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A trong solution of a mixed problem with bounrdary integral conditions for a certain parabolic fractional equation using Fourier's Method

Research Article

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Abstract: In this paper, we study a mixed problem with non boundary conditions for a certain singular fractional parabolic equation. The functional analysis and Fourier methods are used. It is important to know that a priori estimates establish in nonclassical function space is necessary tool to prove the uniqueness and dependance continue of a strong solution of the studied problem.

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Keywords: Fractional equation • Non boundary conditions • Strong solution

1. Introduction

In the doamain $\Omega = (0,\ \ell) \times (0,\ T)$, where $\ell < +\infty$ and $T < \infty$. We shall determine a solution u, in Ω, of the fractional parabolic differential equation

$$D_t^{\alpha} u - \frac{1}{x}\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) + ku = F(x,t), \quad k \geq 0,\ (x,t). \in \Omega. \tag{1}$$

To equation (1), we attache the Cauchy condition

$$u(x,0) = \varphi(x), \quad x \in (0,\ell), \tag{2}$$

and non-local conditions

$$\int_0^{\ell} u(x,t)\,dx = 0, \tag{3}$$

$$\int_0^{\ell} xu(x,t)\,dx = 0, \tag{4}$$

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where $\varphi \in L_2(0,1)$ is a know function and satisfies the compability conditions for consitency, he have

$$\int_0^\ell \varphi(x)\,dx = \int_0^\ell x\varphi(x)\,dx = 0,$$

$f(x,t)$ is a known function, $0<\alpha<1$, the left Caputo derivative D_t^α and the gamma function Γ are respectively defined as

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u}{\partial \tau}\frac{d\tau}{(t-\tau)^\alpha},$$

$$\Gamma(\alpha) = \int_0^{+\infty} x^{1-\alpha}e^{-x}\,dx.$$

The existence and uniqueness of solutions to initial and boundary-value problems for fractional differential equations has been extensively studied by many authors ; see for example [[1], [2], [3], [4], [9], [13], [14], [15], [16], [21], [25], [26]]. Some of the existence and uniqueness results have been obtained by using the well-known Lax-Milgram theorem, by fixed point theorem and energy-integral method [[1], [2], [5], [6], [7], [8], [10], [11], [24], [25]] .
A suitable variational formulation is the starting point of many numerical methods, such as finite element methods, spectral methods, Laplace transform method [[9], [22], [23], [26]], Sumudu transform method [19]. Thus the construction of a variational formulation is essential, and relies strongly on the choice of spaces and their norms.

Motivated by this, we extend and generalize the study for PDEs with integral conditions to the study of fractional PDEs with integral conditions. Also we expand the works in classical problems of fractional PDEs to non standard problems.

In this paper, we extend a energy-integral method and Fourier's method to the study of a mixed-type fractional differential equations.

The general difficult which arises to us is the presence of integral conditions which complicates the application of standard methods. It may, however, be worth while if this type of problems can be transformed into another equivalent problem which involves no integral conditions. For this, we convert (1)-(4) to the following classical problem.

Theorem 1.1.
The boundary value problem with nonlocal conditions (1), (2), (3) and (4) is equivalent to the following Neumann problem :

$$D_t^\alpha u - \frac{1}{x}\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right) + ku = F(x,t), \quad k\geq 0,\ (x,t).\in\Omega, \tag{5}$$

$$u(x,0) = \varphi(x), \quad 0\leq x\leq \ell, \tag{6}$$

$$u(\ell,t) - u(0,t) = \frac{1}{\ell}\int_0^\ell xF(x,t)\,dx - \int_0^\ell F(x,t)\,dx, \tag{7}$$

$$\frac{\partial u}{\partial x}(\ell,t) = -\frac{1}{\ell^2}\int_0^\ell xF(x,t)\,dx. \tag{8}$$

Proof. Integrating (1) over $(0,\ \ell)$, and taking in account of (3), we get

$$\ell\frac{\partial u}{\partial x}(\ell,t) + u(\ell,t) - u(0,t) = -\int_0^\ell F(x,t)\,dx, \tag{9}$$

$$D_t^\alpha\left(\int_0^\ell u(x,t)\,dx\right) + k\int_0^\ell u(x,t) = 0, \quad 0\leq t\leq T. \tag{10}$$

Multiplying the both side of (1) by x and integrating the resulting over $(0,\ \ell)$, and taking account of the condition (4), we have

$$\frac{\partial u}{\partial x}(\ell,t) = -\frac{1}{\ell}\int_0^\ell xF(x,t)\,dx, \tag{11}$$

$$D_t^\alpha\left(\int_0^\ell xu(x,t)\,dx\right) + k\int_0^\ell xu(x,t)\,dx = 0, \quad 0 \le t \le T. \tag{12}$$

Substuting (11) in (9), we get

$$u(\ell,t) - u(0,t) = \frac{1}{\ell}\int_0^\ell xF(x,t)\,dx - \int_0^\ell xF(x,t)\,dx. \tag{13}$$

By virtue of the compatibility conditions, and the equations (10) and (12), we obtain

$$\int_0^\ell u(x,t)\,dx = \int_0^\ell xu(x,t)\,dx = 0$$

□

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2. Preliminaries

We introduce now a new function $v(x,t) = u(x,t) - w(x,t) - \varphi(x)$. Then problem (5)-(8) can be formulated as

$$D_t^\alpha v - \frac{1}{x}\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right) + ku = f(x,t), \quad k \ge 0,\ (x,t). \in \Omega, \tag{14}$$

$$v(x,0) = 0, \quad 0 \le x \le \ell, \tag{15}$$

$$v(\ell,t) = v(0,t), \tag{16}$$

$$\frac{\partial v}{\partial x}(\ell,t) = 0. \tag{17}$$

where

$$w(x,t) = -\frac{(1+\ell)x^2 + \ell(2+\ell)x + 1}{\ell^2}\int_0^\ell xF(x,t)\,dx - \frac{x^2 - 2\ell x + 1}{\ell^2}\int_0^\ell F(x,t)\,dx$$

$$f(x,t) = F(x,t) + \frac{(1+\ell)x^2 + \ell(2+\ell)x + 1}{\ell^2}\int_0^\ell xD_t^\alpha F(x,t)\,dx$$

$$+\frac{x^2 - 2\ell x + 1}{\ell^2}\int_0^\ell D_t^\alpha F(x,t)\,dx + \frac{kx^2 - 2(k\ell + 3)x + 4\ell + k}{\ell^2}\int_0^\ell F(x,t)\,dx$$

$$+\frac{k(1+\ell)x^2 + (2k\ell + k\ell^2 - 6\ell - 6)x + k - 4\ell - 2\ell^2}{\ell^2}\int_0^\ell xF(x,t)\,dx$$

Instead of searching for the function u, we search for the function v. So the solution of problem (14), (15), (16) and (17) will be given by $u(x,t) = v(x,t) + w(x,t) + \varphi(x)$.

In this paper, we prove existence and uniqueness of a strong solution of the problem stated in equations (14), (15), (16) and (17). For this, we consider the problem (14)-(17) as a solution of the operator equation

$$Lv = \mathscr{F} = f, \tag{18}$$

with domain of defintion $D(L)$ consisting of function $u \in L_2(\Omega)$ such that

$$\sqrt{x}v, x\sqrt{x}\frac{\partial v}{\partial x}, \sqrt{x}D_t^\alpha v,\ x\frac{\partial v}{\partial x},\ \frac{\partial}{\partial}\left(\frac{\partial v}{\partial x}\right) \in L_2(\Omega)$$

and u satisfies conditions (16) and (17). The operator L is considered from E to F, where E is the Banach space consisting of function $u \in L_2(\Omega)$, satisfying (16) and (17) with the finite norm

$$\|u\|_E^2 = \int_0^\ell \int_0^T xv^2\,dxdt + \int_0^\ell \int_0^T x\left(\frac{\partial v}{\partial x}\right)^2 dxdt + \int_0^\ell \int_0^T x\left(D_t^\alpha v\right)^2 dxdt, \tag{19}$$

Here F is the Hilbert space of vector-valued functions $\mathscr{F} = f$ obtained by completing of the space $L_2(\Omega)$ with respect to the norm

$$\|\mathscr{F}\|_F^2 = \int_0^\ell \int_0^T f^2(x,t)\,dxdt. \tag{20}$$

Definition 2.1.
A solution of the operator equation $\overline{L}u = \mathscr{F}$ is called a strong solution of the problem (14), (15), (16) and (17).

3. A energy-integral and its consequences

Theorem 3.1.
Let conditions (14)-(17) *be fulfilled. For any function $v \in E$, there is the a priori estimate*

$$\|v\|_E \le c\|Lv\|_F, \tag{21}$$

where c is a constant which may depend on T but not depend on v.

Proof. Firstly, multiplying (14) by $xD_t^\alpha v$ and integrating over Ω, we have

$$\int_0^\ell \int_o^T x(D_t^\alpha v)^2\,dxdt = \int_0^\ell \int_0^T 2x^2\frac{\partial v}{\partial x}D_t^\alpha v\,dxdt + \int_0^\ell \int_0^T x^3\frac{\partial^2 v}{\partial x^2}D_t^\alpha\,dxdt$$
$$-\int_0^\ell \int_0^T kxvD_t^\alpha v\,dxdt + \int_0^\ell \int_o^T xf(x,t)D_t^\alpha v\,dxdt. \tag{22}$$

We estimate the terms of right-hand side of (22). By applying the $\varepsilon-$ elementary inequality, we obtain

$$\int_0^\ell \int_0^T x(D_t^\alpha v)^2\,dxdt \le \frac{2\varepsilon_1 + \varepsilon_2 + k\varepsilon_3 + \varepsilon_4}{2}\int_0^\ell \int_0^T x\left(D_t^\alpha v\right)^2 dxdt + \frac{1}{\varepsilon_1}\int_0^\ell \int_o^T x\left(\frac{\partial v}{\partial x}\right)^2 dxdt$$
$$+\frac{1}{2\varepsilon_2}\int_0^\ell \int_0^T x^3\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dxdt + \frac{k}{2\varepsilon_3}\int_0^\ell \int_0^T xv^2\,dxdt + \frac{1}{2\varepsilon_4}\int_0^\ell \int_0^T xf^2(x,t)\,dxdt. \tag{23}$$

Secondly, similarly, multiplying the equation (14) by xv and integrating over Ω, we get

$$\int_0^\ell \int_0^T xvD_t^\alpha v\,dxdt - \int_0^\ell \int_0^T v\frac{\partial}{\partial x}\left(x^2\frac{\partial v}{\partial x}\right) dxdt + k\int_0^\ell \int_0^T xv^2\,dxdt = \int_0^\ell \int_0^T xvf(x,t)\,dxdt. \tag{24}$$

Integrating by parts the second terms on the left-hand side in (24) with the use of boundary conditions (16) and (17), we get

$$-\int_0^\ell \int_0^T v\frac{\partial}{\partial x}\left(x^2\frac{\partial v}{\partial x}\right) dxdt = \int_0^\ell \int_0^T x^2\left(\frac{\partial v}{\partial x}\right)^2 dxdt. \tag{25}$$

Substituting (25) in (24), and applying an elementary inequality, we have

$$\int_0^\ell \int_0^T x^2\left(\frac{\partial v}{\partial x}\right)^2 dxdt + k\int_0^\ell \int_0^T xv^2\,dxdt \le \frac{\varepsilon_5 + \varepsilon_6}{2\varepsilon_5\varepsilon_6}\int_0^\ell \int_0^T xv^2\,dxdt + \frac{\varepsilon_5}{2}\int_0^\ell \int_0^T x(D_t^\alpha v)^2\,dxdt$$
$$+\frac{\varepsilon_6}{2}\int_0^\ell \int_0^T xf^2(x,t)\,dxdt. \tag{26}$$

Now, adding (23) and (26), it follows that

$$\frac{2-2\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5}{2}\int_0^\ell\int_0^T x(D_t^\alpha v)^2\,dxdt+\left(k-\frac{k}{2\varepsilon_3}-\frac{1}{2\varepsilon_5}-\frac{1}{2\varepsilon_6}\right)\int_0^\ell\int_0^T xv^2\,dxdt$$
$$+\left(1-\frac{1}{\varepsilon_1}-\frac{\ell^3}{2\varepsilon_2}\right)\int_0^\ell\int_0^T x^2\left(\frac{\partial v}{\partial x}\right)^2 dxdt\le\frac{1+\varepsilon_4\varepsilon_6}{2\varepsilon_4}\int_0^\ell\int_0^T xf^2(x,t)\,dxdt. \tag{27}$$

Hence, if $\varepsilon_i>0$ for $i=1,2,3,4,5,6$ satisfies $2-2\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5>0$, $k-\frac{k}{2\varepsilon_3}-\frac{1}{2\varepsilon_5}-\frac{1}{2\varepsilon_6}>0$ and $1-\frac{1}{\varepsilon_1}-\frac{\ell^3}{2\varepsilon_2}>0$, then inequality (27) implies

$$\int_0^\ell\int_0^T x(D_t^\alpha v)^2\,dxdt+\int_0^\ell\int_0^T xv^2\,dxdt+\int_0^\ell\int_0^T x\left(\frac{\partial v}{\partial x}\right)^2 dxdt\le\eta\int_0^\ell\int_0^T xf^2(x,t)\,dxdt. \tag{28}$$

where

$$\eta=\frac{(1+\varepsilon_4\varepsilon_6)\ell}{2\varepsilon_4\min\left(\frac{2-2\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5}{2},\ k-\frac{k}{2\varepsilon_3}-\frac{1}{2\varepsilon_5}-\frac{1}{2\varepsilon_6},\ 1-\frac{1}{\varepsilon_1}-\frac{\ell^3}{2\varepsilon_2}\right)}.$$

Thus, inequality (21) holds, where $c=\sqrt{\eta}$.

This completes the proof.

□

Corollary 3.1.
Under the conditions of Theorem 3.1, there is a constant $C>0$ independent of v such that

$$\|v\|_E\le C\|\overline{L}v\|_F,\qquad v\in\mathbf{D}'(\Omega). \tag{29}$$

Corollary 3.2.
Assert that, if a strong solution exists, it is unique and depends continuously on f, if v is considered in the topology of E and f is considered in the topology of F.

4. Existence of solution

We use the Fourier's method to establish the existence of the solution of the problem (14)-(17).
For this, we consider the function $v_n(x,t)=X_n(x)T_n(t)$ where $X_n(x)$ is a eigenfunction of the boundary value problem

$$\frac{1}{x}\frac{d}{dx}\left(x^2\left(\frac{dX_n(x)}{dx}\right)\right)-kX_n(x)=\lambda_nX_n(x), \tag{30}$$

$$X_n(0)=X_n(\ell), \tag{31}$$

$$\frac{dX_n(\ell)}{dx}=0, \tag{32}$$

$(\lambda_n, n=1,2,\cdots)$ is called the eigenvalue corresponding to the eigenfunction $X_n(x)$ and $T_n(t)$ is satisfying the initial following problem

$$D_t^\alpha T_n(t)-\lambda_nT_n(t)=f_n(t), \tag{33}$$

$$T_n(0)=0, \tag{34}$$

where $f(x,t)$ is expanded in Fourier series in terms of the system $X_1,\ X_2,\ \cdots$ of eigenfunctions

$$f(x,t)=\sum_{n=1}^{+\infty}f_n(t)X_n(x), \tag{35}$$

and by the Parseval-Steklov equality

$$\frac{1}{\ell}\int_0^\ell f(x,t)\,dx = \sum_{n=1}^{+\infty} f_n^2(t). \tag{36}$$

Hence

$$\sum_{n=1}^{+\infty}\int_0^T f_n^2(t)\,dt = \frac{1}{\ell}\int_\Omega f^2(x,t)\,dxdt. \tag{37}$$

By principe of superposition, the solution of the problem (14)-(17) is given by the series

$$v(x,t) = \sum_{n=1}^{+\infty} X_n(x)T_n(t). \tag{38}$$

Theorem 4.1.
Let $f \in L_2(\Omega)$. Then the solution $v(x,t)$ of (14)-(17) exists and its represented by series (38) which converge in E

Proof. Consider the partial sum $S_N(x,t) = \sum_{k=1}^{N} X_k(x)T_k(t)$ of the series (38).
Then by the first theorem 3.1

$$\left\| \sum_{k=1}^{N} X_k(x)T_k(t) \right\|_E^2 \leq C\sum_{k=1}^{N}\int_0^T f_k^2(t)\,dt. \tag{39}$$

The serie $\sum\int_0^T f_n^2(t)$ converge. Thus, from (39), the series (38) converge in E and accordingly it sum $v \in E$. □

References

[1] B. Ahmad, J. Nieto ; Existence results for nonlinear boundary value problems of fractional integro-differential equations with integral boundary conditions, *Boundary Value Problems* Vol. 2009 (2009), Article ID 708576, 11 pages.

[2] A. Anguraj, P. Karthikeyan ; Existence of solutions for fractional semilinear evolution bound- ary value problem, *Commun. Appl. Anal.* 14 (2010) 505–514.

[3] M. Belmekki, M. Benchohra ; Existence results for fractional order semilinear functional differential equations, *Proc. A. Razmadze Math. Inst.* 146 (2008) 9–20.

[4] M. Benchohra, J. R. Graef, S. Hamani ; Existence results for boundary value problems with nonlinear fractional differential equations, *Appl. Anal.* 87 (2008) 851–863.

[5] A. Bouziani and N-E Benouar Mixed problem with integral conditions for a third order parabolic equation, Kobe J. Math.15 (1998), no.1,47-58.

[6] A. Bouziani and N.E. Benouar, Probléme mixte avec conditions intégrales pour une classe déquations paraboliques, comptes rendus de l'Academie des Sciences, Paris t. 321, Série I (1995), 1177-1182.

[7] A. Bouziani, Mixed problem for certain nonclassical equations with a small parameter, Bulletin de la Classe des Sciences, Académie Royale de Belgique, 5 (1994). 389-400.

[8] A. Bouziani, Mixted problem with integral conditions for a certain parabolic equation, J.of App. Math. and Stoch. Anal. 9 (1996) 323-330.

[9] DJIBIBE Moussa Zakari and Ahcene Merad,On solvability of the third pseudo-parabolic fractional equation with purely nonlocal conditions, Advances in Differential Equations and Control Processes Vol. 23, Number 1, 2020, p. 87-104

[10] M. Z. Djibibe, K. Tcharie and N. I. Yurchuk ; Existence, Uniqueness and Continuous Dependance of Solution of Nonlocal Boundary Conditions of Mixed Problem for Singular Parabolic Equation in Nonclassical Function Spaces, *Pioneer Journal of Advances in Applied Mathematics* Vol 7, number 1, 2013, p-7-16.

[11] M. Z Djibibe, K. Tcharie, On the Solvability of an Evolution Problem with Weighted Integral Boundary Conditions in Sobolev Function Spaces with a Priori Estimate and Fourier's Method, *British Journal of Mathematics & Computer Science* , 3(4): 801-810, 2013.

[12] M. Z. Djibibe, K. Tcharie and N. I. Yurchuk ; Continuous dependence of solutions to mixed boundary value problems for a parabolic equation, *Electronic Journal of Differential Equations*, Vol. 2008(2008), No. 17, p. 1-10.

[13] N. J. Ford,J. Xiao, Y. Yan ; A finite element method for time fractional partial differential equations. *Fractional Calculus and Applied Analysis*.14(3) (2011), 454-474. doi : 10.2478/s13540-011-0028-2.

[14] J. H. He ; Nonlinear oscillation with fractional derivative and its applications. *In: International Conference on Vibrating Engineering'98, Dalian, China,* pp. 288-291 (1998).

[15] J. H. He ; Some applications of nonlinear fractional differential equations and their approximations. *Bull Sci Technol15,* 86-90 (1999).

[16] X. J. Li, C. J. Xu; Existence and uniqueness of the weak solution of the space-time fractional diffusion equation and a spectral method approximation, *Communications in Computational Physics,* vol. 8, no. 5, pp. 1016–1051, 2010.

[17] Kaya D, Inan I E. A numerical application of the decomposition method for the combined KdV-MKdV equation. Appl Math Computat, 2005, 168: 915–926.

[18] Mesloub S. A nonlinear nonlocal mixed problem for a second order pseudoparabolic equation. J. Math. Anal. Appl. 2006, 316: 189–209.

[19] Kiliçman A, Gadain H E. On the applications of Laplace and Sumudu transforms. J Frankl Inst, 2010, 347(5): 848–862.

[20] F. Dubois, A. C. Galucio et N. Point. Introduction à la dérivée fractionnaire: Théorie et applications, 29 mars 2010.

[21] Tarasov V.E. Fractional integro-differential equations for electromagnetic waves in dielectric media, Theoretical and Mathematical Physics, 158 number 3 (2009), 355-359.

[22] Atangana A, Oukouomi Noutchie S C. On multi-Laplace transform for solving nonlinear partial differential equations with mixed derivatives. Math Probl Eng, Volume 2014, Article ID 267843, 9 pages.

[23] Eltayeb H, Kiliçman A. A note on solutions of wave, Laplace's and heat equations with convolution terms by using double Laplace transform. Appl Math Lett, 2008, 21(12): 1324–1329.

[24] Mesloub S. A nonlinear nonlocal mixed problem for a second order pseudoparabolic equation. J. Math. Anal. Appl. 2006, 316: 189–209.

[25] SOAMPA Bangan and DJIBIBE Moussa Zakari, Mixed problem with an pure integral two-space-variables condition for a third order fractional parabolic equation, MJM, Vol. 8, No. 1, pp. 258-271, 2020.

[26] SOAMPA Bangan, DJIBIBE Moussa Zakari and Kokou TCHARIE, Analytical approximation solution of pseudo-parabolic Fractional Equation using a modified double laplace decomposition method, Theoretical Mathelatics and Applications, Vol. 10, No. 1, (2020), pp. 17-31.