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Sums of Cubes of Generalized Fibonacci Numbers with Indices in Arithmetic Progression: the Sum Formulas $\sum_{k=0}^{n} x^k W_{mk+j}^3$

Research Article

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Abstract: In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} x^k W_{mk+j}^3$ for generalized Fibonacci numbers are given. As special cases, we give sum formulas of Fibonacci, Lucas, Pell,Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. We present the proofs to indicate how these formulas, in general, were discovered. Of course, all the listed formulas may be proved by induction, but that method of proof gives no clue about their discovery.

MSC: 11B37 • 11B39 • 11B83

Keywords: Fibonacci numbers • Lucas numbers • Pell numbers • Jacobsthal numbers • sum formulas

1. Introduction

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. The sequence of Fibonacci numbers (OEIS:A000045, [28]) $\{F_n\}_{n\geq 0}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \ \ n \geq 2, \quad F_0 = 0, \ F_1 = 1,$$

and the sequence of Lucas numbers (OEIS: A000032, [28]) $\{L_n\}_{n\geq 0}$ is defined by

$$L_n = L_{n-1} + L_{n-2}, \ \ n \geq 2, \quad L_0 = 2, \ L_1 = 1.$$

Here, OEIS stands for On-line Encyclopedia of Integer Sequences. Pell sequence $\{P_n\}_{n\geq 0}$ (OEIS: A000129, [28]) and Pell-Lucas sequence $\{Q_n\}_{n\geq 0}$ (OEIS: A002203, [28]) are defined by the second-order recurrence relations

$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, P_1 = 1$$

and

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad Q_0 = 2, Q_1 = 2.$$

Jacobsthal sequence $\{J_n\}_{n\geq 0}$ (OEIS: A001045, [28]) and Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ (OEIS: A014551, [28]) are defined by the second-order recurrence relations

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, J_1 = 1$$

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and

$$j_n = j_{n-1} + 2j_{n-2}, \quad j_0 = 2, j_1 = 1.$$

Jacobsthal sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1, 2, 4–8, 11, 17, 18, 23, 24, 27, 35, 36].

Pell sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [3, 9, 10, 12, 16, 20, 25]. For higher order Pell sequences, see [21, 22, 29–32].

The generalized Fibonacci sequence (or generalized (r,s)-sequence or Horadam sequence or 2-step Fibonacci sequence) $\{W_n(W_0,W_1;r,s)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined (by Horadam [14]) as follows:

$$W_n = rW_{n-1} + sW_{n-2}, \qquad W_0 = a, W_1 = b, \quad n \geq 2 \tag{1}$$

where W_0, W_1 are arbitrary complex (or real) numbers and r, s are real numbers, see also Horadam [15, 19**?**] and Soykan [33].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1,2,3,...$ when $s \neq 0$. Therefore, recurrence (1) holds for all integer n.

Now we define two special cases of the sequence $\{W_n\}$. (r,s) sequence $\{G_n(0,1;r,s)\}_{n\geq 0}$ and Lucas (r,s) sequence $\{H_n(2,r;r,s)\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = rG_{n+1} + sG_n, \quad G_0 = 0, G_1 = 1, \tag{2}$$

$$H_{n+2} = rH_{n+1} + sH_n, \quad H_0 = 2, H_1 = r, \tag{3}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{r}{s}G_{-(n-1)} + \frac{1}{s}G_{-(n-2)}, \\ H_{-n} &= -\frac{r}{s}H_{-(n-1)} + \frac{1}{s}H_{-(n-2)}, \end{aligned}$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences (2)-(3) hold for all integer n.

Some special cases of (r,s) sequence $\{G_n(0,1;r,s)\}_{n\geq 0}$ and Lucas (r,s) sequence $\{H_n(2,r;r,s)\}_{n\geq 0}$ are as follows:

1. $G_n(0,1;1,1) = F_n$, Fibonacci sequence,
2. $H_n(2,1;1,1) = L_n$, Lucas sequence,
3. $G_n(0,1;2,1) = P_n$, Pell sequence,
4. $H_n(2,2;2,1) = Q_n$, Pell-Lucas sequence,
5. $G_n(0,1;1,2) = J_n$, Jacobsthal sequence,
6. $H_n(2,1;1,2) = j_n$, Jacobsthal-Lucas sequence.

We give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 1.1.
Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ *is the ordinary generating function of the generalized Fibonacci sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} W_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x}{1 - rx - sx^2}. \tag{4}$$

Binet's formula of generalized Fibonacci sequence can be calculated using its characteristic equation (the quadratic equation) which is given as

$$x^2 - rx - s = 0. \tag{5}$$

The roots of characteristic equation are

$$\alpha = \frac{r + \sqrt{\Delta}}{2}, \quad \beta = \frac{r - \sqrt{\Delta}}{2}. \tag{6}$$

where

$$\Delta = r^2 + 4s$$

and the followings hold

$$\begin{aligned} \alpha + \beta &= r, \\ \alpha\beta &= -s, \\ (\alpha - \beta)^2 &= (\alpha + \beta)^2 - 4\alpha\beta = r^2 + 4s. \end{aligned}$$

1.1. Binet's Formula for the Distinct Roots Case

In this subsection, we assume that the roots α and β of characteristic equation (5) are distinct. Using these roots and the recurrence relation, Binet's formula can be given as follows:

Theorem 1.1 (Distinct Roots Case).
Binet's formula of generalized Fibonacci numbers is

$$W_n = \frac{b_1\alpha^n}{\alpha-\beta} + \frac{b_2\beta^n}{\beta-\alpha} = \frac{b_1\alpha^n - b_2\beta^n}{\alpha-\beta} \tag{7}$$

where

$$b_1 = W_1 - \beta W_0,\ b_2 = W_1 - \alpha W_0.$$

(7) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n \tag{8}$$

where

$$A_1 = \frac{W_1-\beta W_0}{\alpha-\beta},\ A_2 = \frac{W_1-\alpha W_0}{\beta-\alpha}.$$

Note that

$$\begin{aligned} A_1A_2 &= \frac{(W_1^2 - sW_0^2 - rW_1W_0)}{-(r^2+4s)}, \\ A_1+A_2 &= W_0. \end{aligned}$$

We next find Binet's formula of generalized Fibonacci numbers $\{W_n\}$ by the use of generating function for W_n.

Theorem 1.2.
(Binet's formula of generalized Fibonacci numbers)

$$W_n = \frac{d_1\alpha^n}{(\alpha-\beta)} + \frac{d_2\beta^n}{(\beta-\alpha)} \tag{9}$$

where

$$\begin{aligned} d_1 &= W_0\alpha + (W_1 - rW_0), \\ d_2 &= W_0\beta + (W_1 - rW_0)\beta. \end{aligned}$$

Proof. For a proof see [[33], Theorem 1.2]. □
Note that from (7) and (9) we have

$$W_1 - \beta W_0 = W_0\alpha + (W_1 - rW_0), \tag{10}$$
$$W_1 - \alpha W_0 = W_0\beta + (W_1 - rW_0)\beta. \tag{11}$$

For all integers n, (r,s) and Lucas (r,s) numbers (using initial conditions in (7) or (9)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)}, \\ H_n &= \alpha^n + \beta^n, \end{aligned}$$

respectively.

1.2. Binet's Formula for the Single Root Case

In this subsection, we assume that the roots α and β of characteristic equation (5) are equal, i.e., $\alpha = \beta$. So (5) can be written as

$$x^2 - rx - s = (x-\alpha)^2 = x^2 - 2\alpha x + \alpha^2 = 0.$$

Note that in this case,

$$\begin{aligned} \alpha &= \frac{r}{2}, \\ r &= 2\alpha, \\ s &= -\alpha^2 = -\frac{r^2}{4}, \\ r^2 + 4s &= 0. \end{aligned}$$

Using the root α and the recurrence relation, Binet's formula can be given as follows:

Theorem 1.3 (Single Root Case).
Binet's formula of generalized Fibonacci numbers is

$$W_n = (D_1 + D_2 n)\alpha^n \tag{12}$$

where

$$\begin{aligned} D_1 &= W_0, \\ D_2 &= \frac{1}{\alpha}(W_1 - \alpha W_0). \end{aligned}$$

Proof. For a proof, see Soykan [34] . □
Note that (12) can be written as

$$W_n = (nW_1 - \frac{r}{2}(n-1)W_0)\left(\frac{r}{2}\right)^{n-1}$$

Note also that

$$\begin{aligned} D_1 D_2 &= \frac{W_0(2W_1 - rW_0)}{r}, \\ D_1 + D_2 &= 2\frac{W_1}{r}. \end{aligned}$$

For all integers n, (r,s) and Lucas (r,s) numbers (using initial conditions in (7) or (9)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= n\alpha^{n-1}, \\ H_n &= 2\alpha^n, \end{aligned}$$

respectively.

2. The Sum Formula $\sum_{k=0}^{n} x^k W_{mk+j}^3$

In this section, we present sum formulas of generalized (r,s) numbers (generalized Fibonacci numbers).

The following theorem presents sum formulas of generalized (r,s) numbers (generalized Fibonacci numbers) in the case the roots α and β of characteristic equation (5) are distinct, i.e. $r^2 + 4s \neq 0$.

Theorem 2.1 (Distinct Roots Case).
Suppose that the roots α and β of characteristic equation (5) are distinct, i.e. $r^2 + 4s \neq 0$. Let x be a real (or complex) number. For all integers m and j, for generalized (r,s) numbers (generalized Fibonacci numbers), we have the following sum formulas:

(a) *If* $((-s)^{3m}x^2 - xH_{3m}+1)((-s)^{3m}x^2 - x(-s)^m H_m+1) \neq 0$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_1}{(r^2+4s)((-s)^{3m}x^2 - xH_{3m}+1)((-s)^{3m}x^2 - x(-s)^m H_m+1)} \quad (13)$$

where

$\Psi_1 = (r^2+4s)x^{n+1}(-s)^{3m}((-s)^{3m}x^2 - (-s)^m xH_m + 1)W_{mn-m+j}^3 + (r^2+4s)x^{n+1}((-s)^{3m}x - H_{3m})((-s)^{3m}x^2 - (-s)^m xH_m+1)W_{mn+j}^3 - (r^2+4s)(-s)^{3m}x((-s)^{3m}x^2 - (-s)^m xH_m+1)W_{j-m}^3 + (r^2+4s)((-s)^{3m}x^2 - (-s)^m xH_m+1) W_j^3 + 3x^n(-s)^{mn+m+j}x((-s)^{3m}x^2 - xH_{3m}+1)(W_1^2 - sW_0^2 - rW_0W_1)W_{mn+m+j} + 3x^n(-s)^{mn+2m+j}x((-s)^{3m}x^2 - (-s)^m xH_m+1)(W_1^2 - sW_0^2 - rW_0W_1)W_{mn-m+j} - 3x^n(-s)^{mn+j}x((-s)^{4m}x^2H_m - ((-s)^m xH_m - 1)H_{3m})(W_1^2 - sW_0^2 - rW_0W_1)W_{mn+j} - 3x(-s)^{m+j}((-s)^{3m}x^2 - xH_{3m}+1)(W_1^2 - sW_0^2 - rW_0W_1)W_{m+j} - 3x(-s)^{2m+j}((-s)^{3m}x^2 - (-s)^m xH_m+1)(W_1^2 - sW_0^2 - rW_0W_1)W_{j-m} + 3x(-s)^j((-s)^{4m}x^2H_m - H_{3m}((-s)^m xH_m - 1))(W_1^2 - sW_0^2 - rW_0W_1)W_j.$

(b) *If* $((-s)^{3m}x^2 - xH_{3m}+1)((-s)^{3m}x^2 - x(-s)^m H_m+1) = u(x-a)(x-b)(x-c)(x-d) = 0$ *for some* $u,a,b,c,d \in \mathbb{C}$ *and* $u \neq 0$ *and* $a \neq b \neq c \neq d$, *i.e.,* $x = a$ *or* $x = b$ *or* $x = c$ *or* $x = d$, *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_2}{\Lambda_1}$$

where

$\Psi_2 = (r^2+4s)(-s)^{3m}x^n((-s)^{3m}x^2(n+3) - x(-s)^m(n+2)H_m + n+1)W_{mn-m+j}^3 + (r^2+4s)((-s)^{6m}(n+4)x^3 - (-s)^{3m}((-s)^mH_m + H_{3m})(n+3)x^2 + (-s)^m(H_mH_{3m} + (-s)^{2m})(n+2)x - (n+1)H_{3m})x^nW_{mn+j}^3 + (r^2+4s)(-s)^{3m}(-3(-s)^{3m}x^2 + 2(-s)^m xH_m - 1)W_{j-m}^3 + (r^2+4s)(2(-s)^{3m}x - (-s)^mH_m)W_j^3 + 3(-s)^{mn+m+j}((-s)^{3m}(n+3)x^2 - x(n+2)H_{3m} + n+1)(W_1^2 - sW_0^2 - rW_0W_1)x^nW_{mn+m+j} + 3(-s)^{mn+2m+j}((-s)^{3m}(n+3)x^2 - x(-s)^m(n+2)H_m + n+1)x^n(W_1^2 - sW_0^2 - rW_0W_1)W_{mn-m+j} + 3(-s)^{mn+j}(-(-s)^{4m}(n+3)x^2H_m + x(-s)^m(n+2)H_{3m}H_m - (n+1)H_{3m})x^n(W_1^2 - sW_0^2 - rW_0W_1)W_{mn+j} + 3(-s)^{m+j}(-3(-s)^{3m}x^2 + 2xH_{3m} - 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{m+j} + 3(-s)^{2m+j}(-3(-s)^{3m}x^2 + 2(-s)^m xH_m - 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{j-m} + 3(-s)^j(3(-s)^{4m}x^2H_m - 2(-s)^m xH_mH_{3m} + H_{3m})(W_1^2 - sW_0^2 - rW_0W_1)W_j$

and

$\Lambda_1 = (r^2+4s)(4(-s)^{6m}x^3 - 3(-s)^{3m}((-s)^mH_m + H_{3m})x^2 + 2(-s)^m(2(-s)^{2m} + H_mH_{3m})x - ((-s)^mH_m + H_{3m})).$

(c) *If* $((-s)^{3m}x^2 - xH_{3m}+1)((-s)^{3m}x^2 - x(-s)^m H_m+1) = u(x-a)^2(x-b)(x-c) = 0$ *for some* $u,a,b,c \in \mathbb{C}$ *and* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $x = a$ *or* $x = b$ *or* $x = c$, *then if* $x = b$ *or* $x = c$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_3}{\Lambda_2}$$

where

$\Psi_3 = (r^2+4s)(-s)^{3m}x^n((-s)^{3m}x^2(n+3) - x(-s)^m(n+2)H_m + n+1)W_{mn-m+j}^3 + (r^2+4s)((-s)^{6m}(n+4)x^3 - (-s)^{3m}((-s)^mH_m + H_{3m})(n+3)x^2 + (-s)^m(H_mH_{3m} + (-s)^{2m})(n+2)x - (n+1)H_{3m})x^nW_{mn+j}^3 + (r^2+4s)(-s)^{3m}(-3(-s)^{3m}x^2 + 2(-s)^m xH_m - 1)W_{j-m}^3 + (r^2+4s)(2(-s)^{3m}x - (-s)^mH_m)W_j^3 + 3(-s)^{mn+m+j}((-s)^{3m}(n+3)x^2 - x(n+2)H_{3m} + n+1)(W_1^2 - sW_0^2 - rW_0W_1)x^nW_{mn+m+j} + 3(-s)^{mn+2m+j}((-s)^{3m}(n+3)x^2 - x(-s)^m(n+2)H_m + n+1)x^n(W_1^2 - sW_0^2 - rW_0W_1)W_{mn-m+j} + 3(-s)^{mn+j}(-(-s)^{4m}(n+3)x^2H_m + x(-s)^m(n+2)H_{3m}H_m - (n+1)H_{3m})x^n(W_1^2 - sW_0^2 - rW_0W_1)W_{mn+j} + 3(-s)^{m+j}(-3(-s)^{3m}x^2 + 2xH_{3m} - 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{m+j} + 3(-s)^{2m+j}(-3(-s)^{3m}x^2 + 2(-s)^m xH_m - 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{j-m} + 3(-s)^j(3(-s)^{4m}x^2H_m - 2(-s)^m xH_mH_{3m} + H_{3m})(W_1^2 - sW_0^2 - rW_0W_1)W_j$

and

$\Lambda_2 = (r^2+4s)(4(-s)^{6m}x^3 - 3(-s)^{3m}((-s)^mH_m + H_{3m})x^2 + 2(-s)^m(2(-s)^{2m} + H_mH_{3m})x - ((-s)^mH_m + H_{3m}))$

and if $x = a$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_4}{2(r^2+4s)(-s)^m(6(-s)^{5m}x^2 - 3x(-s)^{2m}((-s)^mH_m + H_{3m}) + 2(-s)^{2m} + H_mH_{3m})}$$

where

$\Psi_4 = (r^2+4s)(-s)^{3m}((-s)^{3m}(n+3)(n+2)x^2 - (-s)^m x(n+2)(n+1)H_m + n(n+1))x^{n-1}W_{mn-m+j}^3 + (r^2+4s)((-s)^{6m}(n+4)(n+3)x^3 - (-s)^{3m}(n+3)(n+2)((-s)^mH_m + H_{3m})x^2 + x(-s)^m(n+2)(n+1)(H_mH_{3m} + (-s)^{2m}) - n(n+1)H_{3m})x^{n-1}W_{mn+j}^3 + 2(r^2+4s)(-s)^{4m}(H_m - 3(-s)^{2m}x)W_{j-m}^3 + 2(r^2+4s)(-s)^{3m}W_j^3 + 3(-s)^{mn+m+j}((-s)^{3m}(n+3)(n+2)x^2 - x(n+2)(n+1)H_{3m} + n(n+1))(W_1^2 - sW_0^2 - rW_0W_1)x^{n-1}W_{mn+m+j} + 3x^{n-1}(-s)^{mn+2m+j}((-s)^{3m}(n+3)(n+2)x^2 - x(-s)^m(n+2)(n+1)H_m + n(n+1))(W_1^2 - sW_0^2 - rW_0W_1)W_{mn-m+j} + 3x^{n-1}(-s)^{mn+j}(-x^2(-s)^{4m}(n+3)(n+2)H_m + x(-s)^m(n+2)(n+1)H_{3m}H_m - n(n+1)H_{3m})(W_1^2 - sW_0^2 - rW_0W_1) W_{mn+j} + 6(-s)^{m+j}(H_{3m} - 3(-s)^{3m}x)(W_1^2 - sW_0^2 - rW_0W_1)W_{m+j} + 6(-s)^{3m+j}(H_m - 3(-s)^{2m}x)(W_1^2 - sW_0^2 - rW_0W_1)W_{j-m} + 6(-s)^{m+j}(3(-s)^{3m}x - H_{3m})H_m(W_1^2 - sW_0^2 - rW_0W_1)W_j.$

(d) *If* $((-s)^{3m}x^2 - xH_{3m}+1)((-s)^{3m}x^2 - x(-s)^m H_m+1) = u(x-a)^3(x-b) = 0$ *for some* $u,a,b\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b$, *i.e.,* $x=a$ *or* $x=b$, *then if* $x=b$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_5}{\Lambda_3}$$

where

$\Psi_5 = (r^2+4s)(-s)^{3m}x^n((-s)^{3m}x^2(n+3) - x(-s)^m(n+2)H_m + n+1)W_{mn-m+j}^3 + (r^2+4s)((-s)^{6m}(n+4)x^3 - (-s)^{3m}((-s)^m H_m + H_{3m})(n+3)x^2 + (-s)^m(H_mH_{3m} + (-s)^{2m})(n+2)x - (n+1)H_{3m})x^nW_{mn+j}^3 + (r^2+4s)(-s)^{3m}(-3(-s)^{3m}x^2 + 2(-s)^m xH_m - 1)W_{j-m}^3 + (r^2+4s)(2(-s)^{3m}x - (-s)^m H_m)W_j^3 + 3(-s)^{mn+m+j}((-s)^{3m}(n+3)x^2 - x(n+2)H_{3m} + n+1)(W_1^2 - sW_0^2 - rW_0W_1)x^nW_{mn+m+j} + 3(-s)^{mn+2m+j}((-s)^{3m}(n+3)x^2 - x(-s)^m(n+2)H_m + n+1)x^n(W_1^2 - sW_0^2 - rW_0W_1)W_{mn-m+j} + 3(-s)^{mn+j}(-(-s)^{4m}(n+3)x^2H_m + x(-s)^m(n+2)H_{3m}H_m - (n+1)H_{3m})x^n(W_1^2 - sW_0^2 - rW_0W_1)W_{mn+j} + 3(-s)^{m+j}(-3(-s)^{3m}x^2 + 2xH_{3m} - 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{m+j} + 3(-s)^{2m+j}(-3(-s)^{3m}x^2 + 2(-s)^m xH_m - 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{j-m} + 3(-s)^j(3(-s)^{4m}x^2H_m - 2(-s)^m xH_mH_{3m} + H_{3m})(W_1^2 - sW_0^2 - rW_0W_1)W_j$

and

$\Lambda_3 = (r^2+4s)(4(-s)^{6m}x^3 - 3(-s)^{3m}((-s)^m H_m + H_{3m})x^2 + 2(-s)^m(2(-s)^{2m} + H_mH_{3m})x - ((-s)^m H_m + H_{3m}))$

and if $x=a$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_6}{6(r^2+4s)(-s)^{3m}(4(-s)^{3m}x - (-s)^m H_m - H_{3m})}$$

where

$\Psi_6 = (r^2+4s)(-s)^{3m}(n+1)((-s)^{3m}(n+3)(n+2)x^2 - x(-s)^m n(n+2)H_m + n(n-1))x^{n-2}W_{mn-m+j}^3 + (r^2+4s)((-s)^{6m}(n+3)(n+2)(n+4)x^3 - (-s)^{3m}(n+3)(n+2)(n+1)((-s)^m H_m + H_{3m})x^2 + (-s)^m n(n+2)(n+1)(H_mH_{3m} + (-s)^{2m})x - n(n-1)(n+1)H_{3m})x^{n-2}W_{mn+j}^3 - 6(r^2+4s)(-s)^{6m}W_{j-m}^3 + 3(-s)^{mn+m+j}(n+1)((-s)^{3m}(n+3)(n+2)x^2 - xn(n+2)H_{3m} + n(n-1))(W_1^2 - sW_0^2 - rW_0W_1)x^{n-2}W_{mn+m+j} + 3(-s)^{mn+2m+j}(n+1)((-s)^{3m}(n+3)(n+2)x^2 - x(-s)^m n(n+2)H_m + n(n-1))(W_1^2 - sW_0^2 - rW_0W_1)x^{n-2}W_{mn-m+j} + 3(-s)^{mn+j}(n+1)(-x^2(-s)^{4m}(n+3)(n+2)H_m + x(-s)^m n(n+2)H_{3m}H_m - n(n-1)H_{3m})(W_1^2 - sW_0^2 - rW_0W_1)x^{n-2}W_{mn+j} - 18(-s)^{4m+j}(W_1^2 - sW_0^2 - rW_0W_1)W_{m+j} - 18(-s)^{5m+j}(W_1^2 - sW_0^2 - rW_0W_1)W_{j-m} + 18(-s)^{4m+j}H_m(W_1^2 - sW_0^2 - rW_0W_1)W_j.$

(e) *If* $((-s)^{3m}x^2 - xH_{3m}+1)((-s)^{3m}x^2 - x(-s)^m H_m+1) = u(x-a)^4 = 0$ *for some* $u,a\in\mathbb{C}, u\neq 0$ *i.e.,* $x=a$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_7}{24(r^2+4s)(-s)^{6m}}$$

where

$\Psi_7 = (r^2+4s)(-s)^{3m}n(n+1)((-s)^{3m}(n+3)(n+2)x^2 - x(-s)^m(n-1)(n+2)H_m + (n-1)(n-2))x^{n-3}W_{mn-m+j}^3 + (r^2+4s)(n+1)(x^3(-s)^{6m}(n+4)(n+3)(n+2) - x^2(-s)^{3m}n(n+3)(n+2)((-s)^m H_m + H_{3m}) + x(-s)^m n(n-1)(n+2)(H_mH_{3m} + (-s)^{2m}) - n(n-1)(n-2)H_{3m})x^{n-3}W_{mn+j}^3 + 3(-s)^{mn+m+j}n(n+1)(x^2(-s)^{3m}(n+3)(n+2) - x(n+2)(n-1)H_{3m} + (n-1)(n-2))(W_1^2 - sW_0^2 - rW_0W_1)x^{n-3}W_{mn+m+j} + 3(-s)^{mn+2m+j}n(n+1)(x^2(-s)^{3m}(n+3)(n+2) - x(-s)^m(n+2)(n-1)H_m + (n-1)(n-2))(W_1^2 - sW_0^2 - rW_0W_1)x^{n-3}W_{mn-m+j} + 3(-s)^{mn+j}n(n+1)(-x^2(-s)^{4m}(n+3)(n+2)H_m + x(-s)^m(n+2)(n-1)H_{3m}H_m - (n-1)(n-2)H_{3m})(W_1^2 - sW_0^2 - rW_0W_1)x^{n-3}W_{mn+j}.$

Proof.

(a) *Note that*

$$\begin{aligned}
\sum_{k=0}^{n-1} x^k W_{mk+j}^3 &= \sum_{k=0}^{n-1}(A_1\alpha^{mk+j} + A_2\beta^{mk+j})^3 x^k \\
&= A_1^3\alpha^{3j}\sum_{k=0}^{n-1}\alpha^{3mk}x^k + A_2^3\beta^{3j}\sum_{k=0}^{n-1}\beta^{3mk}x^k \\
&\quad +3A_1A_2^2\alpha^j\beta^{2j}\sum_{k=0}^{n-1}\alpha^{mk}\beta^{2mk}x^k + 3A_1^2A_2\alpha^{2j}\beta^j\sum_{k=0}^{n-1}\alpha^{2mk}\beta^{mk}x^k \\
&= A_1^3\alpha^{3j}\left(\frac{\alpha^{3mn}x^n-1}{\alpha^{3m}x-1}\right) + A_2^3\beta^{3j}\left(\frac{\beta^{3mn}x^n-1}{\beta^{3m}x-1}\right) \\
&\quad +3A_1A_2^2(-s)^j\beta^j\frac{(-s)^{mn}\beta^{mn}x^n-1}{(-s)^m\beta^m x-1} + 3A_1^2A_2(-s)^j\alpha^j\frac{(-s)^{mn}\alpha^{mn}x^n-1}{(-s)^m\alpha^m x-1}.
\end{aligned}$$

Simplifying the last equalities in the last two expression imply (13) as required.

(b) We use (13). For $x = a$, the right hand side of the above sum formula (13) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

$$\begin{aligned}\sum_{k=0}^{n} a^k W_{mk+j}^3 &= \left.\frac{\frac{d}{dx}(\Psi_1)}{\frac{d}{dx}(r^2+4s)((-s)^{3m}x^2 - xH_{3m}+1)((-s)^{3m}x^2 - x(-s)^m H_m + 1)}\right|_{x=a} \\ &= \frac{\Psi_2}{\Psi}.\end{aligned}$$

The proof for the case $x = b$, $x = c$ and $x = d$ are the same.

(c) If $x = b$ or $x = c$ then the required result is obtained by (b). Now suppose that $x = a$. We use (13). For $x = a$, the right hand side of the above sum formula (13) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get the required result by using

$$\begin{aligned}\sum_{k=0}^{n} a^k W_{mk+j}^3 &= \left.\frac{\frac{d^2}{dx^2}(\Psi_1)}{\frac{d^2}{dx^2}(r^2+4s)((-s)^{3m}x^2 - xH_{3m}+1)((-s)^{3m}x^2 - x(-s)^m H_m + 1)}\right|_{x=a} \\ &= \frac{\Psi_3}{2(r^2+4s)(-s)^m(6(-s)^{5m}x^2 - 3x(-s)^{2m}((-s)^m H_m + H_{3m}) + 2(-s)^{2m} + H_m H_{3m})}.\end{aligned}$$

(d) If $x = b$ then the required result is obtained by (b). Now suppose that $x = a$. We use (13). For $x = a$, the right hand side of the above sum formula (13) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get the required result by using

$$\begin{aligned}\sum_{k=0}^{n} a^k W_{mk+j}^3 &= \left.\frac{\frac{d^3}{dx^3}(\Psi_1)}{\frac{d^3}{dx^3}(r^2+4s)((-s)^{3m}x^2 - xH_{3m}+1)((-s)^{3m}x^2 - x(-s)^m H_m + 1)}\right|_{x=a} \\ &= \frac{\Psi_4}{6(r^2+4s)(-s)^{3m}(4(-s)^{3m}x - (-s)^m H_m - H_{3m})}.\end{aligned}$$

(e) We use (13). For $x = a$, the right hand side of the above sum formula (13) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get the required result by using

$$\begin{aligned}\sum_{k=0}^{n} a^k W_{mk+j}^3 &= \left.\frac{\frac{d^4}{dx^4}(\Psi_1)}{\frac{d^4}{dx^4}(r^2+4s)((-s)^{3m}x^2 - xH_{3m}+1)((-s)^{3m}x^2 - x(-s)^m H_m + 1)}\right|_{x=a} \\ &= \frac{\Psi_8}{24(r^2+4s)(-s)^{6m}}.\end{aligned}$$

□

Note that (13) can be written in the following form:

$$\sum_{k=1}^{n} x^k W_{mk+j}^3 = \frac{\Psi_8}{(r^2+4s)((-s)^{3m}x^2 - xH_{3m}+1)((-s)^{3m}x^2 - x(-s)^m H_m + 1)}$$

where

$\Psi_8 = (r^2+4s)x^{n+1}(-s)^{3m}((-s)^{3m}x^2 - (-s)^m xH_m + 1)W_{mn-m+j}^3 + (r^2+4s)x^{n+1}((-s)^{3m}x - H_{3m})((-s)^{3m}x^2 - (-s)^m xH_m + 1)W_{mn+j}^3 - (r^2+4s)(-s)^{3m}x((-s)^{3m}x^2 - (-s)^m xH_m + 1)W_{j-m}^3 + (4s+r^2)(H_{3m} - (-s)^{3m}x)((-s)^{3m}x^2 - (-s)^m xH_m + 1)xW_j^3 + 3x^n(-s)^{mn+m+j}x((-s)^{3m}x^2 - xH_{3m} + 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{mn+m+j} + 3x^n(-s)^{mn+2m+j} x((-s)^{3m}x^2 - (-s)^m xH_m + 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{mn-m+j} - 3x^n(-s)^{mn+j}x((-s)^{4m}x^2H_m - ((-s)^m xH_m - 1)H_{3m})(W_1^2 - sW_0^2 - rW_0W_1)W_{mn+j} - 3x(-s)^{m+j}((-s)^{3m}x^2 - xH_{3m} + 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{m+j} - 3x(-s)^{2m+j}((-s)^{3m}x^2 - (-s)^m xH_m + 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{j-m} + 3x(-s)^j((-s)^{4m}x^2H_m - H_{3m}((-s)^m xH_m - 1))(W_1^2 - sW_0^2 - rW_0W_1)W_j.$

The following theorem presents sum formulas of generalized (r,s) numbers (generalized Fibonacci numbers) in the case the roots α and β of characteristic equation (5) are equal, i.e., $\alpha = \beta$ so that $r^2+4s=0$.

Theorem 2.2 (Single Root Case).

Assume that the roots α and β of characteristic equation (5) are equal, i.e., $\alpha = \beta$ so that $r^2+4s=0$. Let x be a real (or complex) number. For all integers m and j, for generalized (r,s) numbers (generalized Fibonacci numbers), we have the following sum formulas:if $(x\left(\frac{r}{2}\right)^{3m} - 1)^4 \neq 0$, i.e., $x \neq \left(\frac{r}{2}\right)^{-3m}$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_9}{(x\left(\frac{r}{2}\right)^{3m} - 1)^4} \tag{14}$$

where

$\Psi_9 = ((x\left(\frac{r}{2}\right)^{3m}-1)^4((D_1+D_2(mn+j))\left(\frac{r}{2}\right)^{mn+j})^3x^n+(D_1^3(x\left(\frac{r}{2}\right)^{3m}-1)^3(x^n\left(\frac{r}{2}\right)^{3mn+3j}-\left(\frac{r}{2}\right)^{3j})+D_2^3(j^3\left(\frac{r}{2}\right)^{3j}-x\left(\frac{r}{2}\right)^{3m+3j}(x^2\left(\frac{r}{2}\right)^{6m}(j-m)^3-x\left(\frac{r}{2}\right)^{3m}(-6j^2m+3j^3+4m^3)-3jm^2-3j^2m+3j^3-m^3)+x^n\left(\frac{r}{2}\right)^{3mn+3j}(x\left(\frac{r}{2}\right)^{3m}(-3jm^2-3j^2m-3m^3n-3m^3n^2+3m^3n^3+3j^3-m^3-6jm^2n+9j^2mn+9jm^2n^2)-x^2\left(\frac{r}{2}\right)^{6m}(-6j^2m-6m^3n^2+3m^3n^3+3j^3+4m^3-12jm^2n+9j^2mn+9jm^2n^2)+x^3\left(\frac{r}{2}\right)^{9m}(j-m+mn)^3-(j+mn)^3)))+3D_1D_2(x\left(\frac{r}{2}\right)^{3m}-1)(D_1(x\left(\frac{r}{2}\right)^{3m}-1)(j\left(\frac{r}{2}\right)^{3j}-\left(\frac{r}{2}\right)^{3m+3j}(j-m)x+\left(\frac{r}{2}\right)^{3mn+3j}(-j-mn+x\left(\frac{r}{2}\right)^{3m}(j-m+mn))x^n)+D_2(-j^2\left(\frac{r}{2}\right)^{3j}+\left(\frac{r}{2}\right)^{3m+3j}(-2jm+2j^2-m^2)x-\left(\frac{r}{2}\right)^{6m+3j}(j-m)^2x^2+\left(\frac{r}{2}\right)^{3mn+3j}(m^2n^2+j^2+2jmn-x\left(\frac{r}{2}\right)^{3m}(-2m^2n+2m^2n^2-2jm+2j^2-m^2+4jmn)+x^2\left(\frac{r}{2}\right)^{6m}(j-m+mn)^2)x^n))).$

Proof. Note that

$$\begin{aligned}\sum_{k=0}^{n-1}x^kW_{mk+j}^3 &= \sum_{k=0}^{n-1}((D_1+D_2(mk+j))\alpha^{mk+j})^3x^k\\ &= \sum_{k=0}^{n-1}\left(D_1^3\alpha^{3mk+3j}x^k+D_2^3(mk+j)^3\alpha^{3mk+3j}x^k\right)\\ &\quad+\sum_{k=0}^{n-1}3D_1D_2(mk+j)\left(D_1+D_2(mk+j)\right)\alpha^{3mk+3j}x^k.\end{aligned}$$

Simplifying the last equalities in the last two expression imply (14) as required. □

2.1. The Case $r=1, s=1$: Generalized Fibonacci Numbers

The following theorem presents sum formulas of generalized Fibonacci numbers (the case $r=1, s=1$).

Theorem 2.3.
Let x be a real (or complex) number. For all integers m and j, for generalized Fibonacci numbers (the case $r=1, s=1$) we have the following sum formulas:

(a) *If $((-1)^{3m}x^2-xL_{3m}+1)((-1)^{3m}x^2-x(-1)^mL_m+1)\neq 0$ then*

$$\sum_{k=0}^{n}x^kW_{mk+j}^3=\frac{\Psi_1}{5((-1)^{3m}x^2-xL_{3m}+1)((-1)^{3m}x^2-x(-1)^mL_m+1)} \tag{15}$$

where

$\Psi_1 = 5x^{n+1}(-1)^{3m}((-1)^{3m}x^2-(-1)^mxL_m+1)W_{mn-m+j}^3+5x^{n+1}((-1)^{3m}x-L_{3m})((-1)^{3m}x^2-(-1)^mxL_m+1)W_{mn+j}^3-5(-1)^{3m}x((-1)^{3m}x^2-(-1)^mxL_m+1)W_{j-m}^3+5((-1)^{3m}x^2-(-1)^mxL_m+1)W_j^3+3x^n(-1)^{mn+m+j}x((-1)^{3m}x^2-xL_{3m}+1)(W_1^2-W_0^2-W_0W_1)W_{mn+m+j}+3x^n(-1)^{mn+2m+j}x((-1)^{3m}x^2-(-1)^mxL_m+1)(W_1^2-W_0^2-W_0W_1)W_{mn-m+j}-3x^n(-1)^{mn+j}x((-1)^{4m}x^2L_m-((-1)^mxL_m-1)L_{3m})(W_1^2-W_0^2-W_0W_1)W_{mn+j}-3x(-1)^{m+j}((-1)^{3m}x^2-xL_{3m}+1)(W_1^2-W_0^2-W_0W_1)W_{m+j}-3x(-1)^{2m+j}((-1)^{3m}x^2-(-1)^mxL_m+1)(W_1^2-W_0^2-W_0W_1)W_{j-m}+3x(-1)^j((-1)^{4m}x^2L_m-L_{3m}((-1)^mxL_m-1))(W_1^2-W_0^2-W_0W_1)W_j.$

(b) *If $((-1)^{3m}x^2-xL_{3m}+1)((-1)^{3m}x^2-x(-1)^mL_m+1)=u(x-a)(x-b)(x-c)(x-d)=0$ for some $u,a,b,c,d\in\mathbb{C}$ with $u\neq 0$ and $a\neq b\neq c\neq d$, i.e., $x=a$ or $x=b$ or $x=c$ or $x=d$, then*

$$\sum_{k=0}^{n}x^kW_{mk+j}^3=\frac{\Psi_2}{\Lambda_1}$$

where

$\Psi_2=5(-1)^{3m}x^n((-1)^{3m}x^2(n+3)-x(-1)^m(n+2)L_m+n+1)W_{mn-m+j}^3+5((-1)^{6m}(n+4)x^3-(-1)^{3m}((-1)^mL_m+L_{3m})(n+3)x^2+(-1)^m(L_mL_{3m}+(-1)^{2m})(n+2)x-(n+1)L_{3m})x^nW_{mn+j}^3+5(-1)^{3m}(-3(-1)^{3m}x^2+2(-1)^mxL_m-1)W_{j-m}^3+5(2(-1)^{3m}x-(-1)^mL_m)W_j^3+3(-1)^{mn+m+j}((-1)^{3m}(n+3)x^2-x(n+2)L_{3m}+n+1)(W_1^2-W_0^2-W_0W_1)x^nW_{mn+m+j}+3(-1)^{mn+2m+j}((-1)^{3m}(n+3)x^2-x(-1)^m(n+2)L_m+n+1)x^n(W_1^2-W_0^2-W_0W_1)W_{mn-m+j}+3(-1)^{mn+j}(-(-1)^{4m}(n+3)x^2L_m+x(-1)^m(n+2)L_{3m}L_m-(n+1)L_{3m})x^n(W_1^2-W_0^2-W_0W_1)W_{mn+j}+3(-1)^{m+j}(-3(-1)^{3m}x^2+2xL_{3m}-1)(W_1^2-W_0^2-W_0W_1)W_{m+j}+3(-1)^{2m+j}(-3(-1)^{3m}x^2+2(-1)^mxL_m-1)(W_1^2-W_0^2-W_0W_1)W_{j-m}+3(-1)^j(3(-1)^{4m}x^2L_m-2(-1)^mxL_mL_{3m}+L_{3m})(W_1^2-W_0^2-W_0W_1)W_j$

and

$\Lambda_1=5(4(-1)^{6m}x^3-3(-1)^{3m}((-1)^mL_m+L_{3m})x^2+2(-1)^m(2(-1)^{2m}+L_mL_{3m})x-((-1)^mL_m+L_{3m})).$

(c) *If* $((-1)^{3m}x^2 - xL_{3m}+1)((-1)^{3m}x^2 - x(-1)^m L_m + 1) = u(x-a)^2(x-b)(x-c) = 0$ *for some* $u,a,b,c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $x = a$ *or* $x = b$ *or* $x = c$, *then if* $x = b$ *or* $x = c$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_3}{\Lambda_2}$$

where

$\Psi_3 = 5(-1)^{3m}x^n((-1)^{3m}x^2(n+3) - x(-1)^m(n+2)L_m + n+1)W_{mn-m+j}^3 + 5((-1)^{6m}(n+4)x^3 - (-1)^{3m}((-1)^m L_m + L_{3m})(n+3)x^2 + (-1)^m(L_m L_{3m} + (-1)^{2m})(n+2)x - (n+1)L_{3m})x^n W_{mn+j}^3 + 5(-1)^{3m}(-3(-1)^{3m}x^2 + 2(-1)^m x L_m - 1)W_{j-m}^3 + 5(2(-1)^{3m}x - (-1)^m L_m)W_j^3 + 3(-1)^{mn+m+j}((-1)^{3m}(n+3)x^2 - x(n+2)L_{3m} + n+1)(W_1^2 - W_0^2 - W_0W_1)x^n W_{mn+m+j} + 3(-1)^{mn+2m+j}((-1)^{3m}(n+3)x^2 - x(-1)^m(n+2)L_m + n+1)x^n(W_1^2 - W_0^2 - W_0W_1)W_{mn-m+j} + 3(-1)^{mn+j}(-(-1)^{4m}(n+3)x^2 L_m + x(-1)^m(n+2)L_{3m}L_m - (n+1)L_{3m})x^n(W_1^2 - W_0^2 - W_0W_1)W_{mn+j} + 3(-1)^{m+j}(-3(-1)^{3m}x^2 + 2xL_{3m} - 1)(W_1^2 - W_0^2 - W_0W_1)W_{m+j} + 3(-1)^{2m+j}(-3(-1)^{3m}x^2 + 2(-1)^m x L_m - 1)(W_1^2 - W_0^2 - W_0W_1)W_{j-m} + 3(-1)^j(3(-1)^{4m}x^2 L_m - 2(-1)^m x L_m L_{3m} + L_{3m})(W_1^2 - W_0^2 - W_0W_1)W_j$

and

$\Lambda_2 = 5(4(-1)^{6m}x^3 - 3(-1)^{3m}((-1)^m L_m + L_{3m})x^2 + 2(-1)^m(2(-1)^{2m} + L_m L_{3m})x - ((-1)^m L_m + L_{3m}))$

and if $x = a$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_4}{10(-1)^m(6(-1)^{5m}x^2 - 3x(-1)^{2m}((-1)^m L_m + L_{3m}) + 2(-1)^{2m} + L_m L_{3m})}$$

where

$\Psi_4 = 5(-1)^{3m}((-1)^{3m}(n+3)(n+2)x^2 - (-1)^m x(n+2)(n+1)L_m + n(n+1))x^{n-1}W_{mn-m+j}^3 + 5((-1)^{6m}(n+4)(n+3)x^3 - (-1)^{3m}(n+3)(n+2)((-1)^m L_m + L_{3m})x^2 + x(-1)^m(n+2)(n+1)(L_m L_{3m} + (-1)^{2m}) - n(n+1)L_{3m})x^{n-1}W_{mn+j}^3 + 10(-1)^{4m}(L_m - 3(-1)^{2m}x)W_{j-m}^3 + 10(-1)^{3m}W_j^3 + 3(-1)^{mn+m+j}((-1)^{3m}(n+3)(n+2)x^2 - x(n+2)(n+1)L_{3m} + n(n+1))(W_1^2 - W_0^2 - W_0W_1)x^{n-1}W_{mn+m+j} + 3x^{n-1}(-1)^{mn+2m+j}((-1)^{3m}(n+3)(n+2)x^2 - x(-1)^m(n+2)(n+1)L_m + n(n+1))(W_1^2 - W_0^2 - W_0W_1)W_{mn-m+j} + 3x^{n-1}(-1)^{mn+j}(-x^2(-1)^{4m}(n+3)(n+2)L_m + x(-1)^m(n+2)(n+1)L_{3m}L_m - n(n+1)L_{3m})(W_1^2 - W_0^2 - W_0W_1)W_{mn+j} + 6(-1)^{m+j}(L_{3m} - 3(-1)^{3m}x)(W_1^2 - W_0^2 - W_0W_1)W_{m+j} + 6(-1)^{3m+j}(L_m - 3(-1)^{2m}x)(W_1^2 - W_0^2 - W_0W_1)W_{j-m} + 6(-1)^{m+j}(3(-1)^{3m}x - L_{3m})L_m(W_1^2 - W_0^2 - W_0W_1)W_j.$

(d) *If* $((-1)^{3m}x^2 - xL_{3m}+1)((-1)^{3m}x^2 - x(-1)^m L_m + 1) = u(x-a)^3(x-b) = 0$ *for some* $u,a,b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $x = a$ *or* $x = b$, *then if* $x = b$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_5}{\Lambda_3}$$

where

$\Psi_5 = 5(-1)^{3m}x^n((-1)^{3m}x^2(n+3) - x(-1)^m(n+2)L_m + n+1)W_{mn-m+j}^3 + 5((-1)^{6m}(n+4)x^3 - (-1)^{3m}((-1)^m L_m + L_{3m})(n+3)x^2 + (-1)^m(L_m L_{3m} + (-1)^{2m})(n+2)x - (n+1)L_{3m})x^n W_{mn+j}^3 + 5(-1)^{3m}(-3(-1)^{3m}x^2 + 2(-1)^m x L_m - 1)W_{j-m}^3 + 5(2(-1)^{3m}x - (-1)^m L_m)W_j^3 + 3(-1)^{mn+m+j}((-1)^{3m}(n+3)x^2 - x(n+2)L_{3m} + n+1)(W_1^2 - W_0^2 - W_0W_1)x^n W_{mn+m+j} + 3(-1)^{mn+2m+j}((-1)^{3m}(n+3)x^2 - x(-1)^m(n+2)L_m + n+1)x^n(W_1^2 - W_0^2 - W_0W_1)W_{mn-m+j} + 3(-1)^{mn+j}(-(-1)^{4m}(n+3)x^2 L_m + x(-1)^m(n+2)L_{3m}L_m - (n+1)L_{3m})x^n(W_1^2 - W_0^2 - W_0W_1)W_{mn+j} + 3(-1)^{m+j}(-3(-1)^{3m}x^2 + 2xL_{3m} - 1)(W_1^2 - W_0^2 - W_0W_1)W_{m+j} + 3(-1)^{2m+j}(-3(-1)^{3m}x^2 + 2(-1)^m x L_m - 1)(W_1^2 - W_0^2 - W_0W_1)W_{j-m} + 3(-1)^j(3(-1)^{4m}x^2 L_m - 2(-1)^m x L_m L_{3m} + L_{3m})(W_1^2 - W_0^2 - W_0W_1)W_j$

and

$\Lambda_3 = 5(4(-1)^{6m}x^3 - 3(-1)^{3m}((-1)^m L_m + L_{3m})x^2 + 2(-1)^m(2(-1)^{2m} + L_m L_{3m})x - ((-1)^m L_m + L_{3m}))$

and if $x = a$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_6}{30(-1)^{3m}(4(-1)^{3m}x - (-1)^m L_m - L_{3m})}$$

where

$\Psi_6 = 5(-1)^{3m}(n+1)((-1)^{3m}(n+3)(n+2)x^2 - x(-1)^m n(n+2)L_m + n(n-1))x^{n-2}W_{mn-m+j}^3 + 5((-1)^{6m}(n+3)(n+2)(n+4)x^3 - (-1)^{3m}(n+3)(n+2)(n+1)((-1)^m L_m + L_{3m})x^2 + (-1)^m n(n+2)(n+1)(L_m L_{3m} + (-1)^{2m})x - n(n-1)(n+1)L_{3m})x^{n-2}W_{mn+j}^3 - 30(-1)^{6m}W_{j-m}^3 + 3(-1)^{mn+m+j}(n+1)((-1)^{3m}(n+3)(n+2)x^2 - xn(n+2)L_{3m} + n(n-1))(W_1^2 - W_0^2 - W_0W_1)x^{n-2}W_{mn+m+j} + 3(-1)^{mn+2m+j}(n+1)((-1)^{3m}(n+3)(n+2)x^2 - x(-1)^m n(n+2)L_m + n(n-1))(W_1^2 - W_0^2 - W_0W_1)x^{n-2}W_{mn-m+j} + 3(-1)^{mn+j}(n+1)(-x^2(-1)^{4m}(n+3)(n+2)L_m + x(-1)^m n(n+2)L_{3m}L_m - n(n-1)L_{3m})(W_1^2 - W_0^2 - W_0W_1)x^{n-2}W_{mn+j} - 18(-1)^{4m+j}(W_1^2 - W_0^2 - W_0W_1)W_{m+j} - 18(-1)^{5m+j}(W_1^2 - W_0^2 - W_0W_1)W_{j-m} + 18(-1)^{4m+j}L_m(W_1^2 - W_0^2 - W_0W_1)W_j.$

(e) *If* $((-1)^{3m}x^2 - xL_{3m}+1)((-1)^{3m}x^2 - x(-1)^m L_m+1) = u(x-a)^4 = 0$ *for some* $u,a\in\mathbb{C}, u\neq 0$ *i.e.,* $x=a$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_7}{120(-1)^{6m}}$$

where

$\Psi_7 = 5(-1)^{3m}n(n+1)((-1)^{3m}(n+3)(n+2)x^2 - x(-1)^m(n-1)(n+2)L_m + (n-1)(n-2))x^{n-3}W_{mn-m+j}^3 + 5(n+1)(x^3(-1)^{6m}(n+4)(n+3)(n+2) - x^2(-1)^{3m}n(n+3)(n+2)((-1)^m L_m + L_{3m}) + x(-1)^m n(n-1)(n+2)(L_mL_{3m}+(-1)^{2m}) - n(n-1)(n-2)L_{3m})x^{n-3}W_{mn+j}^3 + 3(-1)^{mn+m+j}n(n+1)(x^2(-1)^{3m}(n+3)(n+2) - x(n+2)(n-1)L_{3m} + (n-1)(n-2))(W_1^2 - W_0^2 - W_0W_1)x^{n-3}W_{mn+m+j} + 3(-1)^{mn+2m+j}n(n+1)(x^2(-1)^{3m}(n+3)(n+2) - x(-1)^m(n+2)(n-1)L_m + (n-1)(n-2))(W_1^2 - W_0^2 - W_0W_1)x^{n-3}W_{mn-m+j} + 3(-1)^{mn+j}n(n+1)(-x^2(-1)^{4m}(n+3)(n+2)L_m + x(-1)^m(n+2)(n-1)L_{3m}L_m - (n-1)(n-2)L_{3m})(W_1^2 - W_0^2 - W_0W_1)x^{n-3}W_{mn+j}$.

Proof. Take $r=1, s=1$ and $H_n = L_n$ in Theorem 2.1. □

Note that (15) can be written in the following form:

$$\sum_{k=1}^{n} x^k W_{mk+j}^2 = \frac{\Psi_8}{5((-1)^{3m}x^2 - xL_{3m}+1)((-1)^{3m}x^2 - x(-1)^m L_m+1)}$$

where

$\Psi_8 = 5x^{n+1}(-1)^{3m}((-1)^{3m}x^2 - (-1)^m xL_m + 1)W_{mn-m+j}^3 + 5x^{n+1}((-1)^{3m}x - L_{3m})((-1)^{3m}x^2 - (-1)^m xL_m + 1)W_{mn+j}^3 - 5(-1)^{3m}x((-1)^{3m}x^2 - (-1)^m xL_m + 1)W_{j-m}^3 + 5(L_{3m} - (-1)^{3m}x)((-1)^{3m}x^2 - (-1)^m xL_m + 1)xW_j^3 + 3x^n(-1)^{mn+m+j}x((-1)^{3m}x^2 - xL_{3m}+1)(W_1^2 - W_0^2 - W_0W_1)W_{mn+m+j} + 3x^n(-1)^{mn+2m+j}x((-1)^{3m}x^2 - (-1)^m xL_m + 1)(W_1^2 - W_0^2 - W_0W_1)W_{mn-m+j} - 3x^n(-1)^{mn+j}x((-1)^{4m}x^2L_m - ((-1)^m xL_m - 1)L_{3m})(W_1^2 - W_0^2 - W_0W_1)W_{mn+j} - 3x(-1)^{m+j}((-1)^{3m}x^2 - xL_{3m}+1)(W_1^2 - W_0^2 - W_0W_1)W_{m+j} - 3x(-1)^{2m+j}((-1)^{3m}x^2 - (-1)^m xL_m + 1)(W_1^2 - W_0^2 - W_0W_1)W_{j-m} + 3x(-1)^j((-1)^{4m}x^2L_m - L_{3m}((-1)^m xL_m - 1))(W_1^2 - W_0^2 - W_0W_1)W_j$.

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 2.1.
For generalized Fibonacci numbers (the case $r=s=1$*) we have the following sum formulas for* $n\geq 0$*:*

(a) $(m=1,\ j=0)$

If $(x^2-x-1)(x^2+4x-1)\neq 0$*, i.e.,* $x\neq -2+\sqrt{5}$, $x\neq -2-\sqrt{5}$, $x\neq \frac{1}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{1}{2}-\frac{1}{2}\sqrt{5}$*, then*

$$\sum_{k=0}^{n} x^k W_k^3 = \frac{\Psi_1}{5(x^2-x-1)(x^2+4x-1)}$$

where

$\Psi_1 = 5x^{n+1}(x+4)(x^2-x-1)W_n^3 + 5(x^2-x-1)x^{n+1}W_{n-1}^3 + 3(-1)^n x^{n+1}(x^2+4x-1)(W_1^2 - W_0^2 - W_0W_1)W_{n+1} + 3(-1)^{n+1}x^{n+1}(x^2-x-1)(W_1^2 - W_0^2 - W_0W_1)W_{n-1} + 3(-1)^{n+1}x^{n+1}(x+2)^2(W_1^2 - W_0^2 - W_0W_1)W_n + 5(-x(x^2+2x-1)W_1^3 + (x^3-5x^2-3x+1)W_0^3 + 3x^2((x+1)W_1 - (x-1)W_0)W_1W_0)$

and

if $(x^2-x-1)(x^2+4x-1)=0$*, i.e.,* $x=-2+\sqrt{5}$ *or* $x=-2-\sqrt{5}$ *or* $x=\frac{1}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{1}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k W_k^3 = \frac{\Psi_2}{5(4x^3+9x^2-12x-3)}$$

where

$\Psi_2 = 5x^n(n(x+4)(x^2-x-1) + 4x^3 + 9x^2 - 10x - 4)W_n^3 + 5x^n(n(x^2-x-1) + 3x^2 - 2x - 1)W_{n-1}^3 + 3(-1)^n x^n(n(x^2+4x-1) + 3x^2 + 8x - 1)(W_1^2 - W_0^2 - W_0W_1)W_{n+1} + 3(-1)^{n+1}x^n(n(x^2-x-1) + 3x^2 - 2x - 1)(W_1^2 - W_0^2 - W_0W_1)W_{n-1} + 3(-1)^{n+1}x^n(x+2)(n(x+2) + 3x + 2)(W_1^2 - W_0^2 - W_0W_1)W_n + 5(-(3x^2+4x-1)W_1^3 + (3x^2-10x-3)W_0^3 + 3x((3x+2)W_1 - (3x-2)W_0)W_1W_0)$.

(b) $(m=2,\ j=0)$

If $(x^2-3x+1)(x^2-18x+1)\neq 0$*, i.e.,* $x\neq 9+4\sqrt{5}$, $x\neq 9-4\sqrt{5}$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$*, then*

$$\sum_{k=0}^{n} x^k W_{2k}^3 = \frac{\Psi_1}{5(x^2-3x+1)(x^2-18x+1)}$$

where

$\Psi_1 = 5x^{n+1}(x-18)(x^2-3x+1)W_{2n}^3 + 5x^{n+1}(x^2-3x+1)W_{2n-2}^3 + 3x^{n+1}(x^2-18x+1)(W_1^2-W_0^2-W_0W_1)W_{2n+2} - 9x^{n+1}(x^2-18x+6)(W_1^2-W_0^2-W_0W_1)W_{2n} + 3x^{n+1}(x^2-3x+1)(W_1^2-W_0^2-W_0W_1)W_{2n-2} + 5(x(x^2+6x+1)W_1^3 - (8x^3-43x^2+20x-1)W_0^3 - 3x(2x^2+3x-1)W_1^2W_0 + 3x(4x^2-9x+1)W_0^2W_1)$

and

if $\left(x^2-3x+1\right)\left(x^2-18x+1\right)=0$, *i.e.,* $x=9+4\sqrt{5}$ *or* $x=9-4\sqrt{5}$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k W_{2k}^3 = \frac{\Psi_2}{5(4x^3-63x^2+112x-21)}$$

where

$\Psi_2 = 5x^n(n(x-18)(x^2-3x+1)+4x^3-63x^2+110x-18)W_{2n}^3 + 5x^n(n(x^2-3x+1)+3x^2-6x+1)W_{2n-2}^3 + 3x^n(n(x^2-18x+1)+3x^2-36x+1)(W_1^2-W_0^2-W_0W_1)W_{2n+2} + 9x^n(-n(x^2-18x+6)-3x^2+36x-6)(W_1^2-W_0^2-W_0W_1)W_{2n} + 3x^n(n(x^2-3x+1)+3x^2-6x+1)(W_1^2-W_0^2-W_0W_1)W_{2n-2} + 5((3x^2+12x+1)W_1^3 - 2(4x-1)(3x-10)W_0^3 - 3(6x^2+6x-1)W_0W_1^2 + 3(12x^2-18x+1)W_0^2W_1)$.

(c) $(m=2,\ j=1)$

If $\left(x^2-3x+1\right)\left(x^2-18x+1\right)\neq 0$, *i.e.,* $x\neq 9+4\sqrt{5}$, $x\neq 9-4\sqrt{5}$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^3 = \frac{\Psi_1}{5\left(x^2-3x+1\right)\left(x^2-18x+1\right)}$$

where

$\Psi_1 = 5x^{n+1}(x-18)(x^2-3x+1)W_{2n+1}^3 + 5x^{n+1}(x^2-3x+1)W_{2n-1}^3 - 3x^{n+1}(x^2-18x+1)(W_1^2-W_0^2-W_0W_1)W_{2n+3} + 9x^{n+1}(x^2-18x+6)(W_1^2-W_0^2-W_0W_1)W_{2n+1} - 3x^{n+1}(x^2-3x+1)(W_1^2-W_0^2-W_0W_1)W_{2n-1} + 5(-(x-1)(x^2-12x+1)W_1^3 + x(x^2+6x+1)W_0^3 + 3x(x^2-9x+4)W_1^2W_0 - 3x(x^2-3x-2)W_0^2W_1)$

and

if $\left(x^2-3x+1\right)\left(x^2-18x+1\right)=0$, *i.e.,* $x=9+4\sqrt{5}$ *or* $x=9-4\sqrt{5}$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^3 = \frac{\Psi_2}{5(4x^3-63x^2+112x-21)}$$

where

$\Psi_2 = 5x^n(n(x-18)(x^2-3x+1)+4x^3-63x^2+110x-18)W_{2n+1}^3 + 5x^n(n(x^2-3x+1)+3x^2-6x+1)W_{2n-1}^3 - 3x^n(n(x^2-18x+1)+3x^2-36x+1)(W_1^2-W_0^2-W_0W_1)W_{2n+3} + 9x^n(n(x^2-18x+6)+3x^2-36x+6)(W_1^2-W_0^2-W_0W_1)W_{2n+1} - 3x^n(n(x^2-3x+1)+3x^2-6x+1)(W_1^2-W_0^2-W_0W_1)W_{2n-1} + 5(-(3x^2-26x+13)W_1^3 + (3x^2+12x+1)W_0^3 + 3(3x^2-18x+4)W_1^2W_0 - 3(3x^2-6x-2)W_0^2W_1)$.

(d) $(m=-1,\ j=0)$

If $\left(x^2-4x-1\right)\left(x^2+x-1\right)\neq 0$, *i.e.,* $x\neq 2+\sqrt{5}$, $x\neq 2-\sqrt{5}$, $x\neq -\frac{1}{2}+\frac{1}{2}\sqrt{5}$, $x\neq -\frac{1}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k}^3 = \frac{\Psi_1}{5\left(x^2-4x-1\right)\left(x^2+x-1\right)}$$

where

$\Psi_1 = 5x^{n+1}(x^2+x-1)W_{-n+1}^3 + 5x^{n+1}(x-4)(x^2+x-1)W_{-n}^3 + 3(-1)^{n+1}x^{n+1}(x^2+x-1)(W_1^2-W_0^2-W_0W_1)W_{-n+1} + 3(-1)^n x^{n+1}(x-2)^2(W_1^2-W_0^2-W_0W_1)W_{-n} + 3(-1)^n x^{n+1}(x^2-4x-1)(W_1^2-W_0^2-W_0W_1)W_{-n-1} - 5(x(x^2-2x-1)W_1^3 + (x^2-2x-1)W_0^3 + 3x(x+1)W_1^2W_0 + 3x(x-1)W_0^2W_1)$

and

if $\left(x^2-4x-1\right)\left(x^2+x-1\right)=0$, *i.e.,* $x=2+\sqrt{5}$ *or* $x=2-\sqrt{5}$ *or* $x=-\frac{1}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=-\frac{1}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k W_{-k}^3 = \frac{\Psi_2}{5\left(4x^3-9x^2-12x+3\right)}$$

where

$\Psi_2 = 5x^n(n(x^2+x-1)+3x^2+2x-1)W_{-n+1}^3 + 5x^n(n(x-4)(x^2+x-1)+4x^3-9x^2-10x+4)W_{-n}^3 + 3(-1)^{n+1}x^n(n(x^2+x-1)+3x^2+2x-1)(W_1^2-W_0^2-W_0W_1)W_{-n+1} + 3(-1)^n x^n(x-2)(nx-2n+3x-2)(W_1^2-W_0^2-W_0W_1)W_{-n} + 3(-1)^n x^n(n(x^2-4x-1)+3x^2-8x-1)(W_1^2-W_0^2-W_0W_1)W_{-n-1} - 5((3x^2-4x-1)W_1^3 + 2(x-1)W_0^3 + 3(2x+1)W_1^2W_0 + 3(2x-1)W_0^2W_1)$.

(e) $(m=-2,\ j=0)$

If $\left(x^2-3x+1\right)\left(x^2-18x+1\right)\neq 0$, *i.e.,* $x\neq 9+4\sqrt{5}$, $x\neq 9-4\sqrt{5}$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^3 = \frac{\Psi_1}{5\left(x^2-3x+1\right)\left(x^2-18x+1\right)}$$

where

$\Psi_1 = 5x^{n+1}(x^2-3x+1)W_{-2n+2}^3 + 5x^{n+1}(x-18)(x^2-3x+1)W_{-2n}^3 + 3x^{n+1}\left(x^2-3x+1\right)(W_1^2-W_0^2-W_0W_1)W_{-2n+2} - 9x^{n+1}(x^2-18x+6)(W_1^2-W_0^2-W_0W_1)W_{-2n} + 3x^{n+1}(x^2-18x+1)(W_1^2-W_0^2-W_0W_1)W_{-2n-2} - 5(x(x^2+6x+1)W_1^3 + (x-1)(x^2-12x+1)W_0^3 + 3x(x^2-3x-2)W_1^2W_0 + 3x(x^2-9x+4)W_0^2W_1)$

and

if $\left(x^2-3x+1\right)\left(x^2-18x+1\right)=0$, *i.e.,* $x=9+4\sqrt{5}$ *or* $x=9-4\sqrt{5}$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^3 = \frac{\Psi_2}{5\left(4x^3-63x^2+112x-21\right)}$$

where

$\Psi_2 = 5x^n(n(x^2-3x+1)+3x^2-6x+1)W_{-2n+2}^3 + 5x^n(n(x-18)(x^2-3x+1)+4x^3-63x^2+110x-18)W_{-2n}^3 + 3x^n(n(x^2-3x+1)+3x^2-6x+1)(W_1^2-W_0^2-W_0W_1)W_{-2n+2} - 9x^n(n(x^2-18x+6)+3x^2-36x+6)(W_1^2-W_0^2-W_0W_1)W_{-2n} + 3x^n(n(x^2-18x+1)+3x^2-36x+1)(W_1^2-W_0^2-W_0W_1)W_{-2n-2} - 5((3x^2+12x+1)W_1^3 + (3x^2-26x+13)W_0^3 + 3(3x^2-6x-2)W_1^2W_0 + 3(3x^2-18x+4)W_0^2W_1)$.

(f) $(m=-2,\ j=1)$

If $\left(x^2-3x+1\right)\left(x^2-18x+1\right)\neq 0$, *i.e.,* $x\neq 9+4\sqrt{5}$, $x\neq 9-4\sqrt{5}$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^3 = \frac{\Psi_1}{5\left(x^2-3x+1\right)\left(x^2-18x+1\right)}$$

where

$\Psi_1 = 5x^{n+1}(x^2-3x+1)W_{-2n+3}^3 + 5x^{n+1}(x-18)(x^2-3x+1)W_{-2n+1}^3 - 3x^{n+1}(x^2-3x+1)(W_1^2-W_0^2-W_0W_1)W_{-2n+3} + 9x^{n+1}(x^2-18x+6)(W_1^2-W_0^2-W_0W_1)W_{-2n+1} - 3x^{n+1}(x^2-18x+1)(W_1^2-W_0^2-W_0W_1)W_{-2n-1} - 5((8x^3-43x^2+20x-1)W_1^3 + x(x^2+6x+1)W_0^3 + 3x(4x^2-9x+1)W_1^2W_0 + 3x(2x^2+3x-1)W_0^2W_1)$

and

if $\left(x^2-3x+1\right)\left(x^2-18x+1\right)=0$, *i.e.,* $x=9+4\sqrt{5}$ *or* $x=9-4\sqrt{5}$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^3 = \frac{\Psi_2}{5\left(4x^3-63x^2+112x-21\right)}$$

where

$\Psi_2 = 5x^n(n(x^2-3x+1)+3x^2-6x+1)W_{-2n+3}^3 + 5x^n(n(x-18)(x^2-3x+1)+4x^3-63x^2+110x-18)W_{-2n+1}^3 - 3x^n(n(x^2-3x+1)+3x^2-6x+1)(W_1^2-W_0^2-W_0W_1)W_{-2n+3} + 9x^n(n(x^2-18x+6)+3x^2-36x+6)(W_1^2-W_0^2-W_0W_1)W_{-2n+1} - 3x^n(n(x^2-18x+1)+3x^2-36x+1)(W_1^2-W_0^2-W_0W_1)W_{-2n-1} - 5(2(4x-1)(3x-10)W_1^3 + (3x^2+12x+1)W_0^3 + 3(12x^2-18x+1)W_0W_1^2 + 3(6x^2+6x-1)W_0^2W_1)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n=F_n$ with $F_0=0, F_1=1$).

Corollary 2.1.

For $n\geq 0$, *Fibonacci numbers have the following properties:*

(a) $(m=1,\ j=0)$

If $\left(x^2-x-1\right)\left(x^2+4x-1\right)\neq 0$, *i.e.,* $x\neq -2+\sqrt{5}$, $x\neq -2-\sqrt{5}$, $x\neq \frac{1}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{1}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_k^3 = \frac{\Psi_1}{5\left(x^2-x-1\right)\left(x^2+4x-1\right)}$$

where

$\Psi_1 = 5x^{n+1}(x+4)(x^2-x-1)F_n^3 + 5(x^2-x-1)x^{n+1}F_{n-1}^3 + 3(-1)^n x^{n+1}\left(x^2+4x-1\right)F_{n+1} + 3(-1)^{n+1}x^{n+1}(x^2-x-1)F_{n-1} + 3(-1)^{n+1}x^{n+1}(x+2)^2F_n - 5x(x^2+2x-1)$

and

if $\left(x^2-x-1\right)\left(x^2+4x-1\right)=0$, *i.e.,* $x=-2+\sqrt{5}$ *or* $x=-2-\sqrt{5}$ *or* $x=\frac{1}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{1}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k F_k^3 = \frac{\Psi_2}{5\left(4x^3+9x^2-12x-3\right)}$$

where

$\Psi_2 = 5x^n(n(x+4)(x^2-x-1)+4x^3+9x^2-10x-4)F_n^3 + 5x^n(n(x^2-x-1)+3x^2-2x-1)F_{n-1}^3 + 3(-1)^n x^n\left(n(x^2+4x-1)+3x^2+8x-1\right)F_{n+1} + 3(-1)^{n+1}x^n(n(x^2-x-1)+3x^2-2x-1)F_{n-1} + 3(-1)^{n+1}x^n(x+2)(n(x+2)+3x+2)F_n - 5(3x^2+4x-1)$.

(b) $(m=2,\ j=0)$

If $\left(x^2-3x+1\right)\left(x^2-18x+1\right)\neq 0$, *i.e.,* $x\neq 9+4\sqrt{5}$, $x\neq 9-4\sqrt{5}$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_{2k}^3 = \frac{\Psi_1}{5\left(x^2-3x+1\right)\left(x^2-18x+1\right)}$$

where

$\Psi_1 = 5x^{n+1}(x-18)(x^2-3x+1)F_{2n}^3 + 5x^{n+1}(x^2-3x+1)F_{2n-2}^3 + 3x^{n+1}(x^2-18x+1)F_{2n+2} - 9x^{n+1}(x^2-18x+6)F_{2n} + 3x^{n+1}(x^2-3x+1)F_{2n-2} + 5x(x^2+6x+1)$

and

if $\left(x^2-3x+1\right)\left(x^2-18x+1\right)=0$, *i.e.,* $x=9+4\sqrt{5}$ *or* $x=9-4\sqrt{5}$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k F_{2k}^3 = \frac{\Psi_2}{5(4x^3-63x^2+112x-21)}$$

where

$\Psi_2 = 5x^n(n(x-18)(x^2-3x+1)+4x^3-63x^2+110x-18)F_{2n}^3 + 5x^n(n(x^2-3x+1)+3x^2-6x+1)F_{2n-2}^3 + 3x^n(n(x^2-18x+1)+3x^2-36x+1)F_{2n+2} + 9x^n\left(-n(x^2-18x+6)-3x^2+36x-6\right)F_{2n} + 3x^n(n(x^2-3x+1)+3x^2-6x+1)F_{2n-2} + 5(3x^2+12x+1)$.

(c) $(m=2,\ j=1)$

If $\left(x^2-3x+1\right)\left(x^2-18x+1\right)\neq 0$, *i.e.,* $x\neq 9+4\sqrt{5}$, $x\neq 9-4\sqrt{5}$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_{2k+1}^3 = \frac{\Psi_1}{5\left(x^2-3x+1\right)\left(x^2-18x+1\right)}$$

where

$\Psi_1 = 5x^{n+1}(x-18)(x^2-3x+1)F_{2n+1}^3 + 5x^{n+1}(x^2-3x+1)F_{2n-1}^3 - 3x^{n+1}(x^2-18x+1)F_{2n+3} + 9x^{n+1}(x^2-18x+6)F_{2n+1} - 3x^{n+1}(x^2-3x+1)F_{2n-1} - 5(x-1)(x^2-12x+1)$

and

if $\left(x^2-3x+1\right)\left(x^2-18x+1\right)=0$, *i.e.,* $x=9+4\sqrt{5}$ *or* $x=9-4\sqrt{5}$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k F_{2k+1}^3 = \frac{\Psi_2}{5(4x^3-63x^2+112x-21)}$$

where

$\Psi_2 = 5x^n(n(x-18)(x^2-3x+1)+4x^3-63x^2+110x-18)F_{2n+1}^3 + 5x^n(n(x^2-3x+1)+3x^2-6x+1)F_{2n-1}^3 - 3x^n(n(x^2-18x+1)+3x^2-36x+1)F_{2n+3} + 9x^n(n(x^2-18x+6)+3x^2-36x+6)F_{2n+1} - 3x^n(n(x^2-3x+1)+3x^2-6x+1)F_{2n-1} - 5(3x^2-26x+13)$.

(d) $(m=-1,\ j=0)$

If $\left(x^2-4x-1\right)\left(x^2+x-1\right)\neq 0$, *i.e.,* $x\neq 2+\sqrt{5}$, $x\neq 2-\sqrt{5}$, $x\neq -\frac{1}{2}+\frac{1}{2}\sqrt{5}$, $x\neq -\frac{1}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_{-k}^3 = \frac{\Psi_1}{5\left(x^2-4x-1\right)\left(x^2+x-1\right)}$$

where

$\Psi_1 = 5x^{n+1}(x^2+x-1)F_{-n+1}^3 + 5x^{n+1}(x-4)(x^2+x-1)F_{-n}^3 + 3(-1)^{n+1}x^{n+1}(x^2+x-1)F_{-n+1} + 3(-1)^n x^{n+1}(x-2)^2(F_1^2-F_0^2-F_0F_1)F_{-n} + 3(-1)^n x^{n+1}(x^2-4x-1)F_{-n-1} - 5x(x^2-2x-1)$

and

if $\left(x^2-4x-1\right)\left(x^2+x-1\right)=0$, *i.e.,* $x=2+\sqrt{5}$ *or* $x=2-\sqrt{5}$ *or* $x=-\frac{1}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=-\frac{1}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k F_{-k}^3 = \frac{\Psi_2}{5\left(4x^3-9x^2-12x+3\right)}$$

where

$\Psi_2 = 5x^n(n(x^2+x-1)+3x^2+2x-1)F_{-n+1}^3+5x^n(n(x-4)(x^2+x-1)+4x^3-9x^2-10x+4)F_{-n}^3+3(-1)^{n+1}x^n(n(x^2+x-1)+3x^2+2x-1)F_{-n+1}+3(-1)^n x^n(x-2)(nx-2n+3x-2)F_{-n}+3(-1)^n x^n(n(x^2-4x-1)+3x^2-8x-1)F_{-n-1}-5(3x^2-4x-1)$.

(e) $(m=-2,\ j=0)$

If $\left(x^2-3x+1\right)\left(x^2-18x+1\right)\neq 0$, *i.e.,* $x\neq 9+4\sqrt{5}$, $x\neq 9-4\sqrt{5}$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_{-2k}^3 = \frac{\Psi_1}{5\left(x^2-3x+1\right)\left(x^2-18x+1\right)}$$

where

$\Psi_1 = 5x^{n+1}(x^2-3x+1)F_{-2n+2}^3+5x^{n+1}(x-18)(x^2-3x+1)F_{-2n}^3+3x^{n+1}\left(x^2-3x+1\right)F_{-2n+2}-9x^{n+1}(x^2-18x+6)F_{-2n}+3x^{n+1}(x^2-18x+1)F_{-2n-2}-5x(x^2+6x+1)$

and

if $\left(x^2-3x+1\right)\left(x^2-18x+1\right)=0$, *i.e.,* $x=9+4\sqrt{5}$ *or* $x=9-4\sqrt{5}$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k F_{-2k}^3 = \frac{\Psi_2}{5\left(4x^3-63x^2+112x-21\right)}$$

where

$\Psi_2 = 5x^n(n(x^2-3x+1)+3x^2-6x+1)F_{-2n+2}^3+5x^n(n(x-18)(x^2-3x+1)+4x^3-63x^2+110x-18)F_{-2n}^3+3x^n(n(x^2-3x+1)+3x^2-6x+1)F_{-2n+2}-9x^n(n(x^2-18x+6)+3x^2-36x+6)F_{-2n}+3x^n(n(x^2-18x+1)+3x^2-36x+1)F_{-2n-2}-5(3x^2+12x+1)$.

(f) $(m=-2,\ j=1)$

If $\left(x^2-3x+1\right)\left(x^2-18x+1\right)\neq 0$, *i.e.,* $x\neq 9+4\sqrt{5}$, $x\neq 9-4\sqrt{5}$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_{-2k+1}^3 = \frac{\Psi_1}{5\left(x^2-3x+1\right)\left(x^2-18x+1\right)}$$

where

$\Psi_1 = 5x^{n+1}(x^2-3x+1)F_{-2n+3}^3+5x^{n+1}(x-18)(x^2-3x+1)F_{-2n+1}^3-3x^{n+1}(x^2-3x+1)F_{-2n+3}+9x^{n+1}(x^2-18x+6)F_{-2n+1}-3x^{n+1}(x^2-18x+1)F_{-2n-1}-5(8x^3-43x^2+20x-1)$

and

if $\left(x^2-3x+1\right)\left(x^2-18x+1\right)=0$, *i.e.,* $x=9+4\sqrt{5}$ *or* $x=9-4\sqrt{5}$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k F_{-2k+1}^3 = \frac{\Psi_2}{5\left(4x^3-63x^2+112x-21\right)}$$

where

$\Psi_2 = 5x^n(n(x^2-3x+1)+3x^2-6x+1)F_{-2n+3}^3+5x^n(n(x-18)(x^2-3x+1)+4x^3-63x^2+110x-18)F_{-2n+1}^3-3x^n(n(x^2-3x+1)+3x^2-6x+1)F_{-2n+3}+9x^n(n(x^2-18x+6)+3x^2-36x+6)F_{-2n+1}-3x^n(n(x^2-18x+1)+3x^2-36x+1)F_{-2n-1}-10(4x-1)(3x-10)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.2.
For $n \geq 0$, *Lucas numbers have the following properties:*

(a) $(m=1,\ j=0)$

If $(x^2-x-1)(x^2+4x-1)\neq 0$, *i.e.*, $x\neq -2+\sqrt{5}$, $x\neq -2-\sqrt{5}$, $x\neq \frac{1}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{1}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_k^3 = \frac{\Psi_1}{(x^2-x-1)(x^2+4x-1)}$$

where

$\Psi_1 = x^{n+1}(x+4)(x^2-x-1)L_n^3 + (x^2-x-1)x^{n+1}L_{n-1}^3 - 3(-1)^n x^{n+1}(x^2+4x-1)L_{n+1} - 3(-1)^{n+1}x^{n+1}(x^2-x-1)L_{n-1} - 3(-1)^{n+1}x^{n+1}(x+2)^2 L_n + (x^3-24x^2-23x+8)$

and

if $(x^2-x-1)(x^2+4x-1)=0$, *i.e.*, $x=-2+\sqrt{5}$ *or* $x=-2-\sqrt{5}$ *or* $x=\frac{1}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{1}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k L_k^3 = \frac{\Psi_2}{(4x^3+9x^2-12x-3)}$$

where

$\Psi_2 = x^n(n(x+4)(x^2-x-1)+4x^3+9x^2-10x-4)L_n^3 + x^n(n(x^2-x-1)+3x^2-2x-1)L_{n-1}^3 - 3(-1)^n x^n(n(x^2+4x-1)+3x^2+8x-1)L_{n+1} - 3(-1)^{n+1}x^n(n(x^2-x-1)+3x^2-2x-1)L_{n-1} - 3(-1)^{n+1}x^n(x+2)(n(x+2)+3x+2)L_n + (3x^2-48x-23)$.

(b) $(m=2,\ j=0)$

If $(x^2-3x+1)(x^2-18x+1)\neq 0$, *i.e.*, $x\neq 9+4\sqrt{5}$, $x\neq 9-4\sqrt{5}$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{2k}^3 = \frac{\Psi_1}{(x^2-3x+1)(x^2-18x+1)}$$

where

$\Psi_1 = x^{n+1}(x-18)(x^2-3x+1)L_{2n}^3 + x^{n+1}(x^2-3x+1)L_{2n-2}^3 - 3x^{n+1}(x^2-18x+1)L_{2n+2} + 9x^{n+1}(x^2-18x+6)L_{2n} - 3x^{n+1}(x^2-3x+1)L_{2n-2} - (27x^3-224x^2+141x-8)$

and

if $(x^2-3x+1)(x^2-18x+1)=0$, *i.e.*, $x=9+4\sqrt{5}$ *or* $x=9-4\sqrt{5}$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k L_{2k}^3 = \frac{\Psi_2}{(4x^3-63x^2+112x-21)}$$

where

$\Psi_2 = x^n(n(x-18)(x^2-3x+1)+4x^3-63x^2+110x-18)L_{2n}^3 + x^n(n(x^2-3x+1)+3x^2-6x+1)L_{2n-2}^3 - 3x^n(n(x^2-18x+1)+3x^2-36x+1)L_{2n+2} - 9x^n(-n(x^2-18x+6)-3x^2+36x-6)L_{2n} - 3x^n(n(x^2-3x+1)+3x^2-6x+1)L_{2n-2} - (81x^2-448x+141)$.

(c) $(m=2,\ j=1)$

If $(x^2-3x+1)(x^2-18x+1)\neq 0$, *i.e.*, $x\neq 9+4\sqrt{5}$, $x\neq 9-4\sqrt{5}$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{2k+1}^3 = \frac{\Psi_1}{(x^2-3x+1)(x^2-18x+1)}$$

where

$\Psi_1 = x^{n+1}(x-18)(x^2-3x+1)L_{2n+1}^3 + x^{n+1}(x^2-3x+1)L_{2n-1}^3 + 3x^{n+1}(x^2-18x+1)L_{2n+3} - 9x^{n+1}(x^2-18x+6)L_{2n+1} + 3x^{n+1}(x^2-3x+1)L_{2n-1} + (x+1)(x^2+42x+1)$

and

if $(x^2-3x+1)(x^2-18x+1)=0$, *i.e.*, $x=9+4\sqrt{5}$ *or* $x=9-4\sqrt{5}$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k L_{2k+1}^3 = \frac{\Psi_2}{(4x^3-63x^2+112x-21)}$$

where

$\Psi_2 = x^n(n(x-18)(x^2-3x+1)+4x^3-63x^2+110x-18)L_{2n+1}^3 + x^n(n(x^2-3x+1)+3x^2-6x+1)L_{2n-1}^3 + 3x^n(n(x^2-18x+1)+3x^2-36x+1)L_{2n+3} - 9x^n(n(x^2-18x+6)+3x^2-36x+6)L_{2n+1} + 3x^n(n(x^2-3x+1)+3x^2-6x+1)L_{2n-1} + (3x^2+86x+43)$.

(d) $(m=-1,\ j=0)$

If $\left(x^2-4x-1\right)\left(x^2+x-1\right)\neq 0$, *i.e.*, $x\neq 2+\sqrt{5}$, $x\neq 2-\sqrt{5}$, $x\neq -\frac{1}{2}+\frac{1}{2}\sqrt{5}$, $x\neq -\frac{1}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{-k}^3 = \frac{\Psi_1}{\left(x^2-4x-1\right)\left(x^2+x-1\right)}$$

where

$\Psi_1 = x^{n+1}(x^2+x-1)L_{-n+1}^3 + x^{n+1}(x-4)(x^2+x-1)L_{-n}^3 - 3(-1)^{n+1}x^{n+1}(x^2+x-1)L_{-n+1} - 3(-1)^n x^{n+1}(x-2)^2 L_{-n} - 3(-1)^n x^{n+1}(x^2-4x-1)L_{-n-1} - (x^3+24x^2-23x-8)$

and

if $\left(x^2-4x-1\right)\left(x^2+x-1\right)=0$, *i.e.*, $x=2+\sqrt{5}$ *or* $x=2-\sqrt{5}$ *or* $x=-\frac{1}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=-\frac{1}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k L_{-k}^3 = \frac{\Psi_2}{\left(4x^3-9x^2-12x+3\right)}$$

where

$\Psi_2 = x^n(n(x^2+x-1)+3x^2+2x-1)L_{-n+1}^3 + x^n(n(x-4)(x^2+x-1)+4x^3-9x^2-10x+4)L_{-n}^3 - 3(-1)^{n+1}x^n(n(x^2+x-1)+3x^2+2x-1)L_{-n+1} - 3(-1)^n x^n(x-2)(nx-2n+3x-2)L_{-n} - 3(-1)^n x^n(n(x^2-4x-1)+3x^2-8x-1)L_{-n-1} - (3x^2+48x-23)$.

(e) $(m=-2,\ j=0)$

If $\left(x^2-3x+1\right)\left(x^2-18x+1\right)\neq 0$, *i.e.*, $x\neq 9+4\sqrt{5}$, $x\neq 9-4\sqrt{5}$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{-2k}^3 = \frac{\Psi_1}{\left(x^2-3x+1\right)\left(x^2-18x+1\right)}$$

where

$\Psi_1 = x^{n+1}(x^2-3x+1)L_{-2n+2}^3 + x^{n+1}(x-18)(x^2-3x+1)L_{-2n}^3 - 3x^{n+1}\left(x^2-3x+1\right)L_{-2n+2} + 9x^{n+1}(x^2-18x+6) L_{-2n} - 3x^{n+1}(x^2-18x+1)L_{-2n-2} - (27x^3-224x^2+141x-8)$

and

if $\left(x^2-3x+1\right)\left(x^2-18x+1\right)=0$, *i.e.*, $x=9+4\sqrt{5}$ *or* $x=9-4\sqrt{5}$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k L_{-2k}^3 = \frac{\Psi_2}{\left(4x^3-63x^2+112x-21\right)}$$

where

$\Psi_2 = x^n(n(x^2-3x+1)+3x^2-6x+1)L_{-2n+2}^3 + x^n(n(x-18)(x^2-3x+1)+4x^3-63x^2+110x-18)L_{-2n}^3 - 3x^n(n(x^2-3x+1)+3x^2-6x+1)L_{-2n+2} + 9x^n(n(x^2-18x+6)+3x^2-36x+6)L_{-2n} - 3x^n(n(x^2-18x+1)+3x^2-36x+1)L_{-2n-2} - (81x^2-448x+141)$.

(f) $(m=-2,\ j=1)$

If $\left(x^2-3x+1\right)\left(x^2-18x+1\right)\neq 0$, *i.e.*, $x\neq 9+4\sqrt{5}$, $x\neq 9-4\sqrt{5}$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{-2k+1}^3 = \frac{\Psi_1}{\left(x^2-3x+1\right)\left(x^2-18x+1\right)}$$

where

$\Psi_1 = x^{n+1}(x^2-3x+1)L_{-2n+3}^3 + x^{n+1}(x-18)(x^2-3x+1)L_{-2n+1}^3 + 3x^{n+1}(x^2-3x+1)L_{-2n+3} - 9x^{n+1}(x^2-18x+6)L_{-2n+1} + 3x^{n+1}(x^2-18x+1)L_{-2n-1} - (64x^3-13x^2+22x-1)$

and

if $\left(x^2-3x+1\right)\left(x^2-18x+1\right)=0$, *i.e.*, $x=9+4\sqrt{5}$ *or* $x=9-4\sqrt{5}$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k L_{-2k+1}^3 = \frac{\Psi_2}{\left(4x^3-63x^2+112x-21\right)}$$

where

$\Psi_2 = x^n(n(x^2-3x+1)+3x^2-6x+1)L_{-2n+3}^3 + x^n(n(x-18)(x^2-3x+1)+4x^3-63x^2+110x-18)L_{-2n+1}^3 + 3x^n (n(x^2-3x+1)+3x^2-6x+1)L_{-2n+3} - 9x^n(n(x^2-18x+6)+3x^2-36x+6)L_{-2n+1} + 3x^n(n(x^2-18x+1)+3x^2-36x+1)L_{-2n-1} - 2(96x^2-13x+11)$.

Taking $x = 1$ in the last two corollaries we get the following corollary.

Corollary 2.3.
For $n \geq 0$, Fibonacci numbers and Lucas numbers have the following properties:

1.

(a) $\sum_{k=0}^{n} F_k^3 = \frac{1}{20}(25F_n^3 + 5F_{n-1}^3 + 3(-1)^n(9F_n - F_{n-1} - 4F_{n+1}) + 10)$.
(b) $\sum_{k=0}^{n} F_{2k}^3 = \frac{1}{80}(85F_{2n}^3 - 5F_{2n-2}^3 - 48F_{2n+2} + 99F_{2n} - 3F_{2n-2} + 40)$.
(c) $\sum_{k=0}^{n} F_{2k+1}^3 = \frac{1}{80}(85F_{2n+1}^3 - 5F_{2n-1}^3 + 48F_{2n+3} - 99F_{2n+1} + 3F_{2n-1})$.
(d) $\sum_{k=0}^{n} F_{-k}^3 = \frac{1}{20}(-5F_{-n+1}^3 + 15F_{-n}^3 + 3(-1)^n(F_{-n+1} - F_{-n} + 4F_{-n-1}) - 10)$.
(e) $\sum_{k=0}^{n} F_{-2k}^3 = \frac{1}{80}(-5F_{-2n+2}^3 + 85F_{-2n}^3 - 3F_{-2n+2} + 99F_{-2n} - 48F_{-2n-2} - 40)$.
(f) $\sum_{k=0}^{n} F_{-2k+1}^3 = \frac{1}{80}(-5F_{-2n+3}^3 + 85F_{-2n+1}^3 + 3F_{-2n+3} - 99F_{-2n+1} + 48F_{-2n-1} + 80)$.

2.

(a) $\sum_{k=0}^{n} L_k^3 = \frac{1}{4}(5L_n^3 + L_{n-1}^3 + 3(-1)^n(4L_{n+1} - 9L_n + L_{n-1}) + 38)$.
(b) $\sum_{k=0}^{n} L_{2k}^3 = \frac{1}{16}(17L_{2n}^3 - L_{2n-2}^3 + 48L_{2n+2} - 99L_{2n} + 3L_{2n-2} + 64)$.
(c) $\sum_{k=0}^{n} L_{2k+1}^3 = \frac{1}{16}(17L_{2n+1}^3 - L_{2n-1}^3 - 48L_{2n+3} + 99L_{2n+1} - 3L_{2n-1} + 88)$.
(d) $\sum_{k=0}^{n} L_{-k}^3 = \frac{1}{4}(-L_{-n+1}^3 + 3L_{-n}^3 + 3(-1)^n(-L_{-n+1} + L_{-n} - 4L_{-n-1}) - 6)$.
(e) $\sum_{k=0}^{n} L_{-2k}^3 = \frac{1}{16}(-L_{-2n+2}^3 + 17L_{-2n}^3 + 3L_{-2n+2} - 99L_{-2n} + 48L_{-2n-2} + 64)$.
(f) $\sum_{k=0}^{n} L_{-2k+1}^3 = \frac{1}{16}(-L_{-2n+3}^3 + 17L_{-2n+1}^3 - 3L_{-2n+3} + 99L_{-2n+1} - 48L_{-2n-1} - 72)$.

2.2. The Case $r = 2, s = 1$: Generalized Pell Numbers

The following theorem presents sum formulas of generalized Pell numbers (the case $r = 2, s = 1$).

Theorem 2.4.
Let x be a real (or complex) number. For all integers m and j, for generalized Pell numbers (the case $r = 2, s = 1$) we have the following sum formulas:

(a) *If $((-1)^{3m}x^2 - xQ_{3m} + 1)((-1)^{3m}x^2 - x(-1)^m Q_m + 1) \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_1}{8((-1)^{3m}x^2 - xQ_{3m} + 1)((-1)^{3m}x^2 - x(-1)^m Q_m + 1)} \tag{16}$$

where

$\Psi_1 = 8x^{n+1}(-1)^{3m}((-1)^{3m}x^2 - (-1)^m xQ_m + 1)W_{mn-m+j}^3 + 8x^{n+1}((-1)^{3m}x - Q_{3m})((-1)^{3m}x^2 - (-1)^m xQ_m + 1)W_{mn+j}^3 - 8(-1)^{3m}x((-1)^{3m}x^2 - (-1)^m xQ_m + 1)W_{j-m}^3 + 8((-1)^{3m}x^2 - (-1)^m xQ_m + 1) W_j^3 + 3x^n(-1)^{mn+m+j}x((-1)^{3m}x^2 - xQ_{3m} + 1)(W_1^2 - W_0^2 - 2W_0W_1)W_{mn+m+j} + 3x^n(-1)^{mn+2m+j}x((-1)^{3m}x^2 - (-1)^m xQ_m + 1)(W_1^2 - W_0^2 - 2W_0W_1)W_{mn-m+j} - 3x^n(-1)^{mn+j}x((-1)^{4m}x^2 Q_m - ((-1)^m xQ_m - 1)Q_{3m})(W_1^2 - W_0^2 - 2W_0W_1)W_{mn+j} - 3x(-1)^{m+j}((-1)^{3m}x^2 - xQ_{3m} + 1)(W_1^2 - W_0^2 - 2W_0W_1)W_{m+j} - 3x(-1)^{2m+j}((-1)^{3m}x^2 - (-1)^m xQ_m + 1)(W_1^2 - W_0^2 - 2W_0W_1)W_{j-m} + 3x(-1)^j((-1)^{4m}x^2 Q_m - Q_{3m}((-1)^m xQ_m - 1))(W_1^2 - W_0^2 - 2W_0W_1)W_j$.

(b) *If $((-1)^{3m}x^2 - xQ_{3m} + 1)((-1)^{3m}x^2 - x(-1)^m Q_m + 1) = u(x-a)(x-b)(x-c)(x-d) = 0$ for some $u, a, b, c, d \in \mathbb{C}$ with $u \neq 0$ and $a \neq b \neq c \neq d$, i.e., $x = a$ or $x = b$ or $x = c$ or $x = d$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_2}{\Lambda_1}$$

where

$\Psi_2 = 8(-1)^{3m}x^n((-1)^{3m}x^2(n+3) - x(-1)^m(n+2)Q_m + n + 1)W_{mn-m+j}^3 + 8((-1)^{6m}(n+4)x^3 - (-1)^{3m}((-1)^m Q_m + Q_{3m})(n+3)x^2 + (-1)^m(Q_mQ_{3m} + (-1)^{2m})(n+2)x - (n+1)Q_{3m})x^n W_{mn+j}^3 + 8(-1)^{3m}(-3(-1)^{3m}x^2 + 2(-1)^m xQ_m - 1)W_{j-m}^3 + 8(2(-1)^{3m}x - (-1)^m Q_m)W_j^3 + 3(-1)^{mn+m+j}((-1)^{3m}(n+3)x^2 - x(n+2)Q_{3m} + n + 1)(W_1^2 - W_0^2 - 2W_0W_1)x^n W_{mn+m+j} + 3(-1)^{mn+2m+j}((-1)^{3m}(n+3)x^2 - x(-1)^m(n+2)Q_m + n + 1)x^n(W_1^2 - W_0^2 - 2W_0W_1)W_{mn-m+j} + 3(-1)^{mn+j}(-(-1)^{4m}(n+3)x^2 Q_m + x(-1)^m(n+2)Q_{3m}Q_m - (n+1)Q_{3m})x^n(W_1^2 - W_0^2 - 2W_0W_1) W_{mn+j} + 3(-1)^{m+j}(-3(-1)^{3m}x^2 + 2xQ_{3m} - 1)(W_1^2 - W_0^2 - 2W_0W_1)W_{m+j} + 3(-1)^{2m+j}(-3(-1)^{3m}x^2 + 2(-1)^m xQ_m - 1)(W_1^2 - W_0^2 - 2W_0W_1)W_{j-m} + 3(-1)^j(3(-1)^{4m}x^2 Q_m - 2(-1)^m xQ_mQ_{3m} + Q_{3m})(W_1^2 - W_0^2 - 2W_0W_1)W_j$

and

$\Lambda_1 = 8(4(-1)^{6m}x^3 - 3(-1)^{3m}((-1)^m Q_m + Q_{3m})x^2 + 2(-1)^m(2(-1)^{2m} + Q_mQ_{3m})x - ((-1)^m Q_m + Q_{3m}))$.

(c) *If* $((-1)^{3m}x^2 - xQ_{3m} + 1)((-1)^{3m}x^2 - x(-1)^m Q_m + 1) = u(x-a)^2(x-b)(x-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$*, i.e.,* $x = a$ *or* $x = b$ *or* $x = c$*, then if* $x = b$ *or* $x = c$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_3}{\Lambda_2}$$

where

$\Psi_3 = 8(-1)^{3m}x^n((-1)^{3m}x^2(n+3) - x(-1)^m(n+2)Q_m + n+1)W_{mn-m+j}^3 + 8((-1)^{6m}(n+4)x^3 - (-1)^{3m}((-1)^m Q_m + Q_{3m})(n+3)x^2 + (-1)^m(Q_m Q_{3m} + (-1)^{2m})(n+2)x - (n+1)Q_{3m})x^n W_{mn+j}^3 + 8(-1)^{3m}(-3(-1)^{3m}x^2 + 2(-1)^m xQ_m - 1)W_{j-m}^3 + 8(2(-1)^{3m}x - (-1)^m Q_m)W_j^3 + 3(-1)^{mn+m+j}((-1)^{3m}(n+3)x^2 - x(n+2)Q_{3m} + n+1)(W_1^2 - W_0^2 - 2W_0W_1)x^n W_{mn+m+j} + 3(-1)^{mn+2m+j}((-1)^{3m}(n+3)x^2 - x(-1)^m(n+2)Q_m + n+1)x^n(W_1^2 - W_0^2 - 2W_0W_1)W_{mn-m+j} + 3(-1)^{mn+j}(-(-1)^{4m}(n+3)x^2 Q_m + x(-1)^m(n+2)Q_{3m}Q_m - (n+1)Q_{3m})x^n(W_1^2 - W_0^2 - 2W_0W_1) W_{mn+j} + 3(-1)^{m+j}(-3(-1)^{3m}x^2 + 2xQ_{3m} - 1)(W_1^2 - W_0^2 - 2W_0W_1)W_{m+j} + 3(-1)^{2m+j}(-3(-1)^{3m}x^2 + 2(-1)^m xQ_m - 1)(W_1^2 - W_0^2 - 2W_0W_1)W_{j-m} + 3(-1)^j(3(-1)^{4m}x^2 Q_m - 2(-1)^m xQ_m Q_{3m} + Q_{3m})(W_1^2 - W_0^2 - 2W_0W_1)W_j$

and

$\Lambda_2 = 8(4(-1)^{6m}x^3 - 3(-1)^{3m}((-1)^m Q_m + Q_{3m})x^2 + 2(-1)^m(2(-1)^{2m} + Q_m Q_{3m})x - ((-1)^m Q_m + Q_{3m}))$

and if $x = a$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_4}{16(-1)^m(6(-1)^{5m}x^2 - 3x(-1)^{2m}((-1)^m Q_m + Q_{3m}) + 2(-1)^{2m} + Q_m Q_{3m})}$$

where

$\Psi_4 = 8(-1)^{3m}((-1)^{3m}(n+3)(n+2)x^2 - (-1)^m x(n+2)(n+1)Q_m + n(n+1))x^{n-1}W_{mn-m+j}^3 + 8((-1)^{6m}(n+4)(n+3) x^3 - (-1)^{3m}(n+3)(n+2)((-1)^m Q_m + Q_{3m})x^2 + x(-1)^m(n+2)(n+1)(Q_m Q_{3m} + (-1)^{2m}) - n(n+1)Q_{3m})x^{n-1}W_{mn+j}^3 + 16(-1)^{4m}(Q_m - 3(-1)^{2m}x)W_{j-m}^3 + 16(-1)^{3m}W_j^3 + 3(-1)^{mn+m+j}((-1)^{3m}(n+3)(n+2)x^2 - x(n+2)(n+1)Q_{3m} + n(n+1))(W_1^2 - W_0^2 - 2W_0W_1)x^{n-1}W_{mn+m+j} + 3x^{n-1}(-1)^{mn+2m+j}((-1)^{3m}(n+3)(n+2)x^2 - x(-1)^m(n+2)(n+1)Q_m + n(n+1))(W_1^2 - W_0^2 - 2W_0W_1)W_{mn-m+j} + 3x^{n-1}(-1)^{mn+j}(-x^2(-1)^{4m}(n+3)(n+2)Q_m + x(-1)^m(n+2)(n+1)Q_{3m}Q_m - n(n+1)Q_{3m})(W_1^2 - W_0^2 - 2W_0W_1)W_{mn+j} + 6(-1)^{m+j}(Q_{3m} - 3(-1)^{3m}x)(W_1^2 - W_0^2 - 2W_0W_1)W_{m+j} + 6(-1)^{3m+j}(Q_m - 3(-1)^{2m}x)(W_1^2 - W_0^2 - 2W_0W_1)W_{j-m} + 6(-1)^{m+j}(3(-1)^{3m}x - Q_{3m})Q_m(W_1^2 - W_0^2 - 2W_0W_1)W_j.$

(d) *If* $((-1)^{3m}x^2 - xQ_{3m} + 1)((-1)^{3m}x^2 - x(-1)^m Q_m + 1) = u(x-a)^3(x-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$*, i.e.,* $x = a$ *or* $x = b$*, then if* $x = b$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_5}{\Lambda_3}$$

where

$\Psi_5 = 8(-1)^{3m}x^n((-1)^{3m}x^2(n+3) - x(-1)^m(n+2)Q_m + n+1)W_{mn-m+j}^3 + 8((-1)^{6m}(n+4)x^3 - (-1)^{3m}((-1)^m Q_m + Q_{3m})(n+3)x^2 + (-1)^m(Q_m Q_{3m} + (-1)^{2m})(n+2)x - (n+1)Q_{3m})x^n W_{mn+j}^3 + 8(-1)^{3m}(-3(-1)^{3m}x^2 + 2(-1)^m xQ_m - 1)W_{j-m}^3 + 8(2(-1)^{3m}x - (-1)^m Q_m)W_j^3 + 3(-1)^{mn+m+j}((-1)^{3m}(n+3)x^2 - x(n+2)Q_{3m} + n+1)(W_1^2 - W_0^2 - 2W_0W_1)x^n W_{mn+m+j} + 3(-1)^{mn+2m+j}((-1)^{3m}(n+3)x^2 - x(-1)^m(n+2)Q_m + n+1)x^n(W_1^2 - W_0^2 - 2W_0W_1)W_{mn-m+j} + 3(-1)^{mn+j}(-(-1)^{4m}(n+3)x^2 Q_m + x(-1)^m(n+2)Q_{3m}Q_m - (n+1)Q_{3m})x^n(W_1^2 - W_0^2 - 2W_0W_1) W_{mn+j} + 3(-1)^{m+j}(-3(-1)^{3m}x^2 + 2xQ_{3m} - 1)(W_1^2 - W_0^2 - 2W_0W_1)W_{m+j} + 3(-1)^{2m+j}(-3(-1)^{3m}x^2 + 2(-1)^m xQ_m - 1)(W_1^2 - W_0^2 - 2W_0W_1)W_{j-m} + 3(-1)^j(3(-1)^{4m}x^2 Q_m - 2(-1)^m xQ_m Q_{3m} + Q_{3m})(W_1^2 - W_0^2 - 2W_0W_1)W_j$

and

$\Lambda_3 = 8(4(-1)^{6m}x^3 - 3(-1)^{3m}((-1)^m Q_m + Q_{3m})x^2 + 2(-1)^m(2(-1)^{2m} + Q_m Q_{3m})x - ((-1)^m Q_m + Q_{3m}))$

and if $x = a$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_6}{48(-1)^{3m}(4(-1)^{3m}x - (-1)^m Q_m - Q_{3m})}$$

where

$\Psi_6 = 8(-1)^{3m}(n+1)((-1)^{3m}(n+3)(n+2)x^2 - x(-1)^m n(n+2)Q_m + n(n-1))x^{n-2}W_{mn-m+j}^3 + 8((-1)^{6m}(n+3)(n+2)(n+4)x^3 - (-1)^{3m}(n+3)(n+2)(n+1)((-1)^m Q_m + Q_{3m})x^2 + (-1)^m n(n+2)(n+1)(Q_m Q_{3m} + (-1)^{2m})x - n(n-1)(n+1)Q_{3m})x^{n-2}W_{mn+j}^3 - 48(-1)^{6m}W_{j-m}^3 + 3(-1)^{mn+m+j}(n+1)((-1)^{3m}(n+3)(n+2)x^2 - xn(n+2)Q_{3m} + n(n-1))(W_1^2 - W_0^2 - 2W_0W_1)x^{n-2}W_{mn+m+j} + 3(-1)^{mn+2m+j}(n+1)((-1)^{3m}(n+3)(n+2)x^2 - x(-1)^m n(n+2)Q_m + n(n-1))(W_1^2 - W_0^2 - 2W_0W_1)x^{n-2}W_{mn-m+j} + 3(-1)^{mn+j}(n+1)(-x^2(-1)^{4m}(n+3)(n+2)Q_m + x(-1)^m n(n+2)Q_{3m}Q_m - n(n-1)Q_{3m})(W_1^2 - W_0^2 - 2W_0W_1)x^{n-2}W_{mn+j} - 18(-1)^{4m+j}(W_1^2 - W_0^2 - 2W_0W_1)W_{m+j} - 18(-1)^{5m+j}(W_1^2 - W_0^2 - 2W_0W_1)W_{j-m} + 18(-1)^{4m+j}Q_m(W_1^2 - W_0^2 - 2W_0W_1)W_j.$

(e) *If* $((-1)^{3m}x^2 - xQ_{3m}+1)((-1)^{3m}x^2 - x(-1)^m Q_m + 1) = u(x-a)^4 = 0$ *for some* $u, a \in \mathbb{C}, u \neq 0$ *i.e.,* $x = a$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_7}{192(-1)^{6m}}$$

where

$\Psi_7 = 8(-1)^{3m}n(n+1)((-1)^{3m}(n+3)(n+2)x^2 - x(-1)^m(n-1)(n+2)Q_m + (n-1)(n-2))x^{n-3}W_{mn-m+j}^3 + 8(n+1)(x^3(-1)^{6m}(n+4)(n+3)(n+2) - x^2(-1)^{3m}n(n+3)(n+2)((-1)^mQ_m + Q_{3m}) + x(-1)^m n(n-1)(n+2)(Q_mQ_{3m} + (-1)^{2m}) - n(n-1)(n-2)Q_{3m})x^{n-3}W_{mn+j}^3 + 3(-1)^{mn+m+j}n(n+1)(x^2(-1)^{3m}(n+3)(n+2) - x(n+2)(n-1)Q_{3m} + (n-1)(n-2))(W_1^2 - W_0^2 - 2W_0W_1)x^{n-3}W_{mn+m+j} + 3(-1)^{mn+2m+j}n(n+1)(x^2(-1)^{3m}(n+3)(n+2) - x(-1)^m(n+2)(n-1)Q_m + (n-1)(n-2))(W_1^2 - W_0^2 - 2W_0W_1)x^{n-3}W_{mn-m+j} + 3(-1)^{mn+j}n(n+1)(-x^2(-1)^{4m}(n+3)(n+2)Q_m + x(-1)^m(n+2)(n-1)Q_{3m}Q_m - (n-1)(n-2)Q_{3m})(W_1^2 - W_0^2 - 2W_0W_1)x^{n-3}W_{mn+j}.$

Proof. Take $r=2, s=1$ and $H_n = Q_n$ in Theorem 2.1. □

Note that (16) can be written in the following form:

$$\sum_{k=1}^{n} x^k W_{mk+j}^2 = \frac{\Psi_8}{8((-1)^{3m}x^2 - xQ_{3m}+1)((-1)^{3m}x^2 - x(-1)^mQ_m+1)}$$

where

$\Psi_8 = 8x^{n+1}(-1)^{3m}((-1)^{3m}x^2 - (-1)^m xQ_m + 1)W_{mn-m+j}^3 + 8x^{n+1}((-1)^{3m}x - Q_{3m})((-1)^{3m}x^2 - (-1)^m xQ_m + 1) W_{mn+j}^3 - 8(-1)^{3m}x((-1)^{3m}x^2 - (-1)^m xQ_m + 1)W_{j-m}^3 + 8(Q_{3m} - (-1)^{3m}x)((-1)^{3m}x^2 - (-1)^m xQ_m + 1)xW_j^3 + 3x^n(-1)^{mn+m+j}x((-1)^{3m}x^2 - xQ_{3m}+1)(W_1^2 - W_0^2 - 2W_0W_1)W_{mn+m+j} + 3x^n(-1)^{mn+2m+j}x((-1)^{3m}x^2 - (-1)^m xQ_m + 1)(W_1^2 - W_0^2 - 2W_0W_1)W_{mn-m+j} - 3x^n(-1)^{mn+j}x((-1)^{4m}x^2Q_m - ((-1)^m xQ_m - 1)Q_{3m})(W_1^2 - W_0^2 - 2W_0W_1) W_{mn+j} - 3x(-1)^{m+j}((-1)^{3m}x^2 - xQ_{3m}+1)(W_1^2 - W_0^2 - 2W_0W_1)W_{m+j} - 3x(-1)^{2m+j}((-1)^{3m}x^2 - (-1)^m xQ_m + 1)(W_1^2 - W_0^2 - 2W_0W_1)W_{j-m} + 3x(-1)^j((-1)^{4m}x^2Q_m - Q_{3m}((-1)^m xQ_m - 1))(W_1^2 - W_0^2 - 2W_0W_1)W_j.$

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 2.2.
For generalized Pell numbers (the case $r=2, s=1$*) we have the following sum formulas for* $n \geq 0$*:*

(a) $(m=1,\ j=0)$

If $(x^2-2x-1)(x^2+14x-1) \neq 0$, *i.e.,* $x \neq -7+5\sqrt{2}$, $x \neq -7-5\sqrt{2}$, $x \neq 1+\sqrt{2}$, $x \neq 1-\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k^3 = \frac{\Psi_1}{8(x^2-2x-1)(x^2+14x-1)}$$

where

$\Psi_1 = 8x^{n+1}(x+14)(x^2-2x-1)W_n^3 + 8x^{n+1}(x^2-2x-1)W_{n-1}^3 + 3(-1)^n x^{n+1}(x^2+14x-1)(W_1^2 - W_0^2 - 2W_0W_1)W_{n+1} + 6(-1)^{n+1}x^{n+1}(x^2+14x+7)(W_1^2 - W_0^2 - 2W_0W_1)W_n + 3(-1)^{n+1}x^{n+1}(x^2-2x-1)(W_1^2 - W_0^2 - 2W_0W_1)W_{n-1} + 8(-x(x^2+4x-1)W_1^3 + (8x^3 - 29x^2 - 12x + 1)W_0^3 + 6x^2(x+2)W_1^2W_0 - 6x^2(2x-1)W_0^2W_1)$

and

if $(x^2-2x-1)(x^2+14x-1) = 0$, *i.e.,* $x = -7+5\sqrt{2}$ *or* $x = -7-5\sqrt{2}$ *or* $x = 1+\sqrt{2}$ *or* $x = 1-\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_k^3 = \frac{\Psi_2}{32(x^3+9x^2-15x-3)}$$

where

$\Psi_2 = 8x^n(n(x+14)(x^2-2x-1) + 4x^3 + 36x^2 - 58x - 14)W_n^3 + 8x^n(n(x^2-2x-1) + 3x^2 - 4x - 1)W_{n-1}^3 + 3(-1)^n x^n(n(x^2+14x-1) + 3x^2 + 28x - 1)(W_1^2 - W_0^2 - 2W_0W_1)W_{n+1} + 6(-1)^{n+1}x^n(n(x^2+14x+7) + 3x^2 + 28x + 7)(W_1^2 - W_0^2 - 2W_0W_1) W_n + 3(-1)^{n+1}x^n(n(x^2-2x-1) + 3x^2 - 4x - 1)(W_1^2 - W_0^2 - 2W_0W_1)W_{n-1} + 8(-(3x^2+8x-1)W_1^3 + 2(12x^2 - 29x - 6)W_0^3 + 6x(3x+4)W_1^2W_0 - 12x(3x-1)W_0^2W_1).$

(b) $(m=2,\ j=0)$

If $(x^2-6x+1)(x^2-198x+1) \neq 0$, *i.e.,* $x \neq 99+70\sqrt{2}$, $x \neq 99-70\sqrt{2}$, $x \neq 3+2\sqrt{2}$, $x \neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k}^3 = \frac{\Psi_1}{8(x^2-6x+1)(x^2-198x+1)}$$

where

$\Psi_1 = 8x^{n+1}(x-198)(x^2-6x+1)W_{2n}^3 + 8x^{n+1}(x^2-6x+1)W_{2n-2}^3 + 3x^{n+1}(x^2-198x+1)(W_1^2-W_0^2-2W_0W_1)W_{2n+2} - 3x^{n+1}(6x^2-1188x+198)(W_1^2-W_0^2-2W_0W_1)W_{2n} + 3x^{n+1}(x^2-6x+1)(W_1^2-W_0^2-2W_0W_1)W_{2n-2} + 8(8x(x^2+12x+1)W_1^3 - (125x^3-1111x^2+203x-1)W_0^3 - 12x(5x^2+24x-1)W_0W_1^2 + 6x(25x^2-54x+1)W_0^2W_1)$

and

if $(x^2-6x+1)(x^2-198x+1)=0$, *i.e.,* $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{2k}^3 = \frac{\Psi_2}{32\left(x^3-153x^2+595x-51\right)}$$

where

$\Psi_2 = 8x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)W_{2n}^3 + 8x^n(n(x^2-6x+1)+3x^2-12x+1)W_{2n-2}^3 + 3x^n(n(x^2-198x+1)+3x^2-396x+1)(W_1^2-W_0^2-2W_0W_1)W_{2n+2} - 18x^n(n(x^2-198x+33)+3x^2-396x+33)(W_1^2-W_0^2-2W_0W_1)W_{2n} + 3x^n(n(x^2-6x+1)+3x^2-12x+1)(W_1^2-W_0^2-2W_0W_1)W_{2n-2} + 8(8(3x^2+24x+1)W_1^3 - (375x^2-2222x+203)W_0^3 - 12(15x^2+48x-1)W_1^2W_0 + 6(75x^2-108x+1)W_0^2W_1)$.

(c) $(m=2,\ j=1)$

If $(x^2-6x+1)(x^2-198x+1)\neq 0$, *i.e.,* $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^3 = \frac{\Psi_1}{8\left(x^2-6x+1\right)\left(x^2-198x+1\right)}$$

where

$\Psi_1 = 8x^{n+1}(x-198)(x^2-6x+1)W_{2n+1}^3 + 8x^{n+1}(x^2-6x+1)W_{2n-1}^3 - 3x^{n+1}(x^2-198x+1)(W_1^2-W_0^2-2W_0W_1)W_{2n+3} + 18x^{n+1}(x^2-198x+33)(W_1^2-W_0^2-2W_0W_1)W_{2n+1} - 3x^{n+1}(x^2-6x+1)(W_1^2-W_0^2-2W_0W_1)W_{2n-1} + 8(-(x-1)(x^2-78x+1)W_1^3 + 8x(x^2+12x+1)W_0^3 + 6x(x^2-54x+25)W_0W_1^2 - 12x(x^2-24x-5)W_0^2W_1)$

and

if $(x^2-6x+1)(x^2-198x+1)=0$, *i.e.,* $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^3 = \frac{\Psi_2}{32\left(x^3-153x^2+595x-51\right)}$$

where

$\Psi_2 = 8x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)W_{2n+1}^3 + 8x^n(n(x^2-6x+1)+3x^2-12x+1)W_{2n-1}^3 - 3x^n(n(x^2-198x+1)+3x^2-396x+1)(W_1^2-W_0^2-2W_0W_1)W_{2n+3} + 18x^n(n(x^2-198x+33)+3x^2-396x+33)(W_1^2-W_0^2-2W_0W_1)W_{2n+1} - 3x^n(n(x^2-6x+1)+3x^2-12x+1)(W_1^2-W_0^2-2W_0W_1)W_{2n-1} + 8(-(3x^2-158x+79)W_1^3 + 8(3x^2+24x+1)W_0^3 + 6\left((3x^2-108x+25)\right)W_0W_1^2 - 12(3x^2-48x-5)W_0^2W_1)$.

(d) $(m=-1,\ j=0)$

If $(x^2-14x-1)(x^2+2x-1)\neq 0$, *i.e.,* $x\neq -1+\sqrt{2}$, $x\neq -1-\sqrt{2}$, $x\neq 7+5\sqrt{2}$, $x\neq 7-5\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k}^3 = \frac{\Psi_1}{8\left(x^2-14x-1\right)\left(x^2+2x-1\right)}$$

where

$\Psi_1 = 8x^{n+1}(x^2+2x-1)W_{-n+1}^3 + 8x^{n+1}(x-14)(x^2+2x-1)W_{-n}^3 + 3(-1)^{n+1}x^{n+1}(x^2+2x-1)(W_1^2-W_0^2-2W_0W_1)W_{-n+1} + 3(-1)^n x^{n+1}\left(2x^2-28x+14\right)(W_1^2-W_0^2-2W_0W_1)W_{-n} + 3(-1)^n x^{n+1}(x^2-14x-1)(W_1^2-W_0^2-2W_0W_1)W_{-n-1} - 8(x(x^2-4x-1)W_1^3 + (x^2-4x-1)W_0^3 + 6x(2x+1)W_1^2W_0 + 6x(x-2)W_0^2W_1)$

and

if $(x^2-14x-1)(x^2+2x-1)=0$, *i.e.,* $x=-1+\sqrt{2}$ *or* $x=-1-\sqrt{2}$ *or* $x=7+5\sqrt{2}$ *or* $x=7-5\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{-k}^3 = \frac{\Psi_2}{32\left(x^3-9x^2-15x+3\right)}$$

where

$\Psi_2 = 8x^n(n(x^2+2x-1)+3x^2+4x-1)W_{-n+1}^3 + 8x^n(n(x-14)(x^2+2x-1)+4x^3-36x^2-58x+14)W_{-n}^3 + 3(-1)^{n+1}x^n(n(x^2+2x-1)+3x^2+4x-1)(W_1^2-W_0^2-2W_0W_1)W_{-n+1} + 6(-1)^n x^n(n(x^2-14x+7)+3x^2-28x+7)(W_1^2-W_0^2-2W_0W_1)W_{-n} + 3(-1)^n x^n(n(x^2-14x-1)+3x^2-28x-1)(W_1^2-W_0^2-2W_0W_1)W_{-n-1} - 8((3x^2-8x-1)W_1^3 + 2(x-2)W_0^3 + 6(4x+1)W_1^2W_0 + 12(x-1)W_0^2W_1)$.

(e) $(m=-2,\ j=0)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.,* $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^3 = \frac{\Psi_1}{8\left(x^2-6x+1\right)\left(x^2-198x+1\right)}$$

where

$\Psi_1 = 8x^{n+1}(x^2-6x+1)W_{-2n+2}^3+8x^{n+1}(x-198)(x^2-6x+1)W_{-2n}^3+3x^{n+1}(x^2-6x+1)(W_1^2-W_0^2-2W_0W_1)W_{-2n+2}-3x^{n+1}(6x^2-1188x+198)(W_1^2-W_0^2-2W_0W_1)W_{-2n}+3x^{n+1}(x^2-198x+1)(W_1^2-W_0^2-2W_0W_1)W_{-2n-2}-8(8x(x^2+12x+1)W_1^3+(x-1)(x^2-78x+1)W_0^3+12x(x^2-24x-5)W_1^2W_0+6x(x^2-54x+25)W_0^2W_1)$

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.,* $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^3 = \frac{\Psi_2}{32\left(x^3-153x^2+595x-51\right)}$$

where

$\Psi_2 = 8x^n(n(x^2-6x+1)+3x^2-12x+1)W_{-2n+2}^3+8x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)W_{-2n}^3+3x^n(n(x^2-6x+1)+3x^2-12x+1)(W_1^2-W_0^2-2W_0W_1)W_{-2n+2}-18x^n(n(x^2-198x+33)+3x^2-396x+33)(W_1^2-W_0^2-2W_0W_1)W_{-2n}+3x^n(n(x^2-198x+1)+3x^2-396x+1)(W_1^2-W_0^2-2W_0W_1)W_{-2n-2}-8(8(3x^2+24x+1)W_1^3+(3x^2-158x+79)W_0^3+12(3x^2-48x-5)W_1^2W_0+6(3x^2-108x+25)W_0^2W_1)$.

(f) $(m=-2,\ j=1)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.,* $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^3 = \frac{\Psi_1}{8\left(x^2-6x+1\right)\left(x^2-198x+1\right)}$$

where

$\Psi_1 = 8x^{n+1}(x^2-6x+1)W_{-2n+3}^3+8x^{n+1}(x-198)(x^2-6x+1)W_{-2n+1}^3-3x^{n+1}(x^2-6x+1)(W_1^2-W_0^2-2W_0W_1)W_{-2n+3}+18x^{n+1}(x^2-198x+33)(W_1^2-W_0^2-2W_0W_1)W_{-2n+1}-3x^{n+1}(x^2-198x+1)(W_1^2-W_0^2-2W_0W_1)W_{-2n-1}-8((125x^3-1111x^2+203x-1)W_1^3+8x(x^2+12x+1)W_0^3+6x(25x^2-54x+1)W_1^2W_0+12x(5x^2+24x-1)W_0^2W_1)$

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.,* $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^3 = \frac{\Psi_2}{32\left(x^3-153x^2+595x-51\right)}$$

where

$\Psi_2 = 8x^n(n(x^2-6x+1)+3x^2-12x+1)W_{-2n+3}^3+8x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)W_{-2n+1}^3-3x^n(n(x^2-6x+1)+3x^2-12x+1)(W_1^2-W_0^2-2W_0W_1)W_{-2n+3}+18x^n(n(x^2-198x+33)+3x^2-396x+33)(W_1^2-W_0^2-2W_0W_1)W_{-2n+1}-3x^n(n(x^2-198x+1)+3x^2-396x+1)(W_1^2-W_0^2-2W_0W_1)W_{-2n-1}-8((375x^2-2222x+203)W_1^3+8(3x^2+24x+1)W_0^3+6(75x^2-108x+1)W_1^2W_0+12(15x^2+48x-1)W_0^2W_1)$.

From the above proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0=0, P_1=1$).

Corollary 2.4.
For $n\geq 0$, *Pell numbers have the following properties:*

(a) $(m=1,\ j=0)$

If $\left(x^2-2x-1\right)\left(x^2+14x-1\right)\neq 0$, *i.e.,* $x\neq -7+5\sqrt{2}$, $x\neq -7-5\sqrt{2}$, $x\neq 1+\sqrt{2}$, $x\neq 1-\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_k^3 = \frac{\Psi_1}{8\left(x^2-2x-1\right)\left(x^2+14x-1\right)}$$

where

$\Psi_1 = 8x^{n+1}(x+14)(x^2-2x-1)P_n^3+8x^{n+1}(x^2-2x-1)P_{n-1}^3+3(-1)^n x^{n+1}(x^2+14x-1)P_{n+1}+6(-1)^{n+1}x^{n+1}(x^2+14x+7)P_n+3(-1)^{n+1}x^{n+1}(x^2-2x-1)P_{n-1}-8x(x^2+4x-1)$

and

if $\left(x^2-2x-1\right)\left(x^2+14x-1\right)=0$, *i.e.*, $x=-7+5\sqrt{2}$ *or* $x=-7-5\sqrt{2}$ *or* $x=1+\sqrt{2}$ *or* $x=1-\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k P_k^3 = \frac{\Psi_2}{32\left(x^3+9x^2-15x-3\right)}$$

where

$\Psi_2 = 8x^n(n(x+14)(x^2-2x-1)+4x^3+36x^2-58x-14)P_n^3+8x^n(n(x^2-2x-1)+3x^2-4x-1)P_{n-1}^3+3(-1)^n x^n(n(x^2+14x-1)+3x^2+28x-1)P_{n+1}+6(-1)^{n+1}x^n(n(x^2+14x+7)+3x^2+28x+7)P_n+3(-1)^{n+1}x^n(n(x^2-2x-1)+3x^2-4x-1)P_{n-1}-8(3x^2+8x-1).$

(b) $(m=2,\ j=0)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.*, $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{2k}^3 = \frac{\Psi_1}{8\left(x^2-6x+1\right)\left(x^2-198x+1\right)}$$

where

$\Psi_1 = 8x^{n+1}(x-198)(x^2-6x+1)P_{2n}^3+8x^{n+1}(x^2-6x+1)P_{2n-2}^3+3x^{n+1}(x^2-198x+1)P_{2n+2}-3x^{n+1}(6x^2-1188x+198)P_{2n}+3x^{n+1}(x^2-6x+1)P_{2n-2}+64x(x^2+12x+1)$

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.*, $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k P_{2k}^3 = \frac{\Psi_2}{32\left(x^3-153x^2+595x-51\right)}$$

where

$\Psi_2 = 8x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)P_{2n}^3+8x^n(n(x^2-6x+1)+3x^2-12x+1)P_{2n-2}^3+3x^n(n(x^2-198x+1)+3x^2-396x+1)P_{2n+2}-18x^n(n(x^2-198x+33)+3x^2-396x+33)P_{2n}+3x^n(n(x^2-6x+1)+3x^2-12x+1)P_{2n-2}+64(3x^2+24x+1).$

(c) $(m=2,\ j=1)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.*, $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{2k+1}^3 = \frac{\Psi_1}{8\left(x^2-6x+1\right)\left(x^2-198x+1\right)}$$

where

$\Psi_1 = 8x^{n+1}(x-198)(x^2-6x+1)P_{2n+1}^3+8x^{n+1}(x^2-6x+1)P_{2n-1}^3-3x^{n+1}(x^2-198x+1)P_{2n+3}+18x^{n+1}(x^2-198x+33)P_{2n+1}-3x^{n+1}(x^2-6x+1)P_{2n-1}-8(x-1)(x^2-78x+1)$

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.*, $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k P_{2k+1}^3 = \frac{\Psi_2}{32\left(x^3-153x^2+595x-51\right)}$$

where

$\Psi_2 = 8x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)P_{2n+1}^3+8x^n(n(x^2-6x+1)+3x^2-12x+1)P_{2n-1}^3-3x^n(n(x^2-198x+1)+3x^2-396x+1)P_{2n+3}+18x^n(n(x^2-198x+33)+3x^2-396x+33)P_{2n+1}-3x^n(n(x^2-6x+1)+3x^2-12x+1)P_{2n-1}-8(3x^2-158x+79).$

(d) $(m=-1,\ j=0)$

If $\left(x^2-14x-1\right)\left(x^2+2x-1\right)\neq 0$, *i.e.*, $x\neq -1+\sqrt{2}$, $x\neq -1-\sqrt{2}$, $x\neq 7+5\sqrt{2}$, $x\neq 7-5\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{-k}^3 = \frac{\Psi_1}{8\left(x^2-14x-1\right)\left(x^2+2x-1\right)}$$

where

$\Psi_1 = 8x^{n+1}(x^2+2x-1)P_{-n+1}^3+8x^{n+1}(x-14)(x^2+2x-1)P_{-n}^3+3(-1)^{n+1}x^{n+1}(x^2+2x-1)P_{-n+1}+3(-1)^n x^{n+1}\left(2x^2-28x+14\right)P_{-n}+3(-1)^n x^{n+1}(x^2-14x-1)P_{-n-1}-8x(x^2-4x-1)$

and

if $\left(x^2-14x-1\right)\left(x^2+2x-1\right)=0$, *i.e.*, $x=-1+\sqrt{2}$ *or* $x=-1-\sqrt{2}$ *or* $x=7+5\sqrt{2}$ *or* $x=7-5\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k P_{-k}^3=\frac{\Psi_2}{32\left(x^3-9x^2-15x+3\right)}$$

where

$\Psi_2 = 8x^n(n(x^2+2x-1)+3x^2+4x-1)P_{-n+1}^3+8x^n(n(x-14)(x^2+2x-1)+4x^3-36x^2-58x+14)P_{-n}^3+3(-1)^{n+1}x^n(n(x^2+2x-1)+3x^2+4x-1)P_{-n+1}+6(-1)^n x^n(n(x^2-14x+7)+3x^2-28x+7)P_{-n}+3(-1)^n x^n(n(x^2-14x-1)+3x^2-28x-1)P_{-n-1}-8(3x^2-8x-1)$.

(e) $(m=-2,\ j=0)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.*, $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{-2k}^3=\frac{\Psi_1}{8\left(x^2-6x+1\right)\left(x^2-198x+1\right)}$$

where

$\Psi_1 = 8x^{n+1}(x^2-6x+1)P_{-2n+2}^3+8x^{n+1}(x-198)(x^2-6x+1)P_{-2n}^3+3x^{n+1}(x^2-6x+1)P_{-2n+2}-3x^{n+1}(6x^2-1188x+198)P_{-2n}+3x^{n+1}(x^2-198x+1)P_{-2n-2}-64x(x^2+12x+1)$

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.*, $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k P_{-2k}^3=\frac{\Psi_2}{32\left(x^3-153x^2+595x-51\right)}$$

where

$\Psi_2 = 8x^n(n(x^2-6x+1)+3x^2-12x+1)P_{-2n+2}^3+8x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)P_{-2n}^3+3x^n(n(x^2-6x+1)+3x^2-12x+1)P_{-2n+2}-18x^n(n(x^2-198x+33)+3x^2-396x+33)P_{-2n}+3x^n(n(x^2-198x+1)+3x^2-396x+1)P_{-2n-2}-64(3x^2+24x+1)$.

(f) $(m=-2,\ j=1)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.*, $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{-2k+1}^3=\frac{\Psi_1}{8\left(x^2-6x+1\right)\left(x^2-198x+1\right)}$$

where

$\Psi_1 = 8x^{n+1}(x^2-6x+1)P_{-2n+3}^3+8x^{n+1}(x-198)(x^2-6x+1)P_{-2n+1}^3-3x^{n+1}(x^2-6x+1)P_{-2n+3}+18x^{n+1}(x^2-198x+33)P_{-2n+1}-3x^{n+1}(x^2-198x+1)P_{-2n-1}-8(125x^3-1111x^2+203x-1)$

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.*, $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k P_{-2k+1}^3=\frac{\Psi_2}{32\left(x^3-153x^2+595x-51\right)}$$

where

$\Psi_2 = 8x^n(n(x^2-6x+1)+3x^2-12x+1)P_{-2n+3}^3+8x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)P_{-2n+1}^3-3x^n(n(x^2-6x+1)+3x^2-12x+1)P_{-2n+3}+18x^n(n(x^2-198x+33)+3x^2-396x+33)P_{-2n+1}-3x^n(n(x^2-198x+1)+3x^2-396x+1)P_{-2n-1}-8(375x^2-2222x+203)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 2.5.

For $n\geq 0$, *Pell-Lucas numbers have the following properties:*

(a) $(m=1,\ j=0)$

If $\left(x^2-2x-1\right)\left(x^2+14x-1\right)\neq 0$, *i.e.*, $x\neq -7+5\sqrt{2}$, $x\neq -7-5\sqrt{2}$, $x\neq 1+\sqrt{2}$, $x\neq 1-\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_k^3=\frac{\Psi_1}{\left(x^2-2x-1\right)\left(x^2+14x-1\right)}$$

where

$\Psi_1=x^{n+1}(x+14)(x^2-2x-1)Q_n^3+x^{n+1}(x^2-2x-1)Q_{n-1}^3-3(-1)^n x^{n+1}(x^2+14x-1)Q_{n+1}-6(-1)^{n+1}x^{n+1}(x^2+14x+7)Q_n-3(-1)^{n+1}x^{n+1}(x^2-2x-1)Q_{n-1}+8(x^3-15x^2-11x+1)$

and

if $\left(x^2-2x-1\right)\left(x^2+14x-1\right)=0$, *i.e.*, $x=-7+5\sqrt{2}$ *or* $x=-7-5\sqrt{2}$ *or* $x=1+\sqrt{2}$ *or* $x=1-\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k Q_k^3=\frac{\Psi_2}{4\left(x^3+9x^2-15x-3\right)}$$

where

$\Psi_2=x^n(n(x+14)(x^2-2x-1)+4x^3+36x^2-58x-14)Q_n^3+x^n(n(x^2-2x-1)+3x^2-4x-1)Q_{n-1}^3-3(-1)^n x^n(n(x^2+14x-1)+3x^2+28x-1)Q_{n+1}-6(-1)^{n+1}x^n(n(x^2+14x+7)+3x^2+28x+7)Q_n-3(-1)^{n+1}x^n(n(x^2-2x-1)+3x^2-4x-1)Q_{n-1}+8(3x^2-30x-11)$.

(b) $(m=2,\ j=0)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.*, $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{2k}^3=\frac{\Psi_1}{\left(x^2-6x+1\right)\left(x^2-198x+1\right)}$$

where

$\Psi_1=x^{n+1}(x-198)(x^2-6x+1)Q_{2n}^3+x^{n+1}(x^2-6x+1)Q_{2n-2}^3-3x^{n+1}(x^2-198x+1)Q_{2n+2}+3x^{n+1}(6x^2-1188x+198)Q_{2n}-3x^{n+1}(x^2-6x+1)Q_{2n-2}-8(27x^3-595x^2+177x-1)$

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.*, $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k Q_{2k}^3=\frac{\Psi_2}{4\left(x^3-153x^2+595x-51\right)}$$

where

$\Psi_2=x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)Q_{2n}^3+x^n(n(x^2-6x+1)+3x^2-12x+1)Q_{2n-2}^3-3x^n(n(x^2-198x+1)+3x^2-396x+1)Q_{2n+2}+18x^n(n(x^2-198x+33)+3x^2-396x+33)Q_{2n}-3x^n(n(x^2-6x+1)+3x^2-12x+1)Q_{2n-2}-8(81x^2-1190x+177)$.

(c) $(m=2,\ j=1)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.*, $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{2k+1}^3=\frac{\Psi_1}{\left(x^2-6x+1\right)\left(x^2-198x+1\right)}$$

where

$\Psi_1=x^{n+1}(x-198)(x^2-6x+1)Q_{2n+1}^3+x^{n+1}(x^2-6x+1)Q_{2n-1}^3+3x^{n+1}(x^2-198x+1)Q_{2n+3}-18x^{n+1}(x^2-198x+33)Q_{2n+1}+3x^{n+1}(x^2-6x+1)Q_{2n-1}+8(x+1)(x^2+138x+1)$

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.*, $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k Q_{2k+1}^3=\frac{\Psi_2}{4\left(x^3-153x^2+595x-51\right)}$$

where

$\Psi_2=x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)Q_{2n+1}^3+x^n(n(x^2-6x+1)+3x^2-12x+1)Q_{2n-1}^3+3x^n(n(x^2-198x+1)+3x^2-396x+1)Q_{2n+3}-18x^n(n(x^2-198x+33)+3x^2-396x+33)Q_{2n+1}+3x^n(n(x^2-6x+1)+3x^2-12x+1)Q_{2n-1}+8(3x^2+278x+139)$.

(d) $(m=-1,\ j=0)$

If $\left(x^2-14x-1\right)\left(x^2+2x-1\right)\neq 0$, *i.e.,* $x\neq -1+\sqrt{2}$, $x\neq -1-\sqrt{2}$, $x\neq 7+5\sqrt{2}$, $x\neq 7-5\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{-k}^3=\frac{\Psi_1}{\left(x^2-14x-1\right)\left(x^2+2x-1\right)}$$

where

$\Psi_1 = x^{n+1}(x^2+2x-1)Q_{-n+1}^3 + x^{n+1}(x-14)(x^2+2x-1)Q_{-n}^3 - 3(-1)^{n+1}x^{n+1}(x^2+2x-1)Q_{-n+1} - 3(-1)^n x^{n+1}\left(2x^2-28x+14\right)Q_{-n} - 3(-1)^n x^{n+1}(x^2-14x-1)Q_{-n-1} - 8(x^3+15x^2-11x-1)$

and

if $\left(x^2-14x-1\right)\left(x^2+2x-1\right)=0$, *i.e.,* $x=-1+\sqrt{2}$ *or* $x=-1-\sqrt{2}$ *or* $x=7+5\sqrt{2}$ *or* $x=7-5\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k Q_{-k}^3=\frac{\Psi_2}{4\left(x^3-9x^2-15x+3\right)}$$

where

$\Psi_2 = x^n(n(x^2+2x-1)+3x^2+4x-1)Q_{-n+1}^3 + x^n(n(x-14)(x^2+2x-1)+4x^3-36x^2-58x+14)Q_{-n}^3 - 3(-1)^{n+1}x^n(n(x^2+2x-1)+3x^2+4x-1)Q_{-n+1} - 6(-1)^n x^n(n(x^2-14x+7)+3x^2-28x+7)Q_{-n} - 3(-1)^n x^n(n(x^2-14x-1)+3x^2-28x-1)Q_{-n-1} - 8(3x^2+30x-11).$

(e) $(m=-2,\ j=0)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.,* $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{-2k}^3=\frac{\Psi_1}{\left(x^2-6x+1\right)\left(x^2-198x+1\right)}$$

where

$\Psi_1 = x^{n+1}(x^2-6x+1)Q_{-2n+2}^3 + x^{n+1}(x-198)(x^2-6x+1)Q_{-2n}^3 - 3x^{n+1}(x^2-6x+1)Q_{-2n+2} + 3x^{n+1}(6x^2-1188x+198)Q_{-2n} - 3x^{n+1}(x^2-198x+1)Q_{-2n-2} - 8(27x^3-595x^2+177x-1)$

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.,* $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k Q_{-2k}^3=\frac{\Psi_2}{4\left(x^3-153x^2+595x-51\right)}$$

where

$\Psi_2 = x^n(n(x^2-6x+1)+3x^2-12x+1)Q_{-2n+2}^3 + x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)Q_{-2n}^3 - 3x^n(n(x^2-6x+1)+3x^2-12x+1)Q_{-2n+2} + 18x^n(n(x^2-198x+33)+3x^2-396x+33)Q_{-2n} - 3x^n(n(x^2-198x+1)+3x^2-396x+1)Q_{-2n-2} - 8(81x^2-1190x+177).$

(f) $(m=-2,\ j=1)$

If $\left(x^2-6x+1\right)\left(x^2-198x+1\right)\neq 0$, *i.e.,* $x\neq 99+70\sqrt{2}$, $x\neq 99-70\sqrt{2}$, $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{-2k+1}^3=\frac{\Psi_1}{\left(x^2-6x+1\right)\left(x^2-198x+1\right)}$$

where

$\Psi_1 = x^{n+1}(x^2-6x+1)Q_{-2n+3}^3 + x^{n+1}(x-198)(x^2-6x+1)Q_{-2n+1}^3 + 3x^{n+1}(x^2-6x+1)Q_{-2n+3} - 18x^{n+1}(x^2-198x+33)Q_{-2n+1} + 3x^{n+1}(x^2-198x+1)Q_{-2n-1} - 8(343x^3-1051x^2+205x-1)$

and

if $\left(x^2-6x+1\right)\left(x^2-198x+1\right)=0$, *i.e.,* $x=99+70\sqrt{2}$ *or* $x=99-70\sqrt{2}$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k Q_{-2k+1}^3=\frac{\Psi_2}{4\left(x^3-153x^2+595x-51\right)}$$

where

$\Psi_2 = x^n(n(x^2-6x+1)+3x^2-12x+1)Q_{-2n+3}^3 + x^n(n(x-198)(x^2-6x+1)+4x^3-612x^2+2378x-198)Q_{-2n+1}^3 + 3x^n(n(x^2-6x+1)+3x^2-12x+1)Q_{-2n+3} - 18x^n(n(x^2-198x+33)+3x^2-396x+33)Q_{-2n+1} + 3x^n(n(x^2-198x+1)+3x^2-396x+1)Q_{-2n-1} - 8(1029x^2-2102x+205).$

Taking $x = 1$ in the last two corollaries we get the following corollary.

Corollary 2.6.
For $n \geq 0$, Pell numbers and Pell-Lucas numbers have the following properties:

1.

(a) $\sum_{k=0}^{n} P_k^3 = \frac{1}{112}(120P_n^3 + 8P_{n-1}^3 + 3(-1)^n(-7P_{n+1} + 22P_n - P_{n-1}) + 16)$.
(b) $\sum_{k=0}^{n} P_{2k}^3 = \frac{1}{1568}(1576P_{2n}^3 - 8P_{2n-2}^3 - 147P_{2n+2} + 738P_{2n} - 3P_{2n-2} + 224)$.
(c) $\sum_{k=0}^{n} P_{2k+1}^3 = \frac{1}{1568}(1576P_{2n+1}^3 - 8P_{2n-1}^3 + 147P_{2n+3} - 738P_{2n+1} + 3P_{2n-1})$.
(d) $\sum_{k=0}^{n} P_{-k}^3 = \frac{1}{112}(-8P_{-n+1}^3 + 104P_{-n}^3 + 3(-1)^n(P_{-n+1} + 6P_{-n} + 7P_{-n-1}) - 16)$.
(e) $\sum_{k=0}^{n} P_{-2k}^3 = \frac{1}{1568}(1576P_{-2n}^3 - 8P_{-2n+2}^3 - 3P_{-2n+2} + 738P_{-2n} - 147P_{-2n-2} - 224)$.
(f) $\sum_{k=0}^{n} P_{-2k+1}^3 = \frac{1}{1568}(-8P_{-2n+3}^3 + 1576P_{-2n+1}^3 + 3P_{-2n+3} - 738P_{-2n+1} + 147P_{-2n-1} + 1568)$.

2.

(a) $\sum_{k=0}^{n} Q_k^3 = \frac{1}{14}(15Q_n^3 + Q_{n-1}^3 + 3(-1)^n(7Q_{n+1} - 22Q_n + Q_{n-1}) + 96)$.
(b) $\sum_{k=0}^{n} Q_{2k}^3 = \frac{1}{196}(197Q_{2n}^3 - Q_{2n-2}^3 + 147Q_{2n+2} - 738Q_{2n} + 3Q_{2n-2} + 784)$.
(c) $\sum_{k=0}^{n} Q_{2k+1}^3 = \frac{1}{196}((197Q_{2n+1}^3 - Q_{2n-1}^3 - 147Q_{2n+3} + 738Q_{2n+1} - 3Q_{2n-1} + 560)$.
(d) $\sum_{k=0}^{n} Q_{-k}^3 = \frac{1}{14}(-Q_{-n+1}^3 + 13Q_{-n}^3 - 3(-1)^n((Q_{-n+1} + 6Q_{-n} + 7Q_{-n-1}) + 16)$.
(e) $\sum_{k=0}^{n} Q_{-2k}^3 = \frac{1}{196}(-Q_{-2n+2}^3 + 197Q_{-2n}^3 + 3Q_{-2n+2} - 738Q_{-2n} + 147Q_{-2n-2} + 784)$.
(f) $\sum_{k=0}^{n} Q_{-2k+1}^3 = \frac{1}{196}(-Q_{-2n+3}^3 + 197Q_{-2n+1}^3 - 3Q_{-2n+3} + 738Q_{-2n+1} - 147Q_{-2n-1} + 1008)$.

2.3. The Case $r = 1, s = 2$: Generalized Jacobsthal Numbers

The following theorem presents sum formulas of generalized Jacobsthal numbers (the case $r = 1, s = 2$).

Theorem 2.5.
Let x be a real (or complex) number. For all integers m and j, for generalized Jacobsthal numbers (the case $r = 1, s = 2$) we have the following sum formulas:

(a) *If* $((-2)^{3m}x^2 - xj_{3m} + 1)((-2)^{3m}x^2 - x(-2)^m j_m + 1) \neq 0$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_1}{9((-2)^{3m}x^2 - xj_{3m} + 1)((-2)^{3m}x^2 - x(-2)^m j_m + 1)} \tag{17}$$

where

$\Psi_1 = 9x^{n+1}(-2)^{3m}((-2)^{3m}x^2 - (-2)^m xj_m + 1)W_{mn-m+j}^3 + 9x^{n+1}((-2)^{3m}x - j_{3m})((-2)^{3m}x^2 - (-2)^m xj_m + 1)W_{mn+j}^3 - 9(-2)^{3m}x((-2)^{3m}x^2 - (-2)^m xj_m + 1)W_{j-m}^3 + 9((-2)^{3m}x^2 - (-2)^m xj_m + 1)W_j^3 + 3x^n(-2)^{mn+m+j}x((-2)^{3m}x^2 - xj_{3m} + 1)(W_1^2 - 2W_0^2 - W_0W_1)W_{mn+m+j} + 3x^n(-2)^{mn+2m+j}x((-2)^{3m}x^2 - (-2)^m xj_m + 1)(W_1^2 - 2W_0^2 - W_0W_1)W_{mn-m+j} - 3x^n(-2)^{mn+j}x((-2)^{4m}x^2 j_m - ((-2)^m xj_m - 1)j_{3m})(W_1^2 - 2W_0^2 - W_0W_1)W_{mn+j} - 3x(-2)^{m+j}((-2)^{3m}x^2 - xj_{3m} + 1)(W_1^2 - 2W_0^2 - W_0W_1)W_{m+j} - 3x(-2)^{2m+j}((-2)^{3m}x^2 - (-2)^m xj_m + 1)(W_1^2 - 2W_0^2 - W_0W_1)W_{j-m} + 3x(-2)^j((-2)^{4m}x^2 j_m - j_{3m}((-2)^m xj_m - 1))(W_1^2 - 2W_0^2 - W_0W_1)W_j$.

(b) *If* $((-2)^{3m}x^2 - xj_{3m} + 1)((-2)^{3m}x^2 - x(-2)^m j_m + 1) = u(x-a)(x-b)(x-c)(x-d) = 0$ *for some* $u, a, b, c, d \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c \neq d$, *i.e.,* $x = a$ *or* $x = b$ *or* $x = c$ *or* $x = d$, *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_2}{\Lambda_1}$$

where

$\Psi_2 = 9(-2)^{3m}x^n((-2)^{3m}x^2(n+3) - x(-2)^m(n+2)j_m + n + 1)W_{mn-m+j}^3 + 9((-2)^{6m}(n+4)x^3 - (-2)^{3m}((-2)^m j_m + j_{3m})(n+3)x^2 + (-2)^m(j_m j_{3m} + (-2)^{2m})(n+2)x - (n+1)j_{3m})x^n W_{mn+j}^3 + 9(-2)^{3m}(-3(-2)^{3m}x^2 + 2(-2)^m xj_m - 1)W_{j-m}^3 + 9(2(-2)^{3m}x - (-2)^m j_m)W_j^3 + 3(-2)^{mn+m+j}((-2)^{3m}(n+3)x^2 - x(n+2)j_{3m} + n + 1)(W_1^2 - 2W_0^2 - W_0W_1)x^n W_{mn+m+j} + 3(-2)^{mn+2m+j}((-2)^{3m}(n+3)x^2 - x(-2)^m(n+2)j_m + n + 1)x^n(W_1^2 - 2W_0^2 - W_0W_1)W_{mn-m+j} + 3(-2)^{mn+j}(-(-2)^{4m}(n+3)x^2 j_m + x(-2)^m(n+2)j_{3m}j_m - (n+1)j_{3m})x^n(W_1^2 - 2W_0^2 - W_0W_1)W_{mn+j} + 3(-2)^{m+j}(-3(-2)^{3m}x^2 + 2xj_{3m} - 1)(W_1^2 - 2W_0^2 - W_0W_1)W_{m+j} + 3(-2)^{2m+j}(-3(-2)^{3m}x^2 + 2(-2)^m xj_m - 1)(W_1^2 - 2W_0^2 - W_0W_1)W_{j-m} + 3(-2)^j(3(-2)^{4m}x^2 j_m - 2(-2)^m xj_m j_{3m} + j_{3m})(W_1^2 - 2W_0^2 - W_0W_1)W_j$

and

$\Lambda_1 = 9(4(-2)^{6m}x^3 - 3(-2)^{3m}((-2)^m j_m + j_{3m})x^2 + 2(-2)^m(2(-2)^{2m} + j_m j_{3m})x - ((-2)^m j_m + j_{3m}))$.

(c) *If* $((-2)^{3m}x^2 - xj_{3m}+1)((-2)^{3m}x^2 - x(-2)^m j_m + 1) = u(x-a)^2(x-b)(x-c) = 0$ *for some* $u,a,b,c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$*, i.e.,* $x = a$ *or* $x = b$ *or* $x = c$*, then if* $x = b$ *or* $x = c$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_3}{\Lambda_2}$$

where

$\Psi_3 = 9(-2)^{3m}x^n((-2)^{3m}x^2(n+3) - x(-2)^m(n+2)j_m + n+1)W_{mn-m+j}^3 + 9((-2)^{6m}(n+4)x^3 - (-2)^{3m}((-2)^m j_m + j_{3m})(n+3)x^2 + (-2)^m(j_m j_{3m} + (-2)^{2m})(n+2)x - (n+1)j_{3m})x^n W_{mn+j}^3 + 9(-2)^{3m}(-3(-2)^{3m}x^2 + 2(-2)^m x j_m - 1)W_{j-m}^3 + 9(2(-2)^{3m}x - (-2)^m j_m)W_j^3 + 3(-2)^{mn+m+j}((-2)^{3m}(n+3)x^2 - x(n+2)j_{3m} + n+1)(W_1^2 - 2W_0^2 - W_0W_1)x^n W_{mn+m+j} + 3(-2)^{mn+2m+j}((-2)^{3m}(n+3)x^2 - x(-2)^m(n+2)j_m + n+1)x^n(W_1^2 - 2W_0^2 - W_0W_1)W_{mn-m+j} + 3(-2)^{mn+j}(-(-2)^{4m}(n+3)x^2 j_m + x(-2)^m(n+2)j_{3m}j_m - (n+1)j_{3m})x^n(W_1^2 - 2W_0^2 - W_0W_1)W_{mn+j} + 3(-2)^{m+j}(-3(-2)^{3m}x^2 + 2xj_{3m} - 1)(W_1^2 - 2W_0^2 - W_0W_1)W_{m+j} + 3(-2)^{2m+j}(-3(-2)^{3m}x^2 + 2(-2)^m x j_m - 1)(W_1^2 - 2W_0^2 - W_0W_1)W_{j-m} + 3(-2)^j(3(-2)^{4m}x^2 j_m - 2(-2)^m x j_m j_{3m} + j_{3m})(W_1^2 - 2W_0^2 - W_0W_1)W_j$

and

$\Lambda_2 = 9(4(-2)^{6m}x^3 - 3(-2)^{3m}((-2)^m j_m + j_{3m})x^2 + 2(-2)^m(2(-2)^{2m} + j_m j_{3m})x - ((-2)^m j_m + j_{3m}))$

and if $x = a$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_4}{18(-2)^m(6(-2)^{5m}x^2 - 3x(-2)^{2m}((-2)^m j_m + j_{3m}) + 2(-2)^{2m} + j_m j_{3m})}$$

where

$\Psi_4 = 9(-2)^{3m}((-2)^{3m}(n+3)(n+2)x^2 - (-2)^m x(n+2)(n+1)j_m + n(n+1))x^{n-1}W_{mn-m+j}^3 + 9((-2)^{6m}(n+4)(n+3)x^3 - (-2)^{3m}(n+3)(n+2)((-2)^m j_m + j_{3m})x^2 + x(-2)^m(n+2)(n+1)(j_m j_{3m} + (-2)^{2m}) - n(n+1)j_{3m})x^{n-1}W_{mn+j}^3 + 18(-2)^{4m}(j_m - 3(-2)^{2m}x)W_{j-m}^3 + 18(-2)^{3m}W_j^3 + 3(-2)^{mn+m+j}((-2)^{3m}(n+3)(n+2)x^2 - x(n+2)(n+1)j_{3m} + n(n+1))(W_1^2 - 2W_0^2 - W_0W_1)x^{n-1}W_{mn+m+j} + 3x^{n-1}(-2)^{mn+2m+j}((-2)^{3m}(n+3)(n+2)x^2 - x(-2)^m(n+2)(n+1)j_m + n(n+1))(W_1^2 - 2W_0^2 - W_0W_1)W_{mn-m+j} + 3x^{n-1}(-2)^{mn+j}(-x^2(-2)^{4m}(n+3)(n+2)j_m + x(-2)^m(n+2)(n+1)j_{3m}j_m - n(n+1)j_{3m})(W_1^2 - 2W_0^2 - W_0W_1)W_{mn+j} + 6(-2)^{m+j}(j_{3m} - 3(-2)^{3m}x)(W_1^2 - 2W_0^2 - W_0W_1)W_{m+j} + 6(-2)^{3m+j}(j_m - 3(-2)^{2m}x)(W_1^2 - 2W_0^2 - W_0W_1)W_{j-m} + 6(-2)^{m+j}(3(-2)^{3m}x - j_{3m})j_m(W_1^2 - 2W_0^2 - W_0W_1)W_j.$

(d) *If* $((-2)^{3m}x^2 - xj_{3m}+1)((-2)^{3m}x^2 - x(-2)^m j_m + 1) = u(x-a)^3(x-b) = 0$ *for some* $u,a,b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$*, i.e.,* $x = a$ *or* $x = b$*, then if* $x = b$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_5}{\Lambda_3}$$

where

$\Psi_5 = 9(-2)^{3m}x^n((-2)^{3m}x^2(n+3) - x(-2)^m(n+2)j_m + n+1)W_{mn-m+j}^3 + 9((-2)^{6m}(n+4)x^3 - (-2)^{3m}((-2)^m j_m + j_{3m})(n+3)x^2 + (-2)^m(j_m j_{3m} + (-2)^{2m})(n+2)x - (n+1)j_{3m})x^n W_{mn+j}^3 + 9(-2)^{3m}(-3(-2)^{3m}x^2 + 2(-2)^m x j_m - 1)W_{j-m}^3 + 9(2(-2)^{3m}x - (-2)^m j_m)W_j^3 + 3(-2)^{mn+m+j}((-2)^{3m}(n+3)x^2 - x(n+2)j_{3m} + n+1)(W_1^2 - 2W_0^2 - W_0W_1)x^n W_{mn+m+j} + 3(-2)^{mn+2m+j}((-2)^{3m}(n+3)x^2 - x(-2)^m(n+2)j_m + n+1)x^n(W_1^2 - 2W_0^2 - W_0W_1)W_{mn-m+j} + 3(-2)^{mn+j}(-(-2)^{4m}(n+3)x^2 j_m + x(-2)^m(n+2)j_{3m}j_m - (n+1)j_{3m})x^n(W_1^2 - 2W_0^2 - W_0W_1)W_{mn+j} + 3(-2)^{m+j}(-3(-2)^{3m}x^2 + 2xj_{3m} - 1)(W_1^2 - 2W_0^2 - W_0W_1)W_{m+j} + 3(-2)^{2m+j}(-3(-2)^{3m}x^2 + 2(-2)^m x j_m - 1)(W_1^2 - 2W_0^2 - W_0W_1)W_{j-m} + 3(-2)^j(3(-2)^{4m}x^2 j_m - 2(-2)^m x j_m j_{3m} + j_{3m})(W_1^2 - 2W_0^2 - W_0W_1)W_j$

and

$\Lambda_3 = 9(4(-2)^{6m}x^3 - 3(-2)^{3m}((-2)^m j_m + j_{3m})x^2 + 2(-2)^m(2(-2)^{2m} + j_m j_{3m})x - ((-2)^m j_m + j_{3m}))$

and if $x = a$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_6}{54(-2)^{3m}(4(-2)^{3m}x - (-2)^m j_m - j_{3m})}$$

where

$\Psi_6 = 9(-2)^{3m}(n+1)((-2)^{3m}(n+3)(n+2)x^2 - x(-2)^m n(n+2)j_m + n(n-1))x^{n-2}W_{mn-m+j}^3 + 9((-2)^{6m}(n+3)(n+2)(n+4)x^3 - (-2)^{3m}(n+3)(n+2)(n+1)((-2)^m j_m + j_{3m})x^2 + (-2)^m n(n+2)(n+1)(j_m j_{3m} + (-2)^{2m})x - n(n-1)(n+1)j_{3m})x^{n-2}W_{mn+j}^3 - 54(-2)^{6m}W_{j-m}^3 + 3(-2)^{mn+m+j}(n+1)((-2)^{3m}(n+3)(n+2)x^2 - xn(n+2)j_{3m} + n(n-1))(W_1^2 - 2W_0^2 - W_0W_1)x^{n-2}W_{mn+m+j} + 3(-2)^{mn+2m+j}(n+1)((-2)^{3m}(n+3)(n+2)x^2 - x(-2)^m n(n+2)j_m + n(n-1))(W_1^2 - 2W_0^2 - W_0W_1)x^{n-2}W_{mn-m+j} + 3(-2)^{mn+j}(n+1)(-x^2(-2)^{4m}(n+3)(n+2)j_m + x(-2)^m n(n+2)j_{3m}j_m - n(n-1)j_{3m})(W_1^2 - 2W_0^2 - W_0W_1)x^{n-2}W_{mn+j} - 18(-2)^{4m+j}(W_1^2 - 2W_0^2 - W_0W_1)W_{m+j} - 18(-2)^{5m+j}(W_1^2 - 2W_0^2 - W_0W_1)W_{j-m} + 18(-2)^{4m+j}j_m(W_1^2 - 2W_0^2 - W_0W_1)W_j.$

(e) *If* $((-2)^{3m}x^2 - xj_{3m}+1)((-2)^{3m}x^2 - x(-2)^m j_m+1) = u(x-a)^4 = 0$ *for some* $u, a \in \mathbb{C}, u \neq 0$ *i.e.,* $x = a$ *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^3 = \frac{\Psi_7}{216\,(-2)^{6m}}$$

where

$\Psi_7 = 9(-2)^{3m}n(n+1)((-2)^{3m}(n+3)(n+2)x^2 - x(-2)^m(n-1)(n+2)j_m + (n-1)(n-2))x^{n-3}W_{mn-m+j}^3 + 9(n+1)(x^3(-2)^{6m}(n+4)(n+3)(n+2) - x^2(-2)^{3m}n(n+3)(n+2)((-2)^m j_m + j_{3m}) + x(-2)^m n(n-1)(n+2)(j_m j_{3m} + (-2)^{2m}) - n(n-1)(n-2)j_{3m})x^{n-3}W_{mn+j}^3 + 3(-2)^{mn+m+j}n(n+1)(x^2(-2)^{3m}(n+3)(n+2) - x(n+2)(n-1)j_{3m} + (n-1)(n-2))(W_1^2 - 2W_0^2 - W_0W_1)x^{n-3}W_{mn+m+j} + 3(-2)^{mn+2m+j}n(n+1)(x^2(-2)^{3m}(n+3)(n+2) - x(-2)^m(n+2)(n-1)j_m + (n-1)(n-2))(W_1^2 - 2W_0^2 - W_0W_1)x^{n-3}W_{mn-m+j} + 3(-2)^{mn+j}n(n+1)(-x^2(-2)^{4m}(n+3)(n+2)j_m + x(-2)^m(n+2)(n-1)j_{3m}j_m - (n-1)(n-2)j_{3m})(W_1^2 - 2W_0^2 - W_0W_1)x^{n-3}W_{mn+j}.$

Proof. *Take* $r=1, s=2$ and $H_n = j_n$ in Theorem 2.1. □

Note that (17) can be written in the following form:

$$\sum_{k=1}^{n} x^k W_{mk+j}^2 = \frac{\Psi_8}{9((-2)^{3m}x^2 - xj_{3m}+1)((-2)^{3m}x^2 - x(-2)^m j_m+1)}$$

where

$\Psi_8 = 9x^{n+1}(-2)^{3m}((-2)^{3m}x^2 - (-2)^m xj_m + 1)W_{mn-m+j}^3 + 9x^{n+1}((-2)^{3m}x - j_{3m})((-2)^{3m}x^2 - (-2)^m xj_m + 1)W_{mn+j}^3 - 9(-2)^{3m}x((-2)^{3m}x^2 - (-2)^m xj_m + 1)W_{j-m}^3 + 9(j_{3m} - (-2)^{3m}x)((-2)^{3m}x^2 - (-2)^m xj_m + 1)xW_j^3 + 3x^n(-2)^{mn+m+j}x((-2)^{3m}x^2 - xj_{3m}+1)(W_1^2 - 2W_0^2 - W_0W_1)W_{mn+m+j} + 3x^n(-2)^{mn+2m+j}x((-2)^{3m}x^2 - (-2)^m xj_m + 1)(W_1^2 - 2W_0^2 - W_0W_1)W_{mn-m+j} - 3x^n(-2)^{mn+j}x((-2)^{4m}x^2 j_m - ((-2)^m xj_m - 1)j_{3m})(W_1^2 - 2W_0^2 - W_0W_1)W_{mn+j} - 3x(-2)^{m+j}((-2)^{3m}x^2 - xj_{3m}+1)(W_1^2 - 2W_0^2 - W_0W_1)W_{m+j} - 3x(-2)^{2m+j}((-2)^{3m}x^2 - (-2)^m xj_m + 1)(W_1^2 - 2W_0^2 - W_0W_1)W_{j-m} + 3x(-2)^j((-2)^{4m}x^2 j_m - j_{3m}((-2)^m xj_m - 1))(W_1^2 - 2W_0^2 - W_0W_1)W_j.$

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 2.3.
For generalized Jacobsthal numbers (the case $r=1, s=2$*) we have the following sum formulas for* $n \geq 0$*:*

(a) $(m=1,\ j=0)$

If $(4x+1)(2x-1)(x+1)(8x-1) \neq 0$*, i.e.,* $x \neq -1,\ x \neq -\frac{1}{4},\ x \neq \frac{1}{8},\ x \neq \frac{1}{2}$*, then*

$$\sum_{k=0}^{n} x^k W_k^3 = \frac{\Psi_1}{9\,(4x+1)\,(2x-1)\,(x+1)\,(8x-1)}$$

where

$\Psi_1 = 9x^{n+1}(8x+7)(4x+1)(2x-1)W_n^3 + 72x^{n+1}(4x+1)(2x-1)W_{n-1}^3 - 3\,(-2)^{n+1}x^{n+1}(x+1)(8x-1)(W_1^2 - 2W_0^2 - W_0W_1)W_{n+1} - 3\,(-2)^n x^{n+1}\left(14x + 16x^2 + 7\right)(W_1^2 - 2W_0^2 - W_0W_1)W_n - 3\,(-2)^{n+2}x^{n+1}(4x+1)(2x-1)(W_1^2 - 2W_0^2 - W_0W_1)W_{n-1} + 9(-x(8x^2+4x-1)W_1^3 + (8x^3 - 22x^2 - 5x + 1)W_0^3 + 6x^2(4x+1)W_1^2W_0 - 12x^2(2x-1)W_0^2W_1)$

and

if $(4x+1)(2x-1)(x+1)(8x-1) = 0$*, i.e.,* $x=-1$ *or* $x = -\frac{1}{4}$ *or* $x = \frac{1}{8}$ *or* $x = \frac{1}{2}$ *then*

$$\sum_{k=0}^{n} x^k W_k^3 = \frac{\Psi_2}{9\left(256x^3 + 120x^2 - 60x - 5\right)}$$

where

$\Psi_2 = 9x^n(n(8x+7)(2x-1)(4x+1) + 256x^3 + 120x^2 - 44x - 7)W_n^3 + 72x^n(n(4x+1)(2x-1) + 24x^2 - 4x - 1)W_{n-1}^3 - 3\,(-2)^{n+1}x^n\left(n(x+1)(8x-1) + 24x^2 + 14x - 1\right)(W_1^2 - 2W_0^2 - W_0W_1)W_{n+1} - 3\,(-2)^n x^n(n(16x^2 + 14x + 7) + 48x^2 + 28x + 7)(W_1^2 - 2W_0^2 - W_0W_1)W_n - 3\,(-2)^{n+2}x^n(n(4x+1)(2x-1) + 24x^2 - 4x - 1)(W_1^2 - 2W_0^2 - W_0W_1)W_{n-1} + 9(-(24x^2 + 8x - 1)W_1^3 + (24x^2 - 44x - 5)W_0^3 + 12x(6x+1)W_1^2W_0 - 24x(3x-1)W_0^2W_1).$

(b) $(m=2,\ j=0)$

If $(16x-1)(4x-1)(64x-1)(x-1) \neq 0$*, i.e.,* $x \neq \frac{1}{64},\ x \neq \frac{1}{16},\ x \neq \frac{1}{4},\ x \neq 1$*, then*

$$\sum_{k=0}^{n} x^k W_{2k}^3 = \frac{\Psi_1}{9\,(16x-1)\,(4x-1)\,(64x-1)\,(x-1)}$$

where

$\Psi_1 = 9x^{n+1}(64x-65)(16x-1)(4x-1)W_{2n}^3 + 576x^{n+1}(16x-1)(4x-1)W_{2n-2}^3 + 3\times 2^{2n+2}x^{n+1}(64x-1)(x-1)(W_1^2 - 2W_0^2 - W_0W_1)W_{2n+2} - 15\times 2^{2n}x^{n+1}\left(-260x+256x^2+13\right)(W_1^2-2W_0^2-W_0W_1)W_{2n} + 3\times 2^{2n+4}x^{n+1}(16x-1)(4x-1)(W_1^2-2W_0^2-W_0W_1)W_{2n-2} + 9(x(64x^2+40x+1)W_1^3 + (-1728x^3+964x^2-77x+1)W_0^3 - 6x(6x+1)(16x-1)W_1^2W_0 + 12x(36x-1)(4x-1)W_0^2W_1)$

and

if $(16x-1)(4x-1)(64x-1)(x-1) = 0$, *i.e.,* $x = \frac{1}{64}$ *or* $x = \frac{1}{16}$ *or* $x = \frac{1}{4}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k W_{2k}^3 = \frac{\Psi_2}{9\left(16384x^3 - 16320x^2 + 2856x - 85\right)}$$

where

$\Psi_2 = 9x^n(n(64x-65)(16x-1)(4x-1) + 16384x^3 - 16320x^2 + 2728x - 65)W_{2n}^3 + 576x^n(n(16x-1)(4x-1) + 192x^2 - 40x+1)W_{2n-2}^3 + 3\times 2^{2n+2}x^n\left(n(64x-1)(x-1) + 192x^2 - 130x + 1\right)(W_1^2-2W_0^2-W_0W_1)W_{2n+2} - 15\times 2^{2n}x^n(n(256x^2 - 260x+13) + 768x^2 - 520x + 13)(W_1^2-2W_0^2-W_0W_1)W_{2n} + 3\times 2^{2n+4}x^n(n(16x-1)(4x-1) + 192x^2 - 40x + 1)(W_1^2 - 2W_0^2 - W_0W_1)W_{2n-2} + 9((192x^2+80x+1)W_1^3 - (5184x^2 - 1928x + 77)W_0^3 - 6(288x^2+20x-1)W_1^2W_0 + 12(432x^2 - 80x+1)W_0^2W_1)$.

(c) $(m = 2,\ j = 1)$

If $(16x-1)(4x-1)(64x-1)(x-1) \neq 0$, *i.e.,* $x \neq \frac{1}{64}$, $x \neq \frac{1}{16}$, $x \neq \frac{1}{4}$, $x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^3 = \frac{\Psi_1}{9(16x-1)(4x-1)(64x-1)(x-1)}$$

where

$\Psi_1 = 9x^{n+1}(64x-65)(16x-1)(4x-1)W_{2n+1}^3 + 576x^{n+1}(16x-1)(4x-1)W_{2n-1}^3 - 24\times 2^{2n}x^{n+1}(64x-1)(x-1)(W_1^2 - 2W_0^2 - W_0W_1)W_{2n+3} + 30\times 2^{2n}x^{n+1}(256x^2 - 260x+13)(W_1^2 - 2W_0^2 - W_0W_1)W_{2n+1} - 96\times 2^{2n}x^{n+1}(16x-1)(4x-1)(W_1^2-2W_0^2-W_0W_1)W_{2n-1} + 9(-(8x-1)(64x^2-50x+1)W_1^3 + 8x(64x^2+40x+1)W_0^3 + 6x(16x-1)(16x-9)W_1^2W_0 - 12x(32x+3)(4x-1)W_0^2W_1)$

and

if $(16x-1)(4x-1)(64x-1)(x-1) = 0$, *i.e.,* $x = \frac{1}{64}$ *or* $x = \frac{1}{16}$ *or* $x = \frac{1}{4}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^3 = \frac{\Psi_2}{9\left(16384x^3 - 16320x^2 + 2856x - 85\right)}$$

where

$\Psi_2 = 9x^nW_{2n+1}^3(n(64x-65)(16x-1)(4x-1) + 16384x^3 - 16320x^2 + 2728x - 65) + 576x^n(n(16x-1)(4x-1) + 192x^2 - 40x+1)W_{2n-1}^3 - 24\times 2^{2n}x^n(n(64x-1)(x-1) + 192x^2 - 130x+1)(W_1^2 - 2W_0^2 - W_0W_1)W_{2n+3} + 30\times 2^{2n}x^n(n(256x^2 - 260x+13) + 768x^2 - 520x + 13)(W_1^2 - 2W_0^2 - W_0W_1)W_{2n+1} - 96\times 2^{2n}x^n\left(n(16x-1)(4x-1) + 192x^2 - 40x+1\right)(W_1^2 - 2W_0^2 - W_0W_1)W_{2n-1} + 18(-(768x^2 - 464x + 29)W_1^3 + 4(192x^2 + 80x + 1)W_0^3 + 3(768x^2 - 320x + 9)W_1^2W_0 - 6W_0^2W_1(384x^2 - 40x - 3))$.

(d) $(m = -1,\ j = 0)$

If $(x+4)(x-2)(x+1)(x-8) \neq 0$, *i.e.,* $x \neq -4$, $x \neq -1$, $x \neq 2$, $x \neq 8$, *then*

$$\sum_{k=0}^{n} x^k W_{-k}^3 = \frac{\Psi_1}{9(x+4)(x-2)(x+1)(x-8)}$$

where

$\Psi_1 = 9x^{n+1}(x+4)(x-2)W_{-n+1}^3 + 9x^{n+1}(x-7)(x+4)(x-2)W_{-n}^3 - 6\times(-2)^{-n}x^{n+1}(x+4)(x-2)(W_1^2 - 2W_0^2 - W_0W_1)W_{-n+1} + 6\times(-2)^{-n}x^{n+1}\left(-7x + x^2 + 28\right)(W_1^2 - 2W_0^2 - W_0W_1)W_{-n} + 12\times(-2)^{-n}x^{n+1}(x+1)(x-8)(W_1^2 - 2W_0^2 - W_0W_1)W_{-n-1} - 9(x(x^2-4x-8)W_1^3 + 8(x^2-4x-8)W_0^3 + 6x(x+4)W_1^2W_0 + 12x(x-2)W_0^2W_1)$

and

if $(x+4)(x-2)(x+1)(x-8) = 0$, *i.e.,* $x = -4$ *or* $x = -1$ *or* $x = 2$ *or* $x = 8$ *then*

$$\sum_{k=0}^{n} x^k W_{-k}^3 = \frac{\Psi_2}{9\left(4x^3 - 15x^2 - 60x + 40\right)}$$

where

$\Psi_2 = 9x^n(n(x+4)(x-2) + 4x + 3x^2 - 8)W_{-n+1}^3 + 9x^n(n(x-7)(x+4)(x-2) + 4x^3 - 15x^2 - 44x + 56)W_{-n}^3 - 6(-2)^{-n}x^n(n(x+4)(x-2) + 3x^2 + 4x - 8)(W_1^2 - 2W_0^2 - W_0W_1)W_{-n+1} + 6(-2)^{-n}x^n(n(x^2-7x+28) + 3x^2 - 14x + 28)(W_1^2 - 2W_0^2 - W_0W_1)W_{-n} + 12(-2)^{-n}x^n(n(x+1)(x-8) + 3x^2 - 14x - 8)(W_1^2 - 2W_0^2 - W_0W_1)W_{-n-1} - 9((3x^2 - 8x - 8)W_1^3 + 16(x-2)W_0^3 + 12(x+2)W_1^2W_0 + 24(x-1)W_0^2W_1)$.

(e) $(m=-2,\ j=0)$

If $(x-4)(x-16)(x-1)(x-64)\neq 0$, *i.e.,* $x\neq 1,\ x\neq 4,\ x\neq 16,\ x\neq 64$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^3 = \frac{\Psi_1}{9(x-4)(x-16)(x-1)(x-64)}$$

where

$\Psi_1 = 9x^{n+1}(x-4)(x-16)W_{-2n+2}^3 + 9x^{n+1}(x-65)(x-4)(x-16)W_{-2n}^3 + 12\times 2^{-2n}x^{n+1}(x-4)(x-16)(W_1^2-2W_0^2-W_0W_1)W_{-2n+2} - 60\times 2^{-2n}x^{n+1}(x^2-65x+208)(W_1^2-2W_0^2-W_0W_1)W_{-2n} + 48\times 2^{-2n}x^{n+1}(x-1)(x-64)(W_1^2-2W_0^2-W_0W_1)W_{-2n-2} - 9(x(x^2+40x+64)W_1^3 + 8(x-8)(x^2-50x+64)W_0^3 + 6x(x+6)(x-16)W_1^2W_0 + 12x(x-4)(x-36)W_0^2W_1)$

and

if $(x-4)(x-16)(x-1)(x-64)=0$, *i.e.,* $x=1$ *or* $x=4$ *or* $x=16$ *or* $x=64$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^3 = \frac{\Psi_2}{9\left(4x^3-255x^2+2856x-5440\right)}$$

where

$\Psi_2 = 9x^n(n(x-4)(x-16)+3x^2-40x+64)W_{-2n+2}^3 + 9x^n(n(x-65)(x-4)(x-16)+4x^3-255x^2+2728x-4160)W_{-2n}^3 + 12\times 2^{-2n}x^n\left(64n-40x+nx^2-20nx+3x^2+64\right)(W_1^2-2W_0^2-W_0W_1)W_{-2n+2} - 60\times 2^{-2n}x^n(n(x^2-65x+208)+3x^2-130x+208)(W_1^2-2W_0^2-W_0W_1)W_{-2n} + 48\times 2^{-2n}x^n(n(x-1)(x-64)+3x^2-130x+64)(W_1^2-2W_0^2-W_0W_1)W_{-2n-2} - 9((3x^2+80x+64)W_1^3 + 8(3x^2-116x+464)W_0^3 + 6(3x^2-20x-96)W_1^2W_0 + 12(3x^2-80x+144)W_0^2W_1)$.

(f) $(m=-2,\ j=1)$

If $(x-4)(x-16)(x-1)(x-64)\neq 0$, *i.e.,* $x\neq 1,\ x\neq 4,\ x\neq 16,\ x\neq 64$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^3 = \frac{\Psi_1}{9(x-4)(x-16)(x-1)(x-64)}$$

where

$\Psi_1 = 9x^{n+1}(x-4)(x-16)W_{-2n+3}^3 + 9x^{n+1}(x-65)(x-4)(x-16)W_{-2n+1}^3 - 24\times 2^{-2n}x^{n+1}(x-4)(x-16)(W_1^2-2W_0^2-W_0W_1)W_{-2n+3} + 120\times 2^{-2n}x^{n+1}(x^2-65x+208)(W_1^2-2W_0^2-W_0W_1)W_{-2n+1} - 96\times 2^{-2n}x^{n+1}(x-1)(x-64)(W_1^2-2W_0^2-W_0W_1)W_{-2n-1} - 9((27x^3-964x^2+4928x-4096)W_1^3 + 8x(x^2+40x+64)W_0^3 + 6x(9x-16)(x-16)W_1^2W_0 + 12x(3x+32)(x-4)W_0^2W_1)$

and

if $(x-4)(x-16)(x-1)(x-64)=0$, *i.e.,* $x=1$ *or* $x=4$ *or* $x=16$ *or* $x=64$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^3 = \frac{\Psi_2}{9\left(4x^3-255x^2+2856x-5440\right)}$$

where

$\Psi_2 = 9x^n(n(x-4)(x-16)+3x^2-40x+64)W_{-2n+3}^3 + 9x^n(n(x-65)(x-4)(x-16)+4x^3-255x^2+2728x-4160)W_{-2n+1}^3 - 24\times 2^{-2n}x^n(n(x-4)(x-16)+3x^2-40x+64)(W_1^2-2W_0^2-W_0W_1)W_{-2n+3} + 120\times 2^{-2n}x^n(n(x^2-65x+208)+3x^2-130x+208)(W_1^2-2W_0^2-W_0W_1)W_{-2n+1} - 96\times 2^{-2n}x^n\left(n(x-1)(x-64)+3x^2-130x+64\right)(W_1^2-2W_0^2-W_0W_1)W_{-2n-1} - 9((81x^2-1928x+4928)W_1^3 + 8(3x^2+80x+64)W_0^3 + 6(27x^2-320x+256)W_1^2W_0 + 12(9x^2+40x-128)W_0^2W_1)$.

From the above proposition, we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n=J_n$ with $J_0=0, J_1=1$).

Corollary 2.7.
For $n\geq 0$, *Jacobsthal numbers have the following properties:*

(a) $(m=1,\ j=0)$

If $(4x+1)(2x-1)(x+1)(8x-1)\neq 0$, *i.e.,* $x\neq -1,\ x\neq -\frac{1}{4},\ x\neq \frac{1}{8},\ x\neq \frac{1}{2}$, *then*

$$\sum_{k=0}^{n} x^k J_k^3 = \frac{\Psi_1}{9(4x+1)(2x-1)(x+1)(8x-1)}$$

where

$\Psi_1 = 9x^{n+1}(8x+7)(4x+1)(2x-1)J_n^3 + 72x^{n+1}(4x+1)(2x-1)J_{n-1}^3 - 3(-2)^{n+1}x^{n+1}(x+1)(8x-1)J_{n+1} - 3(-2)^n x^{n+1}\left(14x+16x^2+7\right)J_n - 3(-2)^{n+2}x^{n+1}(4x+1)(2x-1)J_{n-1} - 9x(8x^2+4x-1)$

and

if $(4x+1)(2x-1)(x+1)(8x-1)=0$, *i.e.*, $x=-1$ *or* $x=-\frac{1}{4}$ *or* $x=\frac{1}{8}$ *or* $x=\frac{1}{2}$ *then*

$$\sum_{k=0}^{n} x^k J_k^3 = \frac{\Psi_2}{9\left(256x^3+120x^2-60x-5\right)}$$

where

$\Psi_2 = 9x^n(n(8x+7)(2x-1)(4x+1)+256x^3+120x^2-44x-7)J_n^3 + 72x^n(n(4x+1)(2x-1)+24x^2-4x-1)J_{n-1}^3 - 3(-2)^{n+1}x^n\left(n(x+1)(8x-1)+24x^2+14x-1\right)J_{n+1} - 3(-2)^n x^n(n(16x^2+14x+7)+48x^2+28x+7)J_n - 3(-2)^{n+2}x^n(n(4x+1)(2x-1)+24x^2-4x-1)J_{n-1} - 9(24x^2+8x-1)$.

(b) $(m=2,\ j=0)$

If $(16x-1)(4x-1)(64x-1)(x-1) \neq 0$, *i.e.*, $x \neq \frac{1}{64}$, $x \neq \frac{1}{16}$, $x \neq \frac{1}{4}$, $x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k J_{2k}^3 = \frac{\Psi_1}{9(16x-1)(4x-1)(64x-1)(x-1)}$$

where

$\Psi_1 = 9x^{n+1}(64x-65)(16x-1)(4x-1)J_{2n}^3 + 576x^{n+1}(16x-1)(4x-1)J_{2n-2}^3 + 3\times 2^{2n+2}x^{n+1}(64x-1)(x-1)J_{2n+2} - 15\times 2^{2n}x^{n+1}\left(-260x+256x^2+13\right)J_{2n} + 3\times 2^{2n+4}x^{n+1}(16x-1)(4x-1)J_{2n-2} + 9x(64x^2+40x+1)$

and

if $(16x-1)(4x-1)(64x-1)(x-1)=0$, *i.e.*, $x=\frac{1}{64}$ *or* $x=\frac{1}{16}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k J_{2k}^3 = \frac{\Psi_2}{9\left(16384x^3-16320x^2+2856x-85\right)}$$

where

$\Psi_2 = 9x^n(n(64x-65)(16x-1)(4x-1)+16384x^3-16320x^2+2728x-65)J_{2n}^3 + 576x^n(n(16x-1)(4x-1)+192x^2-40x+1)J_{2n-2}^3 + 3\times 2^{2n+2}x^n\left(n(64x-1)(x-1)+192x^2-130x+1\right)J_{2n+2} - 15\times 2^{2n}x^n(n(256x^2-260x+13)+768x^2-520x+13)J_{2n} + 3\times 2^{2n+4}x^n(n(16x-1)(4x-1)+192x^2-40x+1)J_{2n-2} + 9(192x^2+80x+1)$.

(c) $(m=2,\ j=1)$

If $(16x-1)(4x-1)(64x-1)(x-1) \neq 0$, *i.e.*, $x \neq \frac{1}{64}$, $x \neq \frac{1}{16}$, $x \neq \frac{1}{4}$, $x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k J_{2k+1}^3 = \frac{\Psi_1}{9(16x-1)(4x-1)(64x-1)(x-1)}$$

where

$\Psi_1 = 9x^{n+1}(64x-65)(16x-1)(4x-1)J_{2n+1}^3 + 576x^{n+1}(16x-1)(4x-1)J_{2n-1}^3 - 24\times 2^{2n}x^{n+1}(64x-1)(x-1)J_{2n+3} + 30\times 2^{2n}x^{n+1}(256x^2-260x+13)J_{2n+1} - 96\times 2^{2n}x^{n+1}(16x-1)(4x-1)J_{2n-1} - 9(8x-1)(64x^2-50x+1)$

and

if $(16x-1)(4x-1)(64x-1)(x-1)=0$, *i.e.*, $x=\frac{1}{64}$ *or* $x=\frac{1}{16}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k J_{2k+1}^3 = \frac{\Psi_2}{9\left(16384x^3-16320x^2+2856x-85\right)}$$

where

$\Psi_2 = 9x^n J_{2n+1}^3(n(64x-65)(16x-1)(4x-1)+16384x^3-16320x^2+2728x-65) + 576x^n(n(16x-1)(4x-1)+192x^2-40x+1)J_{2n-1}^3 - 24\times 2^{2n}x^n(n(64x-1)(x-1)+192x^2-130x+1)J_{2n+3} + 30\times 2^{2n}x^n(n(256x^2-260x+13)+768x^2-520x+13)J_{2n+1} - 96\times 2^{2n}x^n\left(n(16x-1)(4x-1)+192x^2-40x+1\right)J_{2n-1} - 18(768x^2-464x+29)$.

(d) $(m=-1,\ j=0)$

If $(x+4)(x-2)(x+1)(x-8) \neq 0$, *i.e.*, $x \neq -4$, $x \neq -1$, $x \neq 2$, $x \neq 8$, *then*

$$\sum_{k=0}^{n} x^k J_{-k}^3 = \frac{\Psi_1}{9(x+4)(x-2)(x+1)(x-8)}$$

where

$\Psi_1 = 9x^{n+1}(x+4)(x-2)J^3_{-n+1} + 9x^{n+1}(x-7)(x+4)(x-2)J^3_{-n} - 6\times(-2)^{-n}x^{n+1}(x+4)(x-2)J_{-n+1} + 6\times(-2)^{-n}x^{n+1}\left(-7x+x^2+28\right)J_{-n} + 12\times(-2)^{-n}x^{n+1}(x+1)(x-8)J_{-n-1} - 9x(x^2-4x-8)$

and

if $(x+4)(x-2)(x+1)(x-8)=0$, *i.e.*, $x=-4$ *or* $x=-1$ *or* $x=2$ *or* $x=8$ *then*

$$\sum_{k=0}^{n} x^k J^3_{-k} = \frac{\Psi_2}{9\left(4x^3-15x^2-60x+40\right)}$$

where

$\Psi_2 = 9x^n(n(x+4)(x-2)+4x+3x^2-8)J^3_{-n+1} + 9x^n(n(x-7)(x+4)(x-2)+4x^3-15x^2-44x+56)J^3_{-n} - 6(-2)^{-n}x^n(n(x+4)(x-2)+3x^2+4x-8)J_{-n+1} + 6(-2)^{-n}x^n(n(x^2-7x+28)+3x^2-14x+28)J_{-n} + 12(-2)^{-n}x^n(n(x+1)(x-8)+3x^2-14x-8)J_{-n-1} - 9(3x^2-8x-8)$.

(e) $(m=-2,\ j=0)$

If $(x-4)(x-16)(x-1)(x-64)\neq 0$, *i.e.*, $x\neq 1$, $x\neq 4$, $x\neq 16$, $x\neq 64$, *then*

$$\sum_{k=0}^{n} x^k J^3_{-2k} = \frac{\Psi_1}{9(x-4)(x-16)(x-1)(x-64)}$$

where

$\Psi_1 = 9x^{n+1}(x-4)(x-16)J^3_{-2n+2} + 9x^{n+1}(x-65)(x-4)(x-16)J^3_{-2n} + 12\times 2^{-2n}x^{n+1}(x-4)(x-16)J_{-2n+2} - 60\times 2^{-2n}x^{n+1}(x^2-65x+208)J_{-2n} + 48\times 2^{-2n}x^{n+1}(x-1)(x-64)J_{-2n-2} - 9x(x^2+40x+64)$

and

if $(x-4)(x-16)(x-1)(x-64)=0$, *i.e.*, $x=1$ *or* $x=4$ *or* $x=16$ *or* $x=64$ *then*

$$\sum_{k=0}^{n} x^k J^3_{-2k} = \frac{\Psi_2}{9\left(4x^3-255x^2+2856x-5440\right)}$$

where

$\Psi_2 = 9x^n(n(x-4)(x-16)+3x^2-40x+64)J^3_{-2n+2} + 9x^n(n(x-65)(x-4)(x-16)+4x^3-255x^2+2728x-4160)J^3_{-2n} + 12\times 2^{-2n}x^n\left(64n-40x+nx^2-20nx+3x^2+64\right)J_{-2n+2} - 60\times 2^{-2n}x^n(n(x^2-65x+208)+3x^2-130x+208)J_{-2n} + 48\times 2^{-2n}x^n(n(x-1)(x-64)+3x^2-130x+64)J_{-2n-2} - 9(3x^2+80x+64)$.

(f) $(m=-2,\ j=1)$

If $(x-4)(x-16)(x-1)(x-64)\neq 0$, *i.e.*, $x\neq 1$, $x\neq 4$, $x\neq 16$, $x\neq 64$, *then*

$$\sum_{k=0}^{n} x^k J^3_{-2k+1} = \frac{\Psi_1}{9(x-4)(x-16)(x-1)(x-64)}$$

where

$\Psi_1 = 9x^{n+1}(x-4)(x-16)J^3_{-2n+3} + 9x^{n+1}(x-65)(x-4)(x-16)J^3_{-2n+1} - 24\times 2^{-2n}x^{n+1}(x-4)(x-16)J_{-2n+3} + 120\times 2^{-2n}x^{n+1}(x^2-65x+208)J_{-2n+1} - 96\times 2^{-2n}x^{n+1}(x-1)(x-64)J_{-2n-1} - 9(27x^3-964x^2+4928x-4096)$

and

if $(x-4)(x-16)(x-1)(x-64)=0$, *i.e.*, $x=1$ *or* $x=4$ *or* $x=16$ *or* $x=64$ *then*

$$\sum_{k=0}^{n} x^k J^3_{-2k+1} = \frac{\Psi_2}{9\left(4x^3-255x^2+2856x-5440\right)}$$

where

$\Psi_2 = 9x^n(n(x-4)(x-16)+3x^2-40x+64)J^3_{-2n+3} + 9x^n(n(x-65)(x-4)(x-16)+4x^3-255x^2+2728x-4160)J^3_{-2n+1} - 24\times 2^{-2n}x^n(n(x-4)(x-16)+3x^2-40x+64)J_{-2n+3} + 120\times 2^{-2n}x^n(n(x^2-65x+208)+3x^2-130x+208)J_{-2n+1} - 96\times 2^{-2n}x^n\left(n(x-1)(x-64)+3x^2-130x+64\right)J_{-2n-1} - 9(81x^2-1928x+4928)$.

From the above proposition, we have the following corollary which gives sum formulas of Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0=2$, $j_1=1$).

Corollary 2.8.
For $n\geq 0$, *Jacobsthal-Lucas numbers have the following properties:*

(a) $(m=1,\ j=0)$

If $(4x+1)(2x-1)(x+1)(8x-1)\neq 0$, *i.e.,* $x\neq -1,\ x\neq -\frac{1}{4},\ x\neq \frac{1}{8},\ x\neq \frac{1}{2}$, *then*

$$\sum_{k=0}^{n} x^k j_k^3 = \frac{\Psi_1}{(4x+1)(2x-1)(x+1)(8x-1)}$$

where

$\Psi_1 = x^{n+1}(8x+7)(4x+1)(2x-1)j_n^3 + 8x^{n+1}(4x+1)(2x-1)j_{n-1}^3 + 3(-2)^{n+1}x^{n+1}(x+1)(8x-1)j_{n+1} + 3(-2)^n x^{n+1}\left(14x+16x^2+7\right)j_n + 3(-2)^{n+2}x^{n+1}(4x+1)(2x-1)j_{n-1} + (8x^3-120x^2-39x+8)$

and

if $(4x+1)(2x-1)(x+1)(8x-1)=0$, *i.e.,* $x=-1$ *or* $x=-\frac{1}{4}$ *or* $x=\frac{1}{8}$ *or* $x=\frac{1}{2}$ *then*

$$\sum_{k=0}^{n} x^k j_k^3 = \frac{\Psi_2}{\left(256x^3+120x^2-60x-5\right)}$$

where

$\Psi_2 = x^n(n(8x+7)(2x-1)(4x+1)+256x^3+120x^2-44x-7)j_n^3 + 8x^n(n(4x+1)(2x-1)+24x^2-4x-1)j_{n-1}^3 + 3(-2)^{n+1}x^n\left(n(x+1)(8x-1)+24x^2+14x-1\right)j_{n+1} + 3(-2)^n x^n(n(16x^2+14x+7)+48x^2+28x+7)j_n + 3(-2)^{n+2}x^n(n(4x+1)(2x-1)+24x^2-4x-1)j_{n-1} + 3(8x^2-80x-13)$.

(b) $(m=2,\ j=0)$

If $(16x-1)(4x-1)(64x-1)(x-1)\neq 0$, *i.e.,* $x\neq \frac{1}{64},\ x\neq \frac{1}{16},\ x\neq \frac{1}{4},\ x\neq 1$, *then*

$$\sum_{k=0}^{n} x^k j_{2k}^3 = \frac{\Psi_1}{(16x-1)(4x-1)(64x-1)(x-1)}$$

where

$\Psi_1 = x^{n+1}(64x-65)(16x-1)(4x-1)j_{2n}^3 + 64x^{n+1}(16x-1)(4x-1)j_{2n-2}^3 - 3\times 2^{2n+2}x^{n+1}(64x-1)(x-1)j_{2n+2} + 15\times 2^{2n}x^{n+1}\left(-260x+256x^2+13\right)j_{2n} - 3\times 2^{2n+4}x^{n+1}(16x-1)(4x-1)j_{2n-2} + (-8000x^3+5712x^2-555x+8)$

and

if $(16x-1)(4x-1)(64x-1)(x-1)=0$, *i.e.,* $x=\frac{1}{64}$ *or* $x=\frac{1}{16}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k j_{2k}^3 = \frac{\Psi_2}{\left(16384x^3-16320x^2+2856x-85\right)}$$

where

$\Psi_2 = x^n(n(64x-65)(16x-1)(4x-1)+16384x^3-16320x^2+2728x-65)j_{2n}^3 + 64x^n(n(16x-1)(4x-1)+192x^2-40x+1)j_{2n-2}^3 - 3\times 2^{2n+2}x^n\left(n(64x-1)(x-1)+192x^2-130x+1\right)j_{2n+2} + 15\times 2^{2n}x^n(n(256x^2-260x+13)+768x^2-520x+13)j_{2n} - 3\times 2^{2n+4}x^n(n(16x-1)(4x-1)+192x^2-40x+1)j_{2n-2} - 3(8000x^2-3808x+185)$.

(c) $(m=2,\ j=1)$

If $(16x-1)(4x-1)(64x-1)(x-1)\neq 0$, *i.e.,* $x\neq \frac{1}{64},\ x\neq \frac{1}{16},\ x\neq \frac{1}{4},\ x\neq 1$, *then*

$$\sum_{k=0}^{n} x^k j_{2k+1}^3 = \frac{\Psi_1}{(16x-1)(4x-1)(64x-1)(x-1)}$$

where

$\Psi_1 = x^{n+1}(64x-65)(16x-1)(4x-1)j_{2n+1}^3 + 64x^{n+1}(16x-1)(4x-1)j_{2n-1}^3 + 24\times 2^{2n}x^{n+1}(64x-1)(x-1)j_{2n+3} - 30\times 2^{2n}x^{n+1}(256x^2-260x+13)j_{2n+1} + 96\times 2^{2n}x^{n+1}(16x-1)(4x-1)j_{2n-1} + (8x+1)(64x^2+250x+1)$

and

if $(16x-1)(4x-1)(64x-1)(x-1)=0$, *i.e.,* $x=\frac{1}{64}$ *or* $x=\frac{1}{16}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k j_{2k+1}^3 = \frac{\Psi_2}{\left(16384x^3-16320x^2+2856x-85\right)}$$

where

$\Psi_2 = x^n j_{2n+1}^3(n(64x-65)(16x-1)(4x-1)+16384x^3-16320x^2+2728x-65) + 64x^n(n(16x-1)(4x-1)+192x^2-40x+1)j_{2n-1}^3 + 24\times 2^{2n}x^n(n(64x-1)(x-1)+192x^2-130x+1)j_{2n+3} - 30\times 2^{2n}x^n(n(256x^2-260x+13)+768x^2-520x+13)j_{2n+1} + 96\times 2^{2n}x^n\left(n(16x-1)(4x-1)+192x^2-40x+1\right)j_{2n-1} + 6(256x^2+688x+43)$.

(d) $(m=-1,\ j=0)$

If $(x+4)(x-2)(x+1)(x-8)\neq 0$, *i.e.,* $x\neq -4,\ x\neq -1,\ x\neq 2,\ x\neq 8$, *then*

$$\sum_{k=0}^{n} x^k j_{-k}^3 = \frac{\Psi_1}{(x+4)(x-2)(x+1)(x-8)}$$

where

$\Psi_1 = x^{n+1}(x+4)(x-2)j_{-n+1}^3 + x^{n+1}(x-7)(x+4)(x-2)j_{-n}^3 + 6\times(-2)^{-n}x^{n+1}(x+4)(x-2)j_{-n+1} - 6\times(-2)^{-n}x^{n+1}\left(-7x+x^2+28\right)j_{-n} - 12\times(-2)^{-n}x^{n+1}(x+1)(x-8)j_{-n-1} - (x^3+120x^2-312x-512)$

and

if $(x+4)(x-2)(x+1)(x-8)=0$, *i.e.,* $x=-4$ *or* $x=-1$ *or* $x=2$ *or* $x=8$ *then*

$$\sum_{k=0}^{n} x^k j_{-k}^3 = \frac{\Psi_2}{\left(4x^3-15x^2-60x+40\right)}$$

where

$\Psi_2 = x^n(n(x+4)(x-2)+4x+3x^2-8)j_{-n+1}^3 + x^n(n(x-7)(x+4)(x-2)+4x^3-15x^2-44x+56)j_{-n}^3 + 6(-2)^{-n}x^n(n(x+4)(x-2)+3x^2+4x-8)j_{-n+1} - 6(-2)^{-n}x^n(n(x^2-7x+28)+3x^2-14x+28)j_{-n} - 12(-2)^{-n}x^n(n(x+1)(x-8)+3x^2-14x-8)j_{-n-1} - 3(x^2+80x-104)$.

(e) $(m=-2,\ j=0)$

If $(x-4)(x-16)(x-1)(x-64)\neq 0$, *i.e.,* $x\neq 1,\ x\neq 4,\ x\neq 16,\ x\neq 64$, *then*

$$\sum_{k=0}^{n} x^k j_{-2k}^3 = \frac{\Psi_1}{(x-4)(x-16)(x-1)(x-64)}$$

where

$\Psi_1 = x^{n+1}(x-4)(x-16)j_{-2n+2}^3 + x^{n+1}(x-65)(x-4)(x-16)j_{-2n}^3 - 12\times 2^{-2n}x^{n+1}(x-4)(x-16)j_{-2n+2} + 60\times 2^{-2n}x^{n+1}(x^2-65x+208)j_{-2n} - 48\times 2^{-2n}x^{n+1}(x-1)(x-64)j_{-2n-2} - (125x^3-5712x^2+35520x-32768)$

and

if $(x-4)(x-16)(x-1)(x-64)=0$, *i.e.,* $x=1$ *or* $x=4$ *or* $x=16$ *or* $x=64$ *then*

$$\sum_{k=0}^{n} x^k j_{-2k}^3 = \frac{\Psi_2}{\left(4x^3-255x^2+2856x-5440\right)}$$

where

$\Psi_2 = x^n(n(x-4)(x-16)+3x^2-40x+64)j_{-2n+2}^3 + x^n(n(x-65)(x-4)(x-16)+4x^3-255x^2+2728x-4160)j_{-2n}^3 - 12\times 2^{-2n}x^n\left(64n-40x+nx^2-20nx+3x^2+64\right)j_{-2n+2} + 60\times 2^{-2n}x^n(n(x^2-65x+208)+3x^2-130x+208)j_{-2n} - 48\times 2^{-2n}x^n(n(x-1)(x-64)+3x^2-130x+64)j_{-2n-2} - 3(125x^2-3808x+11840)$.

(f) $(m=-2,\ j=1)$

If $(x-4)(x-16)(x-1)(x-64)\neq 0$, *i.e.,* $x\neq 1,\ x\neq 4,\ x\neq 16,\ x\neq 64$, *then*

$$\sum_{k=0}^{n} x^k j_{-2k+1}^3 = \frac{\Psi_1}{(x-4)(x-16)(x-1)(x-64)}$$

where

$\Psi_1 = x^{n+1}(x-4)(x-16)j_{-2n+3}^3 + x^{n+1}(x-65)(x-4)(x-16)j_{-2n+1}^3 + 24\times 2^{-2n}x^{n+1}(x-4)(x-16)j_{-2n+3} - 120\times 2^{-2n}x^{n+1}(x^2-65x+208)j_{-2n+1} + 96\times 2^{-2n}x^{n+1}(x-1)(x-64)j_{-2n-1} - (343x^3+636x^2+5952x-4096)$

and

if $(x-4)(x-16)(x-1)(x-64)=0$, *i.e.,* $x=1$ *or* $x=4$ *or* $x=16$ *or* $x=64$ *then*

$$\sum_{k=0}^{n} x^k j_{-2k+1}^3 = \frac{\Psi_2}{\left(4x^3-255x^2+2856x-5440\right)}$$

where

$\Psi_2 = x^n(n(x-4)(x-16)+3x^2-40x+64)j_{-2n+3}^3 + x^n(n(x-65)(x-4)(x-16)+4x^3-255x^2+2728x-4160)j_{-2n+1}^3 + 24\times 2^{-2n}x^n(n(x-4)(x-16)+3x^2-40x+64)j_{-2n+3} - 120\times 2^{-2n}x^n(n(x^2-65x+208)+3x^2-130x+208)j_{-2n+1} + 96\times 2^{-2n}x^n\left(n(x-1)(x-64)+3x^2-130x+64\right)j_{-2n-1} - 3(343x^2+424x+1984)$.

Taking $x=1$ in the last two corollaries we get the following corollary.

Corollary 2.9.
For $n \geq 0$, Jacobsthal numbers and Jacobsthal-Lucas numbers have the following properties:

1.

(a) $\sum_{k=0}^{n} J_k^3 = \frac{1}{210}(225J_n^3 + 120J_{n-1}^3 + (-2)^n(28J_{n+1} - 37J_n - 20J_{n-1}) - 33).$

(b) $\sum_{k=0}^{n} J_{2k}^3 = \frac{1}{945}(-3(5n-303)J_{2n}^3 + 192(5n+17)J_{2n-2}^3 + 28\times 2^{2n}J_{2n+2} - 5\times 2^{2n}(n+29)J_{2n} + 2^{2n+4}(5n+17)J_{2n-2} + 91).$

(c) $\sum_{k=0}^{n} J_{2k+1}^3 = \frac{1}{945}(-3(5n-303)J_{2n+1}^3 + 192(5n+17)J_{2n-1}^3 - 56\times 2^{2n}J_{2n+3} + 10\times 2^{2n}(n+29)J_{2n+1} - 2^{2n+5}(5n+17)J_{2n-1} - 222).$

(d) $\sum_{k=0}^{n} J_{-k}^3 = \frac{1}{210}(-15J_{-n+1}^3 + 90J_{-n}^3 + 2(5J_{-n+1} + 22J_{-n} - 28J_{-n-1})(-2)^{-n} + 33).$

(e) $\sum_{k=0}^{n} J_{-2k}^3 = \frac{1}{945}(-3(5n+3)J_{-2n+2}^3 + 3(320n+187)J_{-2n}^3 - 2^{-2n+2}(5n+3)J_{-2n+2} + 5\times 2^{-2n+2}(16n+9)J_{-2n} + 7\times 2^{-2n+4}J_{-2n-2} + 49).$

(f) $\sum_{k=0}^{n} J_{-2k+1}^3 = \frac{1}{945}(-3(5n+3)J_{-2n+3}^3 + 3(320n+187)J_{-2n+1}^3 + 2^{-2n+3}(5n+3)J_{-2n+3} - 5\times 2^{-2n+3}(16n+9)J_{-2n+1} - 7\times 2^{-2n+5}J_{-2n-1} + 1027).$

2.

(a) $\sum_{k=0}^{n} j_k^3 = \frac{1}{70}(75j_n^3 + 40j_{n-1}^3 - 3(-2)^n(28j_{n+1} - 37j_n - 20j_{n-1}) - 143).$

(b) $\sum_{k=0}^{n} j_{2k}^3 = \frac{1}{315}(-(5n-303)j_{2n}^3 + 64(5n+17)j_{2n-2}^3 - 21\times 2^{2n+2}j_{2n+2} + 15\times 2^{2n}(n+29)j_{2n} - 3\times 2^{2n+4}(5n+17)j_{2n-2} - 1459).$

(c) $\sum_{k=0}^{n} j_{2k+1}^3 = \frac{1}{315}(-(5n-303)j_{2n+1}^3 + 64(5n+17)j_{2n-1}^3 + 168\times 2^{2n}j_{2n+3} - 15\times 2^{2n+1}(n+29)j_{2n+1} + 3\times 2^{2n+5}(5n+17)j_{2n-1} + 658).$

(d) $\sum_{k=0}^{n} j_{-k}^3 = \frac{1}{70}(-5j_{-n+1}^3 + 30j_{-n}^3 - 6\times(-2)^{-n}(5j_{-n+1} - 28j_{-n-1} + 22j_{-n}) + 703).$

(e) $\sum_{k=0}^{n} j_{-2k}^3 = \frac{1}{315}(-(5n+3)j_{-2n+2}^3 + (320n+187)j_{-2n}^3 + 3\times 2^{-2n+2}(5n+3)j_{-2n+2} - 15\times 2^{-2n+2}(16n+9)j_{-2n} - 21\times 2^{-2n+4}j_{-2n-2} + 2719).$

(f) $\sum_{k=0}^{n} j_{-2k+1}^3 = \frac{1}{315}(-(5n+3)j_{-2n+3}^3 + (320n+187)j_{-2n+1}^3 - 3\times 2^{-2n+3}(5n+3)j_{-2n+3} + 15\times 2^{-2n+3}(16n+9)j_{-2n+1} + 21\times 2^{-2n+5}j_{-2n-1} + 917).$

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