3.1.
Let $(X,\mu,\nu,*,\triangle)$ be an IFNLS. In this case, for all $z>0$, $\varepsilon>0$,

(a) $st_{\theta}^{\gamma}\left(\hat{F}\right)-\lim x=L_{\hat{F}}(\mu,\nu)$.

(b) $\delta_{\theta}^{\gamma}\left(\left\{k\in\mathbb{N}:\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\leq 1-\varepsilon\right\}\right)=\delta_{\theta}^{\gamma}\left(\left\{k\in\mathbb{N}:\nu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\geq\varepsilon\right\}\right)=0$.

(c) $\delta_{\theta}^{\gamma}\left(\left\{k\in\mathbb{N}:\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)>1-\varepsilon \text{ and } \nu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)<\varepsilon\right\}\right)=1$.

(d) $\delta_{\theta}^{\gamma}\left(\left\{k\in\mathbb{N}:\mu(\hat{F}(x_k)-L_{\hat{F}},z)>1-\varepsilon\right\}\right)=\delta_{\theta}^{\gamma}\left(\left\{k\in\mathbb{N}:\nu(\hat{F}(x_k)-L_{\hat{F}},z)<\varepsilon\right\}\right)=1$.

(e) $S_{\theta}^{\gamma}-\lim\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)=1$ and $S_{\theta}^{\gamma}-\lim\upsilon\left(\hat{F}(x_k)-L_{\hat{F}},z\right)=0$.

are equivalent. Using Definition 3.1, properties of the S_{θ}^{γ}, equivalence of properties is easily obtained.

Lemma 3.2.
Let $(X,\mu,\nu,*,\triangle)$ be an IFNLS. If $0<\gamma<\sigma\leq 1$, then $S_{\theta}^{\gamma}\left(\hat{F}\right)\subset S_{\theta}^{\sigma}\left(\hat{F}\right)$

Proof. The proof is obtained from the next

$$\begin{aligned}&\frac{1}{h_r^{\sigma}}\left|\left\{k\in I_r:\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\leqslant 1-\varepsilon \text{ or } \upsilon\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\geqslant\varepsilon\right\}\right|\leqslant\\ &\frac{1}{h_r^{\gamma}}\left|\left\{k\in I_r:\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\leqslant 1-\varepsilon \text{ or } \upsilon\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\geqslant\varepsilon\right\}\right|\end{aligned}$$

□

Theorem 3.2.
Let, (x_k) be a Fibonacci lacunary statistical convergent of order γ according to (μ,ν) in this case $st_{\theta}^{\gamma}\left(\hat{F}\right)-\lim x$ is unique in an IFNLS.

Proof. Let $st_{\theta}^{\gamma}(\hat{F})(\mu,\nu)-\lim x=(L_{\hat{F}})_1$, $st_{\theta}^{\gamma}(\hat{F})(\mu,\nu)-\lim x=(L_{\hat{F}})_2$ and $(L_{\hat{F}})_1\neq(L_{\hat{F}})_2$. Given $\varepsilon>0$ if $w>0$ is taken : $w\triangle w<\varepsilon,(1-w)*(1-w)>1-\varepsilon$. In this case, for each $z>0$, if next sets is defined:

$$\begin{aligned}\Im_{\mu,1}(w,z) &= \left\{k\in\mathbb{N}:\mu\left(\hat{F}(x_k)-(L_{\hat{F}})_1,\frac{z}{2}\right)\le 1-w\right\},\\ \Im_{\mu,2}(w,z) &= \left\{k\in\mathbb{N}:\mu\left(\hat{F}(x_k)-(L_{\hat{F}})_2,\frac{z}{2}\right)\le 1-w\right\},\\ \Im_{\nu,1}(w,z) &= \left\{k\in\mathbb{N}:\nu\left(\hat{F}(x_k)-(L_{\hat{F}})_1,\frac{z}{2}\right)\ge w\right\},\\ \Im_{\nu,2}(w,z) &= \left\{k\in\mathbb{N}:\nu\left(\hat{F}(x_k)-(L_{\hat{F}})_2,\frac{z}{2}\right)\ge w\right\}.\end{aligned}$$

Using $st_{\theta}^{\gamma}(\hat{F})(\mu,\nu)-\lim x=(L_{\hat{F}})_1$, Lemma 3.1, for every $z>0$ it can be write

$$\delta_{\theta}^{\gamma}\left(\Im_{\mu,1}(\varepsilon,z)\right)=\delta_{\theta}^{\gamma}(\Im_{\nu,1}(\varepsilon,z))=0.$$

Also, since $st_{\theta}^{\gamma}(\hat{F})(\mu,\nu)-\lim x=(L_{\hat{F}})_2$,

$$\delta_{\theta}^{\gamma}(\Im_{\mu,2}(\varepsilon,z))=\delta_{\theta}^{\gamma}(\Im_{\nu,2}(\varepsilon,z))=0,(\text{ for each } z>0).$$

can be obtained. In additionally, let $\Im_{\mu,\nu}(\varepsilon,z)=(\Im_{\nu,2}(\varepsilon,z)\cup\Im_{\nu,1}(\varepsilon,z))\cap(\Im_{\mu,2}(\varepsilon,z)\cup\Im_{\mu,1}(\varepsilon,z))$. Thus situated, $st_{\theta}^{\gamma}(\hat{F})(\Im_{\mu,\nu}(\varepsilon,z))=0$ which implies $\delta_{\theta}^{\gamma}(\mathbb{N}\backslash\Im_{\mu,\nu}(\varepsilon,z))=1$. Let $k\in(\mathbb{N}\backslash\Im_{\mu,\nu}(\varepsilon,z))$, now there are two cases.
(a) $k\in(\mathbb{N}\backslash(\Im_{\mu,2}(\varepsilon,z)\cup\Im_{\mu,1}(\varepsilon,z)))$, (b) $k\in(\mathbb{N}\backslash(\Im_{\nu,2}(\varepsilon,z)\cup\Im_{\nu,1}(\varepsilon,z)))$. Firstly, let $k\in(\mathbb{N}\backslash(\Im_{\mu,2}(\varepsilon,z)\cup\Im_{\mu,1}(\varepsilon,z)))$. Therefore,

$$\mu\left((L_{\hat{F}})_1-(L_{\hat{F}})_2,z\right)\ge\mu\left(\hat{F}(x_k)-(L_{\hat{F}})_1,\frac{z}{2}\right)*\mu\left(\hat{F}(x_k)-(L_{\hat{F}})_2,\frac{z}{2}\right)>(1-w)*(1-w)$$

is obtained. Using $(1-w)*(1-w)>1-\varepsilon$, we get

$$\mu\left((L_{\hat{F}})_1-(L_{\hat{F}})_2,z\right)>1-\varepsilon.$$

For arbitrary $\varepsilon>0$ and all $z>0$, we get $\mu((L_{\hat{F}})_1-(L_{\hat{F}})_2,z)=1$, from Lemma 3.1-(d), which yields $(L_{\hat{F}})_1=(L_{\hat{F}})_2$. Moreover, let $k\in(\mathbb{N}\backslash(\Im_{\nu,1}(\varepsilon,z)\cup\Im_{\nu,2}(\varepsilon,z)))$ in this case,

$$\nu\left((L_{\hat{F}})_1-(L_{\hat{F}})_2,z\right)\le\nu\left(\hat{F}(x_k)-(L_{\hat{F}})_1,\frac{z}{2}\right)\triangle\nu\left(\hat{F}(x_k)-(L_{\hat{F}})_2,\frac{z}{2}\right)<w\triangle w$$

can be written. Then, using $w\triangle w<\varepsilon$ we can write

$$\nu\left((L_{\hat{F}})_1-(L_{\hat{F}})_2,z\right)<\varepsilon.$$

So for every $z>0$, $\nu((L_{\hat{F}})_1-(L_{\hat{F}})_2,z)=0$ is obtained, that is $(L_{\hat{F}})_1=(L_{\hat{F}})_2$. Then, the uniqueness of $st_{\theta}^{\gamma}(\hat{F})(\mu,\nu)-\lim x$ is. □

Similar as at [9] the next theorems can be given.

Theorem 3.3.
Let $(X,\mu,\nu,*,\triangle)$ be an IFNLS. If $\liminf_{r\to\infty}\left(\frac{h_r}{k_r}\right)^{\gamma}>0$, in this case, $S^{\gamma}(\hat{F})(\mu,\nu)\subset S_{\theta}^{\gamma}(\hat{F})(\mu,\nu)$.

Proof. For fixed $\varepsilon>0$ and $t>0$, we have

$$\begin{aligned}&\frac{1}{k_r^{\gamma}}\left|\left\{k\le k_r:\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\le 1-\varepsilon \text{ or } \nu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\ge\varepsilon\right\}\right|\\ \ge\ &\frac{1}{k_r^{\gamma}}\left|\left\{k\in I_r:\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\le 1-\varepsilon \text{ or } \nu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\ge\varepsilon\right\}\right|\\ =\ &\frac{h_r^{\gamma}}{k_r^{\gamma}}.\frac{1}{h_r^{\gamma}}\left|\left\{k\in I_r:\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\le 1-\varepsilon \text{ or } \nu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\ge\varepsilon\right\}\right|.\end{aligned}$$

Using $\liminf_{r\to\infty}\left(\frac{h_r}{k_r}\right)^{\gamma}=\gamma>0$, $S^{\gamma}(\hat{F})(\mu,\nu)\subset S_{\theta}^{\gamma}(\hat{F})(\mu,\nu)$ is obtained. □

Theorem 3.4.
Let θ be lacunary sequence, $(X,\mu,\nu,*,\triangle)$ be an IFNLS, if $\limsup_{r}q_r<\infty$ then $S_{\theta}^{\gamma}(\hat{F})(\mu,\nu)\subset S(\hat{F})(\mu,\nu)$.

Proof. Let $\limsup_r q_r < \infty$, in this case for all r there is a $M > 0$ which $q_r < M$. If $S_\theta^\gamma(\mu,\nu) - \lim x = L_{\hat{F}}$ is taken as $z > 0$, $\varepsilon \in (0,1)$, let

$$G_r = \left|\left\{k \in I_r : \mu\left(\hat{F}(x_k) - L_{\hat{F}}, z\right) \le 1-\varepsilon \text{ or } \nu\left(\hat{F}(x_k) - L_{\hat{F}}, z\right) \ge \varepsilon\right\}\right|.$$

Let $\varepsilon > 0$, from the assumption of the theorem there exists $r_0 \in \mathbb{N}$ which

$$\frac{G_r}{h_r^\gamma} < \varepsilon \Rightarrow \frac{G_r}{h_r} < \varepsilon$$

for every $r > r_0$.Now let $G = \max\{G_r : 1 \le r \le r_0\}$, $n \in \mathbb{N}$ where $k_{r-1} < n \le k_r$. So,

$$\begin{aligned}
&\frac{1}{n}\left|\left\{k \le n : \mu\left(\hat{F}(x_k) - L_{\hat{F}}, z\right) \le 1-\varepsilon \text{ or } \nu\left(\hat{F}(x_k) - L_{\hat{F}}, z\right) \ge \varepsilon\right\}\right| \\
\le\ & \frac{1}{k_{r-1}}\left|\left\{k \le k_r : \mu\left(\hat{F}(x_k) - L_{\hat{F}}, z\right) \le 1-\varepsilon \text{ or } \nu\left(\hat{F}(x_k) - L_{\hat{F}}, z\right) \ge \varepsilon\right\}\right| \\
\le\ & \frac{G}{k_{r-1}} r_0 + \frac{1}{k_{r-1}}\left\{h_{r_0+1}\frac{G_{r_0+1}}{h_{r_0+1}} + \cdots + h_r \frac{G_r}{h_r}\right\} \\
\le\ & \frac{r_0 G}{k_{r-1}} + \frac{1}{k_{r-1}}\left(\sup_{r>r_0}\frac{G_r}{h_r}\right)\left\{h_{r_0+1} + \cdots + h_r\right\} \\
\le\ & \frac{r_0 G}{k_{r-1}} + \varepsilon q_r \\
\le\ & \frac{r_0 G}{k_{r-1}} + \varepsilon M
\end{aligned}$$

can be written. □

Theorem 3.5.
For $\gamma \in (0,1]$, $\liminf_r q_r > 1$ then $S^\gamma\left(\hat{F}\right)(\mu,\nu) \subset S_\theta^\gamma\left(\hat{F}\right)(\mu,\nu)$ in an IFNLS.

Proof. Assume that $\liminf_r q_r > 1$. So there exists a $\delta > 0$ where $1+\delta \le q_r$ for sufficiently large r hence $\frac{h_r}{k_r} \ge \frac{\delta}{1+\delta}$ is obtained. So,

$$\left(\frac{h_r}{k_r}\right)^\gamma \ge \left(\frac{\delta}{1+\delta}\right)^\gamma \Rightarrow \frac{1}{k_r^\gamma} \ge \frac{\delta^\gamma}{(1+\delta)^\gamma}\frac{1}{h_r^\gamma}$$

can be written. If (x_k) is Fibonacci statistically convergent to $L_{\hat{F}}$ order γ with regards to (μ,ν), then for each $\varepsilon > 0$, $u > 0$ and sufficiently large r, so

$$\begin{aligned}
&\frac{\delta^\gamma}{(1+\delta)^\gamma}\frac{1}{h_r^\gamma}\left|\left\{k \in I_r : \mu\left(\hat{F}(x_k) - L_{\hat{F}}, u\right) \le 1-\varepsilon \text{ and } \nu\left(\hat{F}(x_k) - L_{\hat{F}}, u\right) \ge \varepsilon\right\}\right| \\
\le\ & \frac{1}{k_r}\left|\left\{k \in I_r : \mu\left(\hat{F}(x_k) - L_{\hat{F}}, u\right) \le 1-\varepsilon \text{ and } \nu\left(\hat{F}(x_k) - L_{\hat{F}}, u\right) \ge \varepsilon\right\}\right| \\
\le\ & \frac{1}{k_r}\left|\left\{k \le k_r : \mu\left(\hat{F}(x_k) - L_{\hat{F}}, u\right) \le 1-\varepsilon \text{ and } \nu\left(\hat{F}(x_k) - L_{\hat{F}}, u\right) \ge \varepsilon\right\}\right|
\end{aligned}$$

is obtained. Thus $x_k \to L_{\hat{F}} S_\theta^\gamma\left(\hat{F}\right)(\mu,\nu)$. So $S^\gamma\left(\hat{F}\right)(\mu,\nu) \subset S_\theta^\gamma\left(\hat{F}\right)(\mu,\nu)$. □

Remark 3.1.
Let $(X,\mu,\nu,*,\triangle)$ be an IFNLS and θ be a lacunary sequences. $S\left(\hat{F}\right)(\mu,\nu) = S_\theta^\gamma\left(\hat{F}\right)(\mu,\nu) \Leftrightarrow 1 < \liminf_r q_r \le \limsup_r q_r < \infty$.

Theorem 3.6.
If $(X,\mu,\nu,*,\triangle)$ be an IFNLS. Then, for any lacunary sequence θ, $st_\theta^\gamma\left(\hat{F}\right)(\mu,\nu) - \lim x = L_{\hat{F}} \Leftrightarrow$ there exists a $J = \left\{j_1 < j_2 < \cdots\right\} \subseteq \mathbb{N}$ and $\delta_\theta^\gamma(J) = 1$, $(\mu,\nu) - \lim_{n\to\infty} x_{j_n} = L_{\hat{F}}$.

Proof. Necessity. Let $st_{\theta}^{\gamma}(\hat{F})(\mu,\nu)-\lim x = L_{\hat{F}}$ and also for all $s\in\mathbb{N}$ and $z>0$

$$A_{\mu,\nu}(s,z)=\left\{k\in\mathbb{N}:\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)>1-\frac{1}{s}\text{ and }\nu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)<\frac{1}{s}\right\}$$

and

$$B_{\mu,\nu}(s,z)=\left\{k\in\mathbb{N}:\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\le 1-\frac{1}{s}\text{ or }\nu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\ge\frac{1}{s}\right\}.$$

Then $\delta_{\theta}^{\gamma}(B_{\mu,\nu}(s,z))=0$ since $st_{\theta}^{\gamma}(\hat{F})(\mu,\nu)-\lim x=L_{\hat{F}}$. Also

$$A_{\mu,\nu}(s+1,z)\subset A_{\mu,\nu}(s,z)$$

and for $z>0$, $s\in\mathbb{N}$, $\gamma\in(0,1]$

$$\delta_{\theta}^{\gamma}(A_{\mu,\nu}(s,z))=1.$$

It must be shown that $(\mu,\nu)-\lim x_k=L_{\hat{F}}$ for $k\in A_{\mu,\nu}(s,z)$. Let for some $k\in A_{\mu,\nu}(s,z)$, $(\mu,\nu)-\lim x_k\neq L_{\hat{F}}$. In this case, there is a $c'>0$, $k_0\in\mathbb{N}$ such that

$$\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\le 1-c'\ \text{ or }\nu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)\ge c'$$

for every $k\ge k_0$. Let

$$\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)>1-c'\ \text{ and }\nu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)<c'$$

for all $k<k_0$. Then

$$\delta_{\theta}^{\gamma}\left(\{k\in\mathbb{N}:\mu(\hat{F}(x_k)-L_{\hat{F}},z)>1-c'\ \text{ and }\nu(\hat{F}(x_k)-L_{\hat{F}},z)<c'\}\right)=0.$$

Using $c>\frac{1}{s}$, we have

$$\delta_{\theta}^{\gamma}(A_{\mu,\nu}(s,z))=0,$$

which contradicts $\delta_{\theta}^{\gamma}(A_{\mu,\nu}(s,z))=1$. Therefore $(\mu,\nu)-\lim x_k=L_{\hat{F}}$.
Sufficiency. Let there exists a $J=\{j_1<j_2<\cdots\}\subseteq\mathbb{N}$ and $\delta_{\theta}^{\gamma}(J)=1$, $(\mu,\nu)-\lim_{n\to\infty}x_{j_n}=L_{\hat{F}}$, so, for every $c'>0$, $z>0$ there exists $p\in\mathbb{N}$ and

$$\mu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)>1-c',\ \nu\left(\hat{F}(x_k)-L_{\hat{F}},z\right)<c'$$

is obtained. Now

$$\begin{aligned}B_{\mu,\nu}(c',z)\ &:=\left\{k\in\mathbb{N}:\mu(\hat{F}(x_k)-L_{\hat{F}},z)\le 1-c'\ \text{ or }\nu(\hat{F}(x_k)-L_{\hat{F}},z)\ge c'\right\}\\ &\subseteq\ \mathbb{N}-\{j_{p+1},j_{p+2},\ldots\}.\end{aligned}$$

Therefore $\delta_{\theta}^{\gamma}(K_{\mu,\nu}(c',z))\le 0$. So, $st_{\theta}^{\gamma}(\hat{F})(\mu,\nu)-\lim x=L_{\hat{F}}$ can be obtained. □

4. γ Fibonacci Lacunary Statistically Cauchy Sequences in IFNLS

Definition 4.1.
Let θ be a lacunary sequence, $(X,\mu,\nu,*,\triangle)$ be an IFNS and $\gamma\in(0,1]$. Then, $x=(x_k)$ is called γ Fibonacci lacunary statistically Cauchy (or $st_{\theta}^{\gamma}(\hat{F})$-Cauchy) according to (μ,ν) if for all $\varepsilon>0$, $z>0$, there exists $m=m(\varepsilon)$:

$$\delta_{\theta,\hat{F}}^{\gamma}\left(\{k\in\mathbb{N}:\mu(\hat{F}(x_k)-\hat{F}(x_m),z)\le 1-\varepsilon\text{ or }\mu(\hat{F}(x_k)-\hat{F}(x_m),z)\ge\varepsilon\}\right)=0.$$

Theorem 4.1.
Let $\gamma\in(0,1]$, $(X,\mu,\nu,*,\triangle)$ be an IFNLS. If (x_k) is $st_{\theta}^{\gamma}(\hat{F})$-convergent in this case (x_k) is $st_{\theta}^{\gamma}(\hat{F})$-Cauchy according to (μ,ν).

Proof. Let $x_k \to L_{\hat{F}} st_\theta^\gamma(\hat{F})$. For a given $\varepsilon > 0$, be chosen $w > 0$ such that $(1-w)*(1-w) > 1-\varepsilon$, $w \triangle w < \varepsilon$. Hence, for any $z > 0$,

$$\delta_\theta^\gamma(A(\varepsilon,z)) = \delta_\theta^\gamma\left(\left\{k \in \mathbb{N} : \mu\left(\hat{F}(x_k) - L_{\hat{F}}, \frac{z}{2}\right) \le 1-\varepsilon \text{ or } \nu\left(\hat{F}(x_k) - L_{\hat{F}}, \frac{z}{2}\right) \ge \varepsilon\right\}\right) = 0$$

is obtained that implies

$$\delta_\theta^\gamma\left(A^c(\varepsilon,z)\right) = \delta_\theta^\gamma\left(\left\{k \in \mathbb{N} : \mu(\hat{F}(x_r) - L_{\hat{F}}, \frac{z}{2}) > 1-\varepsilon \text{ and } \nu(\hat{F}(x_r) - L_{\hat{F}}, \frac{z}{2}) < \varepsilon\right\}\right) = 1.$$

Let $m \in A^c(\varepsilon,z)$. Then

$$\mu\left(\hat{F}(x_m) - L_{\hat{F}}, z\right) > 1-\varepsilon \text{ and } \nu\left(\hat{F}(x_m) - L_{\hat{F}}, z\right) < \varepsilon.$$

Now, let

$$B(\varepsilon,z) = \left\{k \in \mathbb{N} : \mu\left(\hat{F}(x_k) - \hat{F}(x_m), z\right) \le 1-w \text{ or } \nu\left(\hat{F}(x_k) - \hat{F}(x_m), z\right) \ge w\right\}$$

Now, It should be shown that $A(\varepsilon,z) \supset B(\varepsilon,z)$. Suppose that $k \in B(\varepsilon,z) \setminus A(\varepsilon,z)$. Thus,

$$\mu\left(\hat{F}(x_k) - \hat{F}(x_m), z\right) \le 1-w \text{ and } \mu\left(\hat{F}(x_k) - L_{\hat{F}}, \frac{z}{2}\right) > 1-\varepsilon,$$

is obtained in particularly $\mu(\hat{F}(x_m) - L_{\hat{F}}, \frac{z}{2}) > 1-\varepsilon$. So

$$1-s \ge \mu\left(\hat{F}(x_k) - \hat{F}(x_m), z\right) \ge \mu\left(\hat{F}(x_k) - L_{\hat{F}}, \frac{z}{2}\right) * \mu\left(\hat{F}(x_m) - L_{\hat{F}}, \frac{z}{2}\right) > (1-\varepsilon)*(1-\varepsilon) > 1-w,$$

but this is not possible. Otherwise,

$$\nu\left(\hat{F}(x_k) - \hat{F}(x_m), z\right) \ge w \text{ and } \nu\left(\hat{F}(x_k) - L_{\hat{F}}, \frac{z}{2}\right) < \varepsilon,$$

in particularly $\nu(\hat{F}(x_m) - L_{\hat{F}}, \frac{z}{2}) < \varepsilon$. Then

$$w \le \nu\left(\hat{F}(x_k) - \hat{F}(x_m), z\right) \le \nu\left(\hat{F}(x_k) - L_{\hat{F}}, \frac{z}{2}\right) \triangle \nu\left(\hat{F}(x_m) - L_{\hat{F}}, \frac{z}{2}\right) < \varepsilon \triangle \varepsilon < w,$$

but this is impossible. Hence $B(\varepsilon,z) \subset A(\varepsilon,z)$ and $\delta_\theta^\gamma(B(\varepsilon,z)) = 0$. Therefore, x is γ Fibonacci lacunary statistically Cauchy according to (μ,ν). □

Definition 4.2.
Let $\gamma \in (0,1]$ and θ be any lacunary sequence. $(X,\mu,\nu,*,\triangle)$ is called to be statistically $(st_\theta^\gamma(\hat{F})$-complete) if each $(st_\theta^\gamma(\hat{F})$-Cauchy) sequence is $(st_\theta^\gamma(\hat{F})$-convergent) according to (μ,ν).

Theorem 4.2.
Let θ be lacunary sequence and $\gamma \in (0,1]$. Then every finite dimensional intuitionistic fuzzy normed space $(X,\mu,\nu,*,\triangle)$ is $st_\theta^\gamma(\hat{F})$-complete.

Proof. Let (x_k) be $st_\theta^\gamma(\hat{F})$-Cauchy but not $st_\theta^\gamma(\hat{F})$-convergent according to (μ,ν). For given $\varepsilon > 0$ and $z > 0$, we are choosing $s > 0$ such that $(1-\varepsilon)*(1-\varepsilon) > 1-s$ and $\varepsilon \triangle \varepsilon < s$. Now a fixed n_0

$$\mu\left(\hat{F}(x_k) - \hat{F}(x_{n_0}), z\right) \ge \mu\left(\hat{F}(x_k) - L_{\hat{F}}, \frac{z}{2}\right) * \mu\left(\hat{F}(x_{n_0}) - L_{\hat{F}}, \frac{z}{2}\right) > (1-\varepsilon)*(1-\varepsilon) > 1-s$$

and

$$\nu\left(\hat{F}(x_k) - \hat{F}(x_{n_0}), z\right) \le \nu\left(\hat{F}(x_k) - L_{\hat{F}}, \frac{z}{2}\right) \triangle \nu\left(\hat{F}(x_{n_0}) - L_{\hat{F}}, \frac{z}{2}\right) < \varepsilon \triangle \varepsilon < s,$$

since x is not $st_\theta^\gamma(\mu,\nu)$-convergent. Let,

$$P(\varepsilon,z) = \left\{k \in \mathbb{N} : \nu\left(\hat{F}(x_k) - \hat{F}(x_{n_0}), z\right)_\varepsilon \le 1-r\right\}$$

then $\delta_\theta^\gamma(P^C(\varepsilon,z)) = 0$ and so $\delta_\theta^\gamma(P(\varepsilon,z)) = 1$, but that is impossible because x was $st_\theta^\gamma(\hat{F})$-Cauchy according to (μ,ν). Hence x must be $st_\theta^\gamma(\hat{F})(\mu,\nu)$-convergent according to (μ,ν). Hence every finite dimensional IFNLS is $st_\theta^\gamma(\hat{F})$-complete. □

Theorem 4.3.
Let $(X,\mu,\nu,*,\triangle)$ be an IFNLS. (x_k) is a γ Fibonacci lacunary statistically Cauchy according to (μ,ν) ⇔ There exists an increasing natural numbers sequence $P = (k_n)$ such that $\delta(P) = 1$, (x_{k_n}) is a γ Fibonacci lacunary statistically Cauchy according to (μ,ν).

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