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On Binomial Transform of the Generalized Tetranacci Sequence

Research Article

Yüksel Soykan *

Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey

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Abstract: In this paper, we define the binomial transform of the generalized Tetranacci sequence and as special cases, the binomial transform of the Tetranacci and Tetranacci-Lucas sequences will be introduced. We investigate their properties in details.

MSC: 11B37 • 11B39 • 11B83

Keywords: Binomial transform • Tetranacci sequence • Binomial transform of Tetranacci sequence • Binomial transform of Tetranacci-Lucas sequence

1. Introduction and Preliminaries

In this paper, we introduce the binomial transform of the generalized Tetranacci sequence and we investigate, in detail, two special cases which we call them the binomial transform of the Tetranacci and Tetranacci-Lucas sequences. We investigate their properties in the next sections. In this section, we present some properties of the generalized (r,s,t,u) sequence (generalized Tetranacci) sequence.

The generalized (r,s,t,u) sequence (or generalized Tetranacci sequence or generalized 4-step Fibonacci sequence) $\{W_n(W_0,W_1,W_2,W_3;r,s,t,u)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \quad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \quad n \geq 4 \tag{1}$$

where W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers and r,s,t,u are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [5],[8],[9],[18],[27],[38],[39]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1,2,3,...$ when $t \neq 0$. Therefore, recurrence (1) holds for all integers n.

In literature, for example, the following names and notations (see Table 1) are used for the special case of r,s,t,u and initial values.

* *E-mail address(es):* yuksel_soykan@hotmail.com

Table 1. A few special case of generalized Tetranacci sequences.

No	Sequences (Numbers)	Notation	OEIS [17]	Ref.
1	Tetranacci	$\{M_n\} = \{W_n(0,1,1,2;1,1,1,1)\}$	A000078	[19]
2	Tetranacci-Lucas	$\{R_n\} = \{W_n(4,1,3,7;1,1,1,1)\}$	A073817	[19]
3	fourth order Pell	$\{P_n^{(4)}\} = \{W_n(0,1,2,5;2,1,1,1)\}$	A103142	[20]
4	fourth order Pell-Lucas	$\{Q_n^{(4)}\} = \{W_n(4,2,6,17;2,1,1,1)\}$	A331413	[20]
5	modified fourth order Pell	$\{E_n^{(4)}\} = \{W_n(0,1,1,3;2,1,1,1)\}$	A190139	[20]
6	fourth order Jacobsthal	$\{J_n^{(4)}\} = \{W_n(0,1,1,1;1,1,1,2)\}$	A007909	[15]
7	fourth order Jacobsthal-Lucas	$\{j_n^{(4)}\} = \{W_n(2,1,5,10;1,1,1,2)\}$	A226309	[15]
8	modified fourth order Jacobsthal	$\{K_n^{(4)}\} = \{W_n(3,1,3,10;1,1,1,2)\}$		[15]
9	fourth-order Jacobsthal Perrin	$\{Q_n^{(4)}\} = \{W_n(3,0,2,8;1,1,1,2)\}$		[15]
10	adjusted fourth-order Jacobsthal	$\{S_n^{(4)}\} = \{W_n(0,1,1,2;1,1,1,2)\}$		[15]
11	modified fourth-order Jacobsthal-Lucas	$\{R_n^{(4)}\} = \{W_n(4,1,3,7;1,1,1,2)\}$		[15]
12	4-primes	$\{G_n\} = \{W_n(0,0,1,2;2,3,5,7)\}$		[21]
13	Lucas 4-primes	$\{H_n\} = \{W_n(4,2,10,41;2,3,5,7)\}$		[21]
14	modified 4-primes	$\{E_n\} = \{W_n(0,0,1,1;2,3,5,7)\}$		[21]

Remark 1.1.
As $\{W_n\}$ is a fourth order recurrence sequence (difference equation), it's characteristic equation is

$$x^4 - rx^3 - sx^2 - tx - u = 0 \tag{2}$$

where the four roots $x_1 = \alpha, x_2 = \beta, x_3 = \gamma, x_4 = \delta$ of the quartic polynomial equation ([**?**]) are given by

$$\begin{aligned}
\alpha &= x_1 = -\frac{g_1}{2} + \sqrt{\frac{g_1^2}{4} - h_1} \\
\beta &= x_2 = -\frac{g_1}{2} - \sqrt{\frac{g_1^2}{4} - h_1} \\
\gamma &= x_3 = -\frac{g_2}{2} + \sqrt{\frac{g_2^2}{4} - h_2} \\
\delta &= x_4 = -\frac{g_2}{2} - \sqrt{\frac{g_2^2}{4} - h_2}
\end{aligned}$$

and

$$\begin{aligned}
g_1 &= \frac{-r}{2} - \sqrt{\frac{r^2}{4} + s + y_1} \\
g_2 &= \frac{-r}{2} + \sqrt{\frac{r^2}{4} + s + y_1} \\
h_1 &= \frac{y_1}{2} + \sum_{sign} \sqrt{\frac{y_1^2}{4} + u} \\
h_2 &= \frac{y_1}{2} - \sum_{sign} \sqrt{\frac{y_1^2}{4} + u} \\
\sum_{sign} &= \begin{cases} 1 & , \text{ if } \frac{1}{2}ry_1 - t > 0 \\ -1 & , \text{ otherwise} \end{cases}
\end{aligned}$$

and y_1 as the greatest real solution of the resolvent cubic equation

$$y^3 + sy^2 + (4u + rt)y + 4su - t^2 + r^2u = 0.$$

Note that we have the following identities

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= r, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -s, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= t, \\
\alpha\beta\gamma\delta &= -u.
\end{aligned}$$

Generalized Tetranacci numbers can be expressed, for all integers n, using Binet's formula

$$W_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \quad (3)$$

where

$$\begin{aligned} p_1 &= W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0, \\ p_2 &= W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0, \\ p_3 &= W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0, \\ p_4 &= W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0. \end{aligned}$$

Usually, it is customary to choose $\alpha,\beta,\gamma,\delta$ so that the Equ. (2) has at least one real (say α) solutions. Note that the Binet form of a sequence satisfying (2) for non-negative integers is valid for all integers n (see [7]).

(3) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n$$

where

$$\begin{aligned} A_1 &= \frac{W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}, \\ A_2 &= \frac{W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ A_3 &= \frac{W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ A_4 &= \frac{W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 1.1.
Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ *is the ordinary generating function of the generalized* (r,s,t,u) *sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} W_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0+(W_1-rW_0)x+(W_2-rW_1-sW_0)x^2+(W_3-rW_2-sW_1-tW_0)x^3}{1-rx-sx^2-tx^3-ux^4}. \quad (4)$$

We next find Binet's formula of generalized (r,s,t,u) numbers $\{W_n\}$ by the use of generating function for W_n.

Theorem 1.1.
(Binet's formula of generalized (r,s,t,u) *numbers)*

$$W_n = \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \quad (5)$$

where

$$\begin{aligned} q_1 &= W_0\alpha^3+(W_1-rW_0)\alpha^2+(W_2-rW_1-sW_0)\alpha+(W_3-rW_2-sW_1-tW_0), \\ q_2 &= W_0\beta^3+(W_1-rW_0)\beta^2+(W_2-rW_1-sW_0)\beta+(W_3-rW_2-sW_1-tW_0), \\ q_3 &= W_0\gamma^3+(W_1-rW_0)\gamma^2+(W_2-rW_1-sW_0)\gamma+(W_3-rW_2-sW_1-tW_0), \\ q_4 &= W_0\delta^3+(W_1-rW_0)\delta^2+(W_2-rW_1-sW_0)\delta+(W_3-rW_2-sW_1-tW_0). \end{aligned}$$

Note that from (3) and (5) we have

$$\begin{aligned} W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0 &= W_0\alpha^3+(W_1-rW_0)\alpha^2+(W_2-rW_1-sW_0)\alpha \\ &\quad +(W_3-rW_2-sW_1-tW_0), \\ W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0 &= W_0\beta^3+(W_1-rW_0)\beta^2+(W_2-rW_1-sW_0)\beta \\ &\quad +(W_3-rW_2-sW_1-tW_0), \\ W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0 &= W_0\gamma^3+(W_1-rW_0)\gamma^2+(W_2-rW_1-sW_0)\gamma \\ &\quad +(W_3-rW_2-sW_1-tW_0), \\ W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0 &= W_0\delta^3+(W_1-rW_0)\delta^2+(W_2-rW_1-sW_0)\delta \\ &\quad +(W_3-rW_2-sW_1-tW_0). \end{aligned}$$

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{6}$$

For matrix formulation (6), see [10]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ r & s & t & u \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix}.$$

Next, we consider two special cases of the generalized (r,s,t,u) sequence $\{W_n\}$ which we call them (r,s,t,u) and Lucas (r,s,t,u) sequences. (r,s,t,u) sequence $\{G_n\}_{n\geq 0}$ and Lucas (r,s,t,u) sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned} G_{n+4} &= rG_{n+3}+sG_{n+2}+tG_{n+1}+uG_n, \\ G_0 &= 0, G_1=1, G_2=r, G_3=r^2+s, \\ H_{n+4} &= rH_{n+3}+sH_{n+2}+tH_{n+1}+uH_n, \\ H_0 &= 4, H_1=r, H_2=2s+r^2, H_3=r^3+3sr+3t. \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{t}{u}G_{-(n-1)}-\frac{s}{u}G_{-(n-2)}-\frac{r}{u}G_{-(n-3)}+\frac{1}{u}G_{-(n-4)}, \tag{7}$$

$$H_{-n} = -\frac{t}{u}H_{-(n-1)}-\frac{s}{u}H_{-(n-2)}-\frac{r}{u}H_{-(n-3)}+\frac{1}{u}H_{-(n-4)}, \tag{8}$$

for $n=1,2,3,\ldots$ respectively. Therefore, recurrences (7) and (8) hold for all integers n.

For more details on the generalized (r,s,t,u) numbers, see Soykan [27].

Some special cases of (r,s,t,u) sequence $\{G_n(0,1,r,r^2+s;r,s,t,u)\}$ and Lucas (r,s,t,u) sequence $\{H_n(4,r,2s+r^2,r^3+3sr+3t;r,s,t,u)\}$ are as follows:

1. $G_n(0,1,1,2;1,1,1,1)=M_n$, Tetranacci sequence,
2. $H_n(4,1,3,7;1,1,1,1)=R_n$, Tetranacci-Lucas sequence,
3. $G_n(0,1,2,5;2,1,1,1)=P_n$, fourth-order Pell sequence,
4. $H_n(4,2,6,17;2,1,1,1)=Q_n$, fourth-order Pell-Lucas sequence,
5. $G_n(0,1,1,2;1,1,1,2)=S_n$, adjusted fourth-order Jacobsthal sequence,
6. $H_n(4,1,3,7;1,1,1,2)=R_n$, modified fourth-order Jacobsthal-Lucas sequence.

For all integers n, (r,s,t,u) and Lucas (r,s,t,u) numbers (using initial conditions in (3) or (5)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}, \\ H_n &= \alpha^n+\beta^n+\gamma^n+\delta^n, \end{aligned}$$

respectively.

Lemma 1.1 gives the following results as particular examples (generating functions of (r,s,t,u), Lucas (r,s,t,u) and modified (r,s,t,u) numbers).

Corollary 1.1.
Generating functions of (r,s,t,u), Lucas (r,s,t,u) and modified (r,s,t,u) numbers are

$$\begin{aligned}\sum_{n=0}^{\infty} G_n x^n &= \frac{x}{1-rx-sx^2-tx^3-ux^4},\\ \sum_{n=0}^{\infty} H_n x^n &= \frac{4-3rx-2sx^2-tx^3}{1-rx-sx^2-tx^3-ux^4},\end{aligned}$$

respectively.

The following theorem shows that the generalized Tetranacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 1.2.
For $n\in\mathbb{Z}$, for the generalized Tetranacci sequence (or generalized (r,s,t,u)-sequence or 4-step Fibonacci sequence) we have the following:

$$\begin{aligned}W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n}+6H_nW_{2n}-3H_n^2W_n+3H_{2n}W_n+W_0H_n^3+2W_0H_{3n}-3W_0H_nH_{2n})\\ &= (-1)^{-n-1}u^{-n}(W_{3n}-H_nW_{2n}+\frac{1}{2}(H_n^2-H_{2n})W_n-\frac{1}{6}(H_n^3+2H_{3n}-3H_{2n}H_n)W_0).\end{aligned}$$

Proof. For the proof, see Soykan [26], Theorem 1.]. □

Using Theorem 1.2, we have the following corollary, see Soykan [[26], Corollary 4].

Corollary 1.2.
For $n\in\mathbb{Z}$, we have

(a) $2(-u)^{n+4}G_{-n} = -(3ru^2+t^3-3stu)^2G_n^3-(2su-t^2)^2G_{n+3}^2G_n-(-rt^2-tu+2rsu)^2G_{n+2}^2G_n-(-st^2+2s^2u+4u^2+rtu)^2G_{n+1}^2G_n+2(3ru^2+t^3-3stu)((-2su+t^2)G_{n+3}+(-rt^2-tu+2rsu)G_{n+2}+(-st^2+2s^2u+4u^2+rtu)G_{n+1})G_n^2+2(2su-t^2)(-rt^2-tu+2rsu)G_{n+3}G_{n+2}G_n+2(2su-t^2)(-st^2+2s^2u+4u^2+rtu)G_{n+3}G_{n+1}G_n-2(-st^2+2s^2u+4u^2+rtu)(-rt^2-tu+2rsu)G_{n+2}G_{n+1}G_n-2G_{3n}u^4+u^2(-2su+t^2)G_{2n+3}G_n+u^2(-rt^2-tu+2rsu)G_{2n+2}G_n+u^2(-st^2+2s^2u+4u^2+rtu)G_{2n+1}G_n-2u^2(2su-t^2)G_{2n}G_{n+3}+2u^2(-rt^2-tu+2rsu)G_{2n}G_{n+2}+2u^2(-st^2+2s^2u+4u^2+rtu)G_{2n}G_{n+1}-3u^2(3ru^2+t^3-3stu)G_{2n}G_n.$

(b) $H_{-n}=\frac{1}{6}(-u)^{-n}\left(H_n^3+2H_{3n}-3H_{2n}H_n\right).$

Note that G_{-n} and H_{-n} can be given as follows by using $G_0=0$ and $H_0=4$ in Theorem 1.2,

$$\begin{aligned}G_{-n} &= \frac{1}{6}(-u)^{-n}(-6G_{3n}+6H_nG_{2n}-3H_n^2G_n+3H_{2n}G_n),\\ H_{-n} &= \frac{1}{6}(-u)^{-n}\left(H_n^3+2H_{3n}-3H_{2n}H_n\right),\end{aligned}$$

respectively.

In this paper, we consider the case $r=1,\ s=1, t=1, u=1$ and in this case we write $V_n=W_n$. So, the generalized Tetranacci sequence $\{V_n\}_{n\geq0}=\{V_n(V_0,V_1,V_2,V_3)\}_{n\geq0}$ is defined by the fourth-order recurrence relation

$$V_n=V_{n-1}+V_{n-2}+V_{n-3}+V_{n-4} \tag{9}$$

with the initial values $V_0=c_0, V_1=c_1, V_2=c_2, V_3=c_3$ not all being zero.

This sequence has been studied by many authors and more details can be found in the extensive literature dedicated to these sequences, see for example [[5],[8],[9],[18],[38],[39]].

The sequence $\{V_n\}_{n\geq0}$ can be extended to negative subscripts by defining

$$V_{-n}=-V_{-(n-1)}-V_{-(n-2)}-V_{-(n-3)}+V_{-(n-4)}$$

for $n=1,2,3,....$ Therefore, recurrence (9) holds for all integer n.

The first few generalized Tetranacci numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few generalized Tetranacci numbers

n	0	1	2	3	4	5	...
V_n	V_0	V_1	V_2	V_3	$V_0+V_1+V_2+V_3$	$V_0+2V_1+2V_2+2V_3$	...
V_{-n}	V_0	$V_3-V_2-V_1-V_0$	$2V_2-V_3$	$2V_1-V_2$	$2V_0-V_1$	$2V_3-2V_2-2V_1-3V_0$	...

(3) can be used to obtain Binet's formula of generalized Tetranacci numbers. Generalized Tetranacci numbers can be expressed, for all integers n, using Binet's formula

$$V_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

where

$$\begin{aligned} p_1 &= V_3-(\beta+\gamma+\delta)V_2+(\beta\gamma+\beta\delta+\gamma\delta)V_1-\beta\gamma\delta V_0, \\ p_2 &= V_3-(\alpha+\gamma+\delta)V_2+(\alpha\gamma+\alpha\delta+\gamma\delta)V_1-\alpha\gamma\delta V_0, \\ p_3 &= V_3-(\alpha+\beta+\delta)V_2+(\alpha\beta+\alpha\delta+\beta\delta)V_1-\alpha\beta\delta V_0, \\ p_4 &= V_3-(\alpha+\beta+\gamma)V_2+(\alpha\beta+\alpha\gamma+\beta\gamma)V_1-\alpha\beta\gamma V_0. \end{aligned}$$

Here, α,β,γ and δ are the roots of the quartic equation

$$x^4-x^3-x^2-x-1=0.$$

Moreover,

$$\begin{aligned} \alpha &= \frac{1}{4}+\frac{1}{2}\omega+\frac{1}{2}\sqrt{\frac{11}{4}-\omega^2+\frac{13}{4}\omega^{-1}}, \\ \beta &= \frac{1}{4}+\frac{1}{2}\omega-\frac{1}{2}\sqrt{\frac{11}{4}-\omega^2+\frac{13}{4}\omega^{-1}}, \\ \gamma &= \frac{1}{4}-\frac{1}{2}\omega+\frac{1}{2}\sqrt{\frac{11}{4}-\omega^2-\frac{13}{4}\omega^{-1}}, \\ \delta &= \frac{1}{4}-\frac{1}{2}\omega-\frac{1}{2}\sqrt{\frac{11}{4}-\omega^2-\frac{13}{4}\omega^{-1}}, \end{aligned}$$

where

$$\omega = \sqrt{\frac{11}{12}+\left(\frac{-65}{54}+\sqrt{\frac{563}{108}}\right)^{1/3}+\left(\frac{-65}{54}-\sqrt{\frac{563}{108}}\right)^{1/3}}.$$

Now, we define two new special cases of the sequence $\{V_n\}$. Tetranacci sequence $\{T_n\}_{n\geq 0}$ and Tetranacci-Lucas sequence $\{K_n\}_{n\geq 0}$, are defined, respectively, by the fourth-order recurrence relations the fourth-order recurrence relations

$$M_n = M_{n-1}+M_{n-2}+M_{n-3}+M_{n-4}, \quad M_0=0, M_1=1, M_2=1, M_3=2 \tag{10}$$

and

$$R_n = R_{n-1}+R_{n-2}+R_{n-3}+R_{n-4}, \quad R_0=4, R_1=1, R_2=3, R_3=7. \tag{11}$$

The sequences $\{M_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} M_{-n} &= -M_{-(n-1)}-M_{-(n-2)}-M_{-(n-3)}+M_{-(n-4)}, \\ R_{-n} &= -R_{-(n-1)}-R_{-(n-2)}-R_{-(n-3)}+R_{-(n-4)}, \end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (10) and (11) hold for all integer n. Next, we present the first few values of the Tetranacci and Tetranacci-Lucas numbers with positive and negative subscripts, in the following Table 3.

M_n is the sequence A000078 in [?], R_n is the sequence A073817 in [?]. For all integers n, Tetranacci and Tetranacci-Lucas numbers (using initial conditions in ((10)-(11)) can be expressed using Binet's formulas as

$$\begin{aligned} M_n &= \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}, \\ R_n &= \alpha^n+\beta^n+\gamma^n+\delta^n, \end{aligned}$$

respectively.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Table 3. A few values of Tetranacci and Tetranacci-Lucas numbers.

n	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
M_n	1	-3	2	0	0	-1	1	0	0	0	1	1	2	4	8	15	29	56	108
R_n	-19	15	-1	-1	-6	7	-1	-1	-1	4	1	3	7	15	26	51	99	191	367

Lemma 1.2.
Suppose that $f_{V_n}(x)=\sum_{n=0}^{\infty}V_nx^n$ is the ordinary generating function of the generalized Tetranacci sequence $\{V_n\}_{n\geq 0}$. Then $f_{V_n}(x)$ is given by

$$f_{V_n}(x)=\frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3}{1-x-x^2-x^3-x^4}. \tag{12}$$

The previous Lemma gives the following results as particular examples: generating function of the Tetranacci sequence M_n is

$$f_{M_n}(x)=\sum_{n=0}^{\infty}M_nx^n=\frac{x}{1-x-x^2-x^3-x^4}$$

and generating function of the Tetranacci-Lucas sequence R_n is

$$f_{R_n}(x)=\sum_{n=0}^{\infty}R_nx^n=\frac{4-3x-2x^2-x^3}{1-x-x^2-x^3-x^4}.$$

2. Binomial Transform of the Generalized Tetranacci Sequence V_n

In [13], p. 137], Knuth introduced the idea of the binomial transform. Given a sequence of numbers (a_n), its binomial transform $(\hat{a}_n)$ may be defined by the rule

$$\hat{a}_n=\sum_{i=0}^{n}\binom{n}{i}a_i,\ \text{ with inversion } a_n=\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\hat{a}_i,$$

or, in the symmetric version

$$\hat{a}_n=\sum_{i=0}^{n}\binom{n}{i}(-1)^{i+1}a_i,\ \text{ with inversion } a_n=\sum_{i=0}^{n}\binom{n}{i}(-1)^{i+1}\hat{a}_i.$$

For more information on binomial transform, see, for example, [4],[6],[16],[37] and references therein. For recent works on binomial transform of well-known sequences, see for example, [2],[3],[11],[12],[14],[29],[30],[31],[32],[33],[34],[35],[36].

In this section, we define the binomial transform of the generalized Tetranacci sequence V_n and as special cases the binomial transform of the Tetranacci and Tetranacci-Lucas sequences will be introduced.

Definition 2.1.
The binomial transform of the generalized Tetranacci sequence V_n is defined by

$$b_n=\widehat{V}_n=\sum_{i=0}^{n}\binom{n}{i}V_i.$$

The few terms of b_n are

$$\begin{aligned}
b_0 &= \sum_{i=0}^{0}\binom{0}{i}V_i=V_0,\\
b_1 &= \sum_{i=0}^{1}\binom{1}{i}V_i=V_0+V_1,\\
b_2 &= \sum_{i=0}^{2}\binom{2}{i}V_i=V_0+2V_1+V_2,\\
b_3 &= \sum_{i=0}^{3}\binom{3}{i}V_i=V_0+3V_1+3V_2+V_3.
\end{aligned}$$

Translated to matrix language, b_n has the nice (lower-triangular matrix) form

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ \vdots \end{pmatrix}.$$

As special cases of $b_n = \widehat{V}_n$, the binomial transforms of the Tetranacci and Tetranacci-Lucas sequences are defined as follows: The binomial transform of the Tetranacci sequence M_n is

$$\widehat{M}_n = \sum_{i=0}^{n} \binom{n}{i} M_i,$$

and the binomial transform of the Tetranacci-Lucas sequence R_n is

$$\widehat{R}_n = \sum_{i=0}^{n} \binom{n}{i} R_i.$$

Lemma 2.1.
For $n \geq 0$, the binomial transform of the generalized Tetranacci sequence V_n satisfies the following relation:

$$b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (V_i + V_{i+1}).$$

Proof. We use the following well-known identity:

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}.$$

Note also that

$$\binom{n+1}{0} = \binom{n}{0} = 1 \text{ and } \binom{n}{n+1} = 0.$$

Then

$$\begin{aligned} b_{n+1} &= V_0 + \sum_{i=1}^{n+1} \binom{n+1}{i} V_i \\ &= V_0 + \sum_{i=1}^{n+1} \binom{n}{i} V_i + \sum_{i=1}^{n+1} \binom{n}{i-1} V_i \\ &= V_0 + \sum_{i=1}^{n} \binom{n}{i} V_i + \sum_{i=0}^{n} \binom{n}{i} V_{i+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} V_i + \sum_{i=0}^{n} \binom{n}{i} V_{i+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} (V_i + V_{i+1}). \end{aligned}$$

This completes the proof. □

Remark 2.1.
From the last Lemma, we see that

$$b_{n+1} = b_n + \sum_{i=0}^{n} \binom{n}{i} V_{i+1}.$$

The following theorem gives recurrent relations of the binomial transform of the generalized Tetranacci sequence.

Theorem 2.1.
For $n \geq 0$, the binomial transform of the generalized Tetranacci sequence V_n satisfies the following recurrence relation:

$$b_{n+4} = 5b_{n+3} - 8b_{n+2} + 6b_{n+1} - b_n \tag{13}$$

Proof. To show (13), writing

$$b_{n+4} = r_1 \times b_{n+3} + s_1 \times b_{n+2} + t_1 \times b_{n+1} + u_1 \times b_n$$

and taking the values $n = 0, 1, 2, 3$ and then solving the system of equations

$$\begin{aligned} b_4 &= r_1 \times b_3 + s_1 \times b_2 + t_1 \times b_1 + u_1 \times b_0 \\ b_5 &= r_1 \times b_4 + s_1 \times b_3 + t_1 \times b_2 + u_1 \times b_1 \\ b_6 &= r_1 \times b_5 + s_1 \times b_4 + t_1 \times b_3 + u_1 \times b_2 \\ b_7 &= r_1 \times b_6 + s_1 \times b_5 + t_1 \times b_4 + u_1 \times b_3 \end{aligned}$$

we find that $r_1 = 5, s_1 = -8, t_1 = 6, u_1 = -1$. □

The sequence $\{b_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$b_{-n} = 6b_{-(n-1)} - 8b_{-(n-2)} + 5b_{-(n-3)} - b_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (13) holds for all integer n.

Note that the recurence relation (13) is independent from initial values. So,

$$\begin{aligned} \widehat{M}_{n+4} &= 5\widehat{M}_{n+3} - 8\widehat{M}_{n+2} + 6\widehat{M}_{n+1} - \widehat{M}_n, \\ \widehat{R}_{n+4} &= 5\widehat{R}_{n+3} - 8\widehat{R}_{n+2} + 6\widehat{R}_{n+1} - \widehat{R}_n. \end{aligned}$$

The first few terms of the binomial transform of the generalized Tetranacci sequence with positive subscript and negative subscript are given in the following Table 4. The first few terms of the binomial transform numbers of the Tetranacci and Tetranacci-Lucas sequences with positive subscript and negative subscript are given in the following Table 5.

Table 4. A few binomial transform (terms) of the generalized Tetranacci sequence.

n	b_n	b_{-n}
0	V_0	V_0
1	$V_0 + V_1$	$2V_0 - V_1 + 2V_2 - V_3$
2	$V_0 + 2V_1 + V_2$	$8V_0 - 3V_1 + 11V_2 - 6V_3$
3	$V_0 + 3V_1 + 3V_2 + V_3$	$36V_0 - 11V_1 + 50V_2 - 28V_3$
4	$2V_0 + 5V_1 + 7V_2 + 5V_3$	$161V_0 - 47V_1 + 222V_2 - 125V_3$
5	$7V_0 + 12V_1 + 17V_2 + 17V_3$	$716V_0 - 208V_1 + 985V_2 - 555V_3$
6	$24V_0 + 36V_1 + 46V_2 + 51V_3$	$3180V_0 - 924V_1 + 4373V_2 - 2464V_3$
7	$75V_0 + 111V_1 + 133V_2 + 148V_3$	$14121V_0 - 4104V_1 + 19418V_2 - 10941V_3$
8	$223V_0 + 334V_1 + 392V_2 + 429V_3$	$62705V_0 - 18225V_1 + 86227V_2 - 48584V_3$
9	$652V_0 + 986V_1 + 1155V_2 + 1250V_3$	$278446V_0 - 80930V_1 + 382898V_2 - 215741V_3$
10	$1902V_0 + 2888V_1 + 3391V_2 + 3655V_3$	$1236461V_0 - 359376V_1 + 1700289V_2 - 958015V_3$
11	$5557V_0 + 8445V_1 + 9934V_2 + 10701V_3$	$5490602V_0 - 1595837V_1 + 7550267V_2 - 4254141V_3$
12	$16258V_0 + 24703V_1 + 29080V_2 + 31336V_3$	$24381449V_0 - 7086439V_1 + 33527553V_2 - 18890847V_3$

Table 5. A few binomial transform (terms).

n	0	1	2	3	4	5	6	7	8	9	10	11
$\widehat{M}_n$	0	1	3	8	22	63	184	540	1584	4641	13589	39781
$\widehat{M}_{-n}$	0	−1	−4	−17	−75	−333	−1479	−6568	−29166	−129514	−575117	−2553852
$\widehat{R}_n$	4	5	9	23	69	210	627	1846	5405	15809	46254	135382
$\widehat{R}_{-n}$	4	6	20	87	388	1726	7667	34047	151188	671361	2981230	13238385

(3) can be used to obtain Binet's formula of the binomial transform of generalized Tetranacci numbers. Binet's formula of the binomial transform of generalized Tetranacci numbers can be given as

$$\begin{aligned} b_n &= \frac{C_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)} + \frac{C_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)} \\ &\quad + \frac{C_3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)} + \frac{C_4\theta_4^n}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)} \end{aligned} \tag{14}$$

where

$$\begin{aligned}
C_1 &= b_3-(\theta_2+\theta_3+\theta_4)b_2+(\theta_2\theta_3+\theta_2\theta_4+\theta_3\theta_4)b_1-\theta_2\theta_3\theta_4 b_0 \\
&= (V_0+3V_1+3V_2+V_3)-(\theta_2+\theta_3+\theta_4)(V_0+2V_1+V_2)+(\theta_2\theta_3+\theta_2\theta_4+\theta_3\theta_4)(V_0+V_1)-\theta_2\theta_3\theta_4 V_0, \\
C_2 &= b_3-(\theta_1+\theta_3+\theta_4)b_2+(\theta_1\theta_3+\theta_1\theta_4+\theta_3\theta_4)b_1-\theta_1\theta_3\theta_4 b_0 \\
&= (V_0+3V_1+3V_2+V_3)-(\theta_1+\theta_3+\theta_4)(V_0+2V_1+V_2)+(\theta_1\theta_3+\theta_1\theta_4+\theta_3\theta_4)(V_0+V_1)-\theta_1\theta_3\theta_4 V_0, \\
C_3 &= b_3-(\theta_1+\theta_2+\theta_4)b_2+(\theta_1\theta_2+\theta_1\theta_4+\theta_2\theta_4)b_1-\theta_1\theta_2\theta_4 b_0 \\
&= (V_0+3V_1+3V_2+V_3)-(\theta_1+\theta_2+\theta_4)(V_0+2V_1+V_2)+(\theta_1\theta_2+\theta_1\theta_4+\theta_2\theta_4)(V_0+V_1)-\theta_1\theta_2\theta_4 V_0, \\
C_4 &= b_3-(\theta_1+\theta_2+\theta_3)b_2+(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)b_1-\theta_1\theta_2\theta_3 b_0 \\
&= (V_0+3V_1+3V_2+V_3)-(\theta_1+\theta_2+\theta_3)(V_0+2V_1+V_2)+(\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3)(V_0+V_1)-\theta_1\theta_2\theta_3 V_0,
\end{aligned}$$

Here, $\theta_1,\theta_2,\theta_3$ and θ_4 are the roots of the quartic equation $x^4-5x^3+8x^2-6x+1=0$. Moreover,

$$\begin{aligned}
\theta_1 &= -\frac{g_1}{2}+\sqrt{\frac{g_1^2}{4}-h_1} \cong 2.92756197548293 \\
\theta_2 &= -\frac{g_1}{2}-\sqrt{\frac{g_1^2}{4}-h_1} \cong 0.225195886784566 \\
\theta_3 &= -\frac{g_2}{2}+\sqrt{\frac{g_2^2}{4}-h_2} \cong 0.923621068866254+0.814703647170387i \\
\theta_4 &= -\frac{g_2}{2}-\sqrt{\frac{g_2^2}{4}-h_2} \cong 0.923621068866254-0.814703647170387i
\end{aligned}$$

where

$$\begin{aligned}
g_1 &= -\frac{5}{2}-\sqrt{\frac{25}{4}-8+y_1} \\
g_2 &= -\frac{5}{2}+\sqrt{\frac{25}{4}-8+y_1} \\
h_1 &= \frac{y_1}{2}-\sqrt{\frac{y_1^2}{4}-1} \\
h_2 &= \frac{y_1}{2}+\sqrt{\frac{y_1^2}{4}-1}
\end{aligned}$$

and $y_1=\frac{8}{3}+\left(-\frac{65}{54}+\sqrt{\frac{563}{108}}\right)^{1/3}-\left(\frac{65}{54}+\sqrt{\frac{563}{108}}\right)^{1/3}$ as the greatest real solution of the resolvent cubic equation

$y^3-8y^2+26y-29=0.$

Note that

$$\begin{aligned}
\theta_1+\theta_2+\theta_3+\theta_4 &= 5, \\
\theta_1\theta_2+\theta_1\theta_3+\theta_1\theta_4+\theta_2\theta_3+\theta_2\theta_4+\theta_3\theta_4 &= 8, \\
\theta_1\theta_2\theta_3+\theta_1\theta_2\theta_4+\theta_1\theta_3\theta_4+\theta_2\theta_3\theta_4 &= 6, \\
\theta_1\theta_2\theta_3\theta_4 &= 1.
\end{aligned}$$

For all integers n, (Binet's formulas of) binomial transforms of Tetranacci and Tetranacci-Lucas numbers (using initial conditions in (14)) can be expressed using Binet's formulas as

$$\begin{aligned}
\widehat{M}_n &= \frac{(\theta_1-1)^2\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)}+\frac{(\theta_2-1)^2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)} \\
&\quad +\frac{(\theta_3-1)^2\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)}+\frac{(\theta_4-1)^2\theta_4^n}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)}, \\
\widehat{R}_n &= \theta_1^n+\theta_2^n+\theta_3^n+\theta_4^n,
\end{aligned}$$

respectively.

3. Generating Functions and Obtaining Binet Formula of Binomial Transform From Generating Function

The generating function of the binomial transform of the generalized Tetranacci sequence V_n is a power series centered at the origin whose coefficients are the binomial transform of the generalized Tetranacci sequence.

Next, we give the ordinary generating function $f_{b_n}(x) = \sum_{n=0}^{\infty} b_n x^n$ of the sequence b_n.

Lemma 3.1.
Suppose that $f_{b_n}(x) = \sum_{n=0}^{\infty} b_n x^n$ is the ordinary generating function of the binomial transform of the generalized Tetranacci sequence $\{V_n\}_{n\geq 0}$. Then, $f_{b_n}(x)$ is given by

$$f_{b_n}(x) = \frac{V_0 + (V_1 - 4V_0)x + (V_2 - 3V_1 + 4V_0)x^2 + (V_3 - 2V_2 + V_1 - 2V_0)x^3}{1 - 5x + 8x^2 - 6x^3 + x^4}. \tag{15}$$

Proof. Using Lemma 1.1, we obtain

$$\begin{aligned} f_{b_n}(x) &= \frac{b_0 + (b_1 - 5b_0)x + (b_2 - 5b_1 + 8b_0)x^2 + (b_3 - 5b_2 + 8b_1 - 6b_0)x^3}{1 - 5x + 8x^2 - 6x^3 + x^4} \\ &= \frac{V_0 + (V_1 - 4V_0)x + (V_2 - 3V_1 + 4V_0)x^2 + (V_3 - 2V_2 + V_1 - 2V_0)x^3}{1 - 5x + 8x^2 - 6x^3 + x^4} \end{aligned}$$

where

$$\begin{aligned} b_0 &= V_0, \\ b_1 &= V_0 + V_1, \\ b_2 &= V_0 + 2V_1 + V_2, \\ b_3 &= V_0 + 3V_1 + 3V_2 + V_3. \end{aligned}$$

□

Note that P. Barry shows in [1] that if $A(x)$ is the generating function of the sequence $\{a_n\}$, then

$$S(x) = \frac{1}{1-x} A(\frac{x}{1-x})$$

is the generating function of the sequence $\{b_n\}$ with $b_n = \sum_{i=0}^{n} \binom{n}{i} a_i$. In our case, since

$$A(x) = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2}{1 - x - x^2 - x^3}, \quad \text{see Lemma 1.2},$$

we obtain

$$\begin{aligned} S(x) &= \frac{1}{1-x} A(\frac{x}{1-x}) \\ &= \frac{1}{1-x} \frac{V_0 + (V_1 - V_0)\frac{x}{1-x} + (V_2 - V_1 - V_0)\left(\frac{x}{1-x}\right)^2 + (V_3 - V_2 - V_1 - V_0)\left(\frac{x}{1-x}\right)^3}{1 - \frac{x}{1-x} - \left(\frac{x}{1-x}\right)^2 - \left(\frac{x}{1-x}\right)^3 - \left(\frac{x}{1-x}\right)^4} \\ &= \frac{V_0 + (V_1 - 4V_0)x + (V_2 - 3V_1 + 4V_0)x^2 + (V_3 - 2V_2 + V_1 - 2V_0)x^3}{1 - 5x + 8x^2 - 6x^3 + x^4}. \end{aligned}$$

The previous lemma gives the following results as particular examples.

Corollary 3.1.
Generating functions of the binomial transform of the Tetranacci, Tetranacci-Lucas numbers are

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{M}_n x^n &= \frac{x - 2x^2 + x^3}{1 - 5x + 8x^2 - 6x^3 + x^4}, \\ \sum_{n=0}^{\infty} \widehat{R}_n x^n &= \frac{4 - 15x + 16x^2 - 6x^3}{1 - 5x + 8x^2 - 6x^3 + x^4}, \end{aligned}$$

respectively.

4. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Tetranacci sequence $\{W_n\}$.

Theorem 4.1 (Simson Formula of Generalized Tetranacci Numbers).
For all integers n, we have

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \tag{16}$$

Proof. (16) is given in Soykan [[28], Theorem 3.1]. □

Taking $\{W_n\} = \{b_n\}$ in the above theorem and considering $b_{n+4} = 5b_{n+3} - 8b_{n+2} + 6b_{n+1} - b_n$, $r = 5, s = -8, t = 6, u = -1$, we have the following proposition.

Proposition 4.1.
For all integers n, Simson formula of binomial transforms of generalized Tetranacci numbers is given as

$$\begin{vmatrix} b_{n+3} & b_{n+2} & b_{n+1} & b_n \\ b_{n+2} & b_{n+1} & b_n & b_{n-1} \\ b_{n+1} & b_n & b_{n-1} & b_{n-2} \\ b_n & b_{n-1} & b_{n-2} & b_{n-3} \end{vmatrix} = \begin{vmatrix} b_3 & b_2 & b_1 & b_0 \\ b_2 & b_1 & b_0 & b_{-1} \\ b_1 & b_0 & b_{-1} & b_{-2} \\ b_0 & b_{-1} & b_{-2} & b_{-3} \end{vmatrix}.$$

The previous proposition gives the following results as particular examples.

Corollary 4.1.
For all integers n, Simson formula of binomial transforms of the Tetranacci and Tetranacci-Lucas numbers are given as

$$\begin{vmatrix} \widehat{M}_{n+3} & \widehat{M}_{n+2} & \widehat{M}_{n+1} & \widehat{M}_n \\ \widehat{M}_{n+2} & \widehat{M}_{n+1} & \widehat{M}_n & \widehat{M}_{n-1} \\ \widehat{M}_{n+1} & \widehat{M}_n & \widehat{M}_{n-1} & \widehat{M}_{n-2} \\ \widehat{M}_n & \widehat{M}_{n-1} & \widehat{M}_{n-2} & \widehat{M}_{n-3} \end{vmatrix} = 1,$$

$$\begin{vmatrix} \widehat{R}_{n+3} & \widehat{R}_{n+2} & \widehat{R}_{n+1} & \widehat{R}_n \\ \widehat{R}_{n+2} & \widehat{R}_{n+1} & \widehat{R}_n & \widehat{R}_{n-1} \\ \widehat{R}_{n+1} & \widehat{R}_n & \widehat{R}_{n-1} & \widehat{R}_{n-2} \\ \widehat{R}_n & \widehat{R}_{n-1} & \widehat{R}_{n-2} & \widehat{R}_{n-3} \end{vmatrix} = -563,$$

respectively.

5. Some Identities

In this section, we obtain some identities of binomial transforms of generalized Tetranacci, Tetranacci and Tetranacci-Lucas numbers. First, we present a few basic relations between $\{b_n\}$ and $\{\widehat{M}_n\}$.

Lemma 5.1.
The following equalities are true:

(a) $b_n = (14V_0 - 3V_1 + 19V_2 - 11V_3)\widehat{M}_{n+5} + (15V_1 - 67V_0 - 90V_2 + 52V_3)\widehat{M}_{n+4} + (97V_0 - 24V_1 + 129V_2 - 74V_3)\widehat{M}_{n+3} + (17V_1 - 61V_0 - 83V_2 + 47V_3)\widehat{M}_{n+2}$.

(b) $b_n = (3V_0+5V_2-3V_3)\widehat{M}_{n+4} + (14V_3-23V_2-15V_0)\widehat{M}_{n+3} + (23V_0-V_1+31V_2-19V_3)\widehat{M}_{n+2} + (3V_1-14V_0-19V_2+11V_3)\widehat{M}_{n+1}$.

(c) $b_n = (2V_2-V_3)\widehat{M}_{n+3} + (5V_3-V_1-9V_2-V_0)\widehat{M}_{n+2} + (4V_0+3V_1+11V_2-7V_3)\widehat{M}_{n+1} + (3V_3-5V_2-3V_0)\widehat{M}_n$.

(d) $b_n = (V_2-V_1-V_0)\widehat{M}_{n+2} + (4V_0+3V_1-5V_2+V_3)\widehat{M}_{n+1} + (7V_2-3V_0-3V_3)\times\widehat{M}_n + (V_3-2V_2)\widehat{M}_{n-1}$.

(e) $b_n = (V_3-2V_1-V_0)\widehat{M}_{n+1} + (5V_0+8V_1-V_2-3V_3)\widehat{M}_n + (4V_2-6V_1-6V_0+V_3)\widehat{M}_{n-1} + (V_0+V_1-V_2)\widehat{M}_{n-2}$.

Proof. Writing

$$b_n = a\times\widehat{M}_{n+5} + b\times\widehat{M}_{n+4} + c\times\widehat{M}_{n+3} + d\times\widehat{M}_{n+2}$$

and solving the system of equations

$$\begin{array}{rcl} b_0 &=& a\times\widehat{M}_5 + b\times\widehat{M}_4 + c\times\widehat{M}_3 + d\times\widehat{M}_2 \\ b_1 &=& a\times\widehat{M}_6 + b\times\widehat{M}_5 + c\times\widehat{M}_4 + d\times\widehat{M}_3 \\ b_2 &=& a\times\widehat{M}_7 + b\times\widehat{M}_6 + c\times\widehat{M}_5 + d\times\widehat{M}_4 \\ b_3 &=& a\times\widehat{M}_8 + b\times\widehat{M}_7 + c\times\widehat{M}_6 + d\times\widehat{M}_5 \end{array}$$

we find that $a = 14V_0-3V_1+19V_2-11V_3$, $b = 15V_1-67V_0-90V_2+52V_3$, $c = 97V_0-24V_1+129V_2-74V_3$, $d = 17V_1-61V_0-83V_2+47V_3$.

The other equalities can be proved similarly. □

Now, we give a few basic relations between $\{b_n\}$ and $\{\widehat{R}_n\}$.

Lemma 5.2.
The following equalities are true:

(a) $563b_n = -(1441V_0-403V_1+2104V_2-1185V_3)\widehat{R}_{n+5} + (6949V_0-1868V_1+10004V_2-5659V_3)\widehat{R}_{n+4} - (10238V_0-2646V_1+14355V_2-8166V_3)\widehat{R}_{n+3} + (6574V_0-1844V_1+9081V_2-5133V_3)\widehat{R}_{n+2}$.

(b) $563b_n = -(256V_0-147V_1+516V_2-266V_3)\widehat{R}_{n+4} + (1290V_0-578V_1+2477V_2-1314V_3)\widehat{R}_{n+3} - (2072V_0-574V_1+3543V_2-1977V_3)\widehat{R}_{n+2} + (1441V_0-403V_1+2104V_2-1185V_3)\widehat{R}_{n+1}$.

(c) $563b_n = (10V_0+157V_1-103V_2+16V_3)\widehat{R}_{n+3} - (24V_0+602V_1-585V_2+151V_3)\widehat{R}_{n+2} - (95V_0-479V_1+992V_2-411V_3)\widehat{R}_{n+1} + (256V_0-147V_1+516V_2-266V_3)\widehat{R}_n$.

(d) $563b_n = (26V_0+183V_1+70V_2-71V_3)\widehat{R}_{n+2} - (175V_0+777V_1+168V_2-283V_3)\widehat{R}_{n+1} + (316V_0+795V_1-102V_2-170V_3)\widehat{R}_n - (10V_0+157V_1-103V_2+16V_3)\widehat{R}_{n-1}$.

(e) $563b_n = -(45V_0-138V_1-182V_2+72V_3)\widehat{R}_{n+1} + (108V_0-669V_1-662V_2+398V_3)\widehat{R}_n + (146V_0+941V_1+523V_2-442V_3)\widehat{R}_{n-1} - (26V_0+183V_1+70V_2-71V_3)\widehat{R}_{n-2}$.

Next, we present a few basic relations between $\{\widehat{R}_n\}$ and $\{\widehat{M}_n\}$.

Lemma 5.3.
The following equalities are true:

$$\begin{array}{rcl} \widehat{R}_n &=& 33\widehat{M}_{n+5} - 159\widehat{M}_{n+4} + 233\widehat{M}_{n+3} - 147\widehat{M}_{n+2}, \\ \widehat{R}_n &=& 6\widehat{M}_{n+4} - 31\widehat{M}_{n+3} + 51\widehat{M}_{n+2} - 33\widehat{M}_{n+1}, \\ \widehat{R}_n &=& -\widehat{M}_{n+3} + 3\widehat{M}_{n+2} + 3\widehat{M}_{n+1} - 6\widehat{M}_n, \\ \widehat{R}_n &=& -2\widehat{M}_{n+2} + 11\widehat{M}_{n+1} - 12\widehat{M}_n + \widehat{M}_{n-1}, \\ \widehat{R}_n &=& \widehat{M}_{n+1} + 4\widehat{M}_n - 11\widehat{M}_{n-1} + 2\widehat{M}_{n-2}, \end{array}$$

and

$$\begin{array}{rcl} 563\widehat{M}_n &=& 669\widehat{R}_{n+5} - 3182\widehat{R}_{n+4} + 4623\widehat{R}_{n+3} - 3029\widehat{R}_{n+2}, \\ 563\widehat{M}_n &=& 163\widehat{R}_{n+4} - 729\widehat{R}_{n+3} + 985\widehat{R}_{n+2} - 669\widehat{R}_{n+1}, \\ 563\widehat{M}_n &=& 86\widehat{R}_{n+3} - 319\widehat{R}_{n+2} + 309\widehat{R}_{n+1} - 163\widehat{R}_n, \\ 563\widehat{M}_n &=& 111\widehat{R}_{n+2} - 379\widehat{R}_{n+1} + 353\widehat{R}_n - 86\widehat{R}_{n-1}, \\ 563\widehat{M}_n &=& 176\widehat{R}_{n+1} - 535\widehat{R}_n + 580\widehat{R}_{n-1} - 111\widehat{R}_{n-2}. \end{array}$$

6. On the Recurrence Properties of Binomial Transform of the Generalized Tetranacci Sequence

Taking $r_1 = 5, s_1 = -8, t_1 = 6, u_1 = -1$ and $H_n = \widehat{R}_n$ in Theorem 1.2, we obtain the following Proposition.

Proposition 6.1.
For $n \in \mathbb{Z}$, binomial Transform of the generalized Tetranacci sequence have the following identity:

$$\begin{aligned} b_{-n} &= \frac{1}{6}(-6b_{3n} + 6\widehat{R}_n b_{2n} - 3\widehat{R}_n^2 b_n + 3\widehat{R}_{2n} b_n + b_0\widehat{R}_n^3 + 2b_0\widehat{R}_{3n} - 3b_0\widehat{R}_n\widehat{R}_{2n}) \\ &= -(b_{3n} - \widehat{R}_n b_{2n} + \frac{1}{2}(\widehat{R}_n^2 - \widehat{R}_{2n})b_n - \frac{1}{6}(\widehat{R}_n^3 + 2\widehat{R}_{3n} - 3\widehat{R}_{2n}\widehat{R}_n)b_0). \end{aligned}$$

Using Proposition 6.1 (and Corollary 1.2 (b)), we obtain the following corollary which gives the connection between the special cases of binomial transform of generalized Tetranacci sequence at the positive index and the negative index: for binomial transform of Tetranacci, Tetranacci-Lucas numbers: take $b_n = \widehat{M}_n$ with $\widehat{M}_0 = 0, \widehat{M}_1 = 1, \widehat{M}_2 = 3, \widehat{M}_3 = 8$, take $b_n = \widehat{R}_n$ with $\widehat{R}_0 = 4, \widehat{R}_1 = 5$, $\widehat{R}_2 = 9, \widehat{R}_3 = 23$, respectively. Note that in this case we have $H_n = \widehat{R}_n$. Note also that $G_n \neq \widehat{M}_n$.

Corollary 6.1.
For $n \in \mathbb{Z}$, we have the following recurrence relations:

(a) *Recurrence relations of binomial transforms of Tetranacci numbers (take $b_n = \widehat{M}_n$ in Proposition 6.1):*

$$\begin{aligned} \widehat{M}_{-n} &= \frac{1}{6}(-6\widehat{M}_{3n} + 6\widehat{R}_n\widehat{M}_{2n} - 3\widehat{R}_n^2\widehat{M}_n + 3\widehat{R}_{2n}\widehat{M}_n + \widehat{M}_0\widehat{R}_n^3 + 2\widehat{M}_0\widehat{R}_{3n} - 3\widehat{M}_0\widehat{R}_n\widehat{R}_{2n}) \\ &= \frac{1}{6}(-6\widehat{M}_{3n} + 6\widehat{R}_n\widehat{M}_{2n} - 3\widehat{R}_n^2\widehat{M}_n + 3\widehat{R}_{2n}\widehat{M}_n). \end{aligned}$$

(b) *Recurrence relations of binomial transforms of Tetranacci-Lucas numbers (take $b_n = \widehat{R}_n$ in Proposition 6.1 or take $H_n = \widehat{R}_n$ in Corollary 1.2 (b)):*

$$\widehat{R}_{-n} = \frac{1}{6}\left(\widehat{R}_n^3 + 2\widehat{R}_{3n} - 3\widehat{R}_{2n}\widehat{R}_n\right).$$

7. Sum Formulas

7.1. Sums of Terms with Positive Subscripts

The following proposition presents some formulas of binomial transform of generalized Tetranacci numbers with positive subscripts.

Proposition 7.1.
If $r = 5, s = -8, t = 6, u = -1$ then for $n \geq 0$, we have the following formulas:

(a) $\sum_{k=0}^{n} b_k = b_{n+4} - 4b_{n+3} + 4b_{n+2} - 2b_{n+1} - b_3 + 4b_2 - 4b_1 + 2b_0$.

(b) $\sum_{k=0}^{n} b_{2k} = \frac{1}{21}(10b_{2n+2} - 39b_{2n+1} + 56b_{2n} - 11b_{2n-1} - 11b_3 + 45b_2 - 49b_1 + 31b_0)$.

(c) $\sum_{k=0}^{n} b_{2k+1} = \frac{1}{21}(11b_{2n+2} - 24b_{2n+1} + 49b_{2n} - 10b_{2n-1} - 10b_3 + 39b_2 - 35b_1 + 11b_0)$.

Proof. Take $r = 5, s = -8, t = 6, u = -1$ in Theorem 2.1 in [22] (or take $x = 1, r = 5, s = -8, t = 6, u = -1$ in Theorem 1 in [23]).

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of Tetranacci numbers (take $b_n = \widehat{M}_n$ with $\widehat{M}_0 = 0, \widehat{M}_1 = 1, \widehat{M}_2 = 3, \widehat{M}_3 = 8$).

Corollary 7.1.
For $n \geq 0$, we have the following formulas:

(a) $\sum_{k=0}^{n} \widehat{M}_k = \widehat{M}_{n+4} - 4\widehat{M}_{n+3} + 4\widehat{M}_{n+2} - 2\widehat{M}_{n+1}$.

(b) $\sum_{k=0}^{n} \widehat{M}_{2k} = \frac{1}{21}(10\widehat{M}_{2n+2} - 39\widehat{M}_{2n+1} + 56\widehat{M}_{2n} - 11\widehat{M}_{2n-1} - 2)$.

(c) $\sum_{k=0}^{n} \widehat{M}_{2k+1} = \frac{1}{21}(11\widehat{M}_{2n+2} - 24\widehat{M}_{2n+1} + 49\widehat{M}_{2n} - 10\widehat{M}_{2n-1} + 2)$.

Taking $b_n = \widehat{R}_n$ with $\widehat{R}_0 = 4, \widehat{R}_1 = 5, \widehat{R}_2 = 9, \widehat{R}_3 = 23$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of Tetranacci-Lucas numbers.

Corollary 7.2.
For $n \geq 0$, we have the following formulas:

(a) $\sum_{k=0}^{n} \widehat{R}_k = \widehat{R}_{n+4} - 4\widehat{R}_{n+3} + 4\widehat{R}_{n+2} - 2\widehat{R}_{n+1} + 1$.

(b) $\sum_{k=0}^{n} \widehat{R}_{2k} = \frac{1}{21}(10\widehat{R}_{2n+2} - 39\widehat{R}_{2n+1} + 56\widehat{R}_{2n} - 11\widehat{R}_{2n-1} + 31)$.

(c) $\sum_{k=0}^{n} \widehat{R}_{2k+1} = \frac{1}{21}(11\widehat{R}_{2n+2} - 24\widehat{R}_{2n+1} + 49\widehat{R}_{2n} - 10\widehat{R}_{2n-1} - 10)$.

7.2. Sums of Terms with Negative Subscripts

The following proposition presents some formulas of binomial transform of generalized Tetranacci numbers with negative subscripts.

Proposition 7.2.
If $r = 5, s = -8, t = 6, u = -1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} b_{-k} = -b_{-n+3} + 4b_{-n+2} - 4b_{-n+1} + 2b_{-n} + b_3 - 4b_2 + 4b_1 - 2b_0$.

(b) $\sum_{k=1}^{n} b_{-2k} = \frac{1}{21}(-10b_{-2n+2} + 39b_{-2n+1} - 35b_{-2n} + 11b_{-2n-1} + 11b_3 - 45b_2 + 49b_1 - 31b_0)$.

(c) $\sum_{k=1}^{n} b_{-2k+1} = \frac{1}{21}(-11b_{-2n+2} + 45b_{-2n+1} - 49b_{-2n} + 10b_{-2n-1} + 10b_3 - 39b_2 + 35b_1 - 11b_0)$.

Proof. Take $r = 5, s = -8, t = 6, u = -1$ in Theorem 3.1 in [22] or (or take $x = 1, r = 5, s = -8, t = 6, u = -1$ in Theorem 8 in [23]).

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of Tetranacci numbers (take $b_n = \widehat{M}_n$ with $\widehat{M}_0 = 0, \widehat{M}_1 = 1, \widehat{M}_2 = 3, \widehat{M}_3 = 8$).

Corollary 7.3.
For $n \geq 1$, binomial transform of Tetranacci numbers have the following properties.

(a) $\sum_{k=1}^{n} \widehat{M}_{-k} = -\widehat{M}_{-n+3} + 4\widehat{M}_{-n+2} - 4\widehat{M}_{-n+1} + 2\widehat{M}_{-n}$.

(b) $\sum_{k=1}^{n} \widehat{M}_{-2k} = \frac{1}{21}(-10\widehat{M}_{-2n+2} + 39\widehat{M}_{-2n+1} - 35\widehat{M}_{-2n} + 11\widehat{M}_{-2n-1} + 2)$.

(c) $\sum_{k=1}^{n} \widehat{M}_{-2k+1} = \frac{1}{21}(-11\widehat{M}_{-2n+2} + 45\widehat{M}_{-2n+1} - 49\widehat{M}_{-2n} + 10\widehat{M}_{-2n-1} - 2)$.

Taking $b_n = \widehat{R}_n$ with $\widehat{R}_0 = 4, \widehat{R}_1 = 5, \widehat{R}_2 = 9, \widehat{R}_3 = 23$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of Tetranacci-Lucas numbers.

Corollary 7.4.
For $n \geq 1$, binomial transform of Tetranacci-Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n} \widehat{R}_{-k} = -\widehat{R}_{-n+3} + 4\widehat{R}_{-n+2} - 4\widehat{R}_{-n+1} + 2\widehat{R}_{-n} - 1$.

(b) $\sum_{k=1}^{n} \widehat{R}_{-2k} = \frac{1}{21}(-10\widehat{R}_{-2n+2} + 39\widehat{R}_{-2n+1} - 35\widehat{R}_{-2n} + 11\widehat{R}_{-2n-1} - 31)$.

(c) $\sum_{k=1}^{n} \widehat{R}_{-2k+1} = \frac{1}{21}(-11\widehat{R}_{-2n+2} + 45\widehat{R}_{-2n+1} - 49\widehat{R}_{-2n} + 10\widehat{R}_{-2n-1} + 10)$.

7.3. Sums of the Squares of Terms with Positive Subscripts

The following proposition presents some formulas of binomial transform of generalized Tetranacci numbers with positive subscripts.

Proposition 7.3.
If $r = 5, s = -8, t = 6, u = -1$ *then for* $n \geq 0$, *we have the following formulas:*

(a) $\sum_{k=0}^{n} b_k^2 = \frac{1}{21}(11b_{n+4}^2 + 186b_{n+3}^2 + 266b_{n+2}^2 - 10b_{n+1}^2 - 90b_{n+3}b_{n+4} + 98b_{n+2}b_{n+4} - 20b_{n+1}b_{n+4} - 420b_{n+2}b_{n+3} + 78b_{n+1}b_{n+3} - 112b_{n+1}b_{n+2} - 11b_3^2 - 186b_2^2 - 266b_1^2 + 10b_0^2 + 90b_2b_3 - 98b_1b_3 + 20b_0b_3 + 420b_1b_2 - 78b_0b_2 + 112b_0b_1)$.

(b) $\sum_{k=0}^{n} b_{k+1}b_k = \frac{1}{21}(10b_{n+4}^2 + 150b_{n+3}^2 + 238b_{n+2}^2 + 10b_{n+1}^2 - 78b_{n+3}b_{n+4} + 91b_{n+2}b_{n+4} - 22b_{n+1}b_{n+4} - 357b_{n+2}b_{n+3} + 69b_{n+1}b_{n+3} - 119b_{n+1}b_{n+2} - 10b_3^2 - 150b_2^2 - 238b_1^2 - 10b_0^2 + 78b_2b_3 - 91b_1b_3 + 22b_0b_3 + 357b_1b_2 - 69b_0b_2 + 119b_0b_1)$.

(c) $\sum_{k=0}^{n} b_{k+2}b_k = \frac{1}{21}(11b_{n+4}^2 + 81b_{n+3}^2 + 161b_{n+2}^2 + 11b_{n+1}^2 - 69b_{n+3}b_{n+4} + 98b_{n+2}b_{n+4} - 231b_{n+2}b_{n+3} - 41b_{n+1}b_{n+4} + 57b_{n+1}b_{n+3} - 91b_{n+1}b_{n+2} - 11b_3^2 - 81b_2^2 - 161b_1^2 - 11b_0^2 + 69b_2b_3 - 98b_1b_3 + 41b_0b_3 + 231b_1b_2 - 57b_0b_2 + 91b_0b_1)$.

(d) $\sum_{k=0}^{n} b_{k+3}b_k = \frac{1}{21}(31b_{n+4}^2 + 171b_{n+3}^2 + 259b_{n+2}^2 + 31b_{n+1}^2 - 183b_{n+3}b_{n+4} + 259b_{n+2}b_{n+4} - 148b_{n+1}b_{n+4} - 462b_{n+2}b_{n+3} + 237b_{n+1}b_{n+3} - 224b_{n+1}b_{n+2} - 31b_3^2 - 171b_2^2 - 259b_1^2 - 31b_0^2 + 183b_2b_3 - 259b_1b_3 + 148b_0b_3 + 462b_1b_2 - 237b_0b_2 + 224b_0b_1)$.

Proof. Take $r = 5, s = -8, t = 6, u = -1$ in Theorem 2.1. in [24] (or take $x = 1, r = 5, s = -8, t = 6, u = -1$ in Theorem 3.1. in Soykan [25]).

From the last proposition, we have the following corollary which gives sum formulas of binomial transform of Tetranacci numbers (take $b_n = \widehat{M}_n$ with $\widehat{M}_0 = 0, \widehat{M}_1 = 1, \widehat{M}_2 = 3, \widehat{M}_3 = 8$).

Corollary 7.5.
For $n \geq 0$, *we have the following formulas:*

(a) $\sum_{k=0}^{n} \widehat{M}_k^2 = \frac{1}{21}(11\widehat{M}_{n+4}^2 + 186\widehat{M}_{n+3}^2 + 266\widehat{M}_{n+2}^2 - 10\widehat{M}_{n+1}^2 - 90\widehat{M}_{n+3}\widehat{M}_{n+4} + 98\widehat{M}_{n+2}\widehat{M}_{n+4} - 20\widehat{M}_{n+1}\widehat{M}_{n+4} - 420\widehat{M}_{n+2}\widehat{M}_{n+3} + 78\widehat{M}_{n+1}\widehat{M}_{n+3} - 112\widehat{M}_{n+1}\widehat{M}_{n+2} - 8)$.

(b) $\sum_{k=0}^{n} \widehat{M}_{k+1}\widehat{M}_k = \frac{1}{21}(10\widehat{M}_{n+4}^2 + 150\widehat{M}_{n+3}^2 + 238\widehat{M}_{n+2}^2 + 10\widehat{M}_{n+1}^2 - 78\widehat{M}_{n+3}\widehat{M}_{n+4} + 91\widehat{M}_{n+2}\widehat{M}_{n+4} - 22\widehat{M}_{n+1}\widehat{M}_{n+4} - 357\widehat{M}_{n+2}\widehat{M}_{n+3} + 69\widehat{M}_{n+1}\widehat{M}_{n+3} - 119\widehat{M}_{n+1}\widehat{M}_{n+2} - 13)$.

(c) $\sum_{k=0}^{n} \widehat{M}_{k+2}\widehat{M}_k = \frac{1}{21}(11\widehat{M}_{n+4}^2 + 81\widehat{M}_{n+3}^2 + 161\widehat{M}_{n+2}^2 + 11\widehat{M}_{n+1}^2 - 69\widehat{M}_{n+3}\widehat{M}_{n+4} + 98\widehat{M}_{n+2}\widehat{M}_{n+4} - 231\widehat{M}_{n+2}\widehat{M}_{n+3} - 41\widehat{M}_{n+1}\widehat{M}_{n+4} + 57\widehat{M}_{n+1}\widehat{M}_{n+3} - 91\widehat{M}_{n+1}\widehat{M}_{n+2} - 29)$.

(d) $\sum_{k=0}^{n} \widehat{M}_{k+3}\widehat{M}_k = \frac{1}{21}(31\widehat{M}_{n+4}^2 + 171\widehat{M}_{n+3}^2 + 259\widehat{M}_{n+2}^2 + 31\widehat{M}_{n+1}^2 - 183\widehat{M}_{n+3}\widehat{M}_{n+4} + 259\widehat{M}_{n+2}\widehat{M}_{n+4} - 148\widehat{M}_{n+1}\widehat{M}_{n+4} - 462\widehat{M}_{n+2}\widehat{M}_{n+3} + 237\widehat{M}_{n+1}\widehat{M}_{n+3} - 224\widehat{M}_{n+1}\widehat{M}_{n+2} - 76)$.

Taking $b_n = \widehat{R}_n$ with $\widehat{R}_0 = 4, \widehat{R}_1 = 5, \widehat{R}_2 = 9, \widehat{R}_3 = 23$ in the last proposition, we have the following corollary which presents sum formulas of binomial transform of Tetranacci-Lucas numbers.

Corollary 7.6.
For $n \geq 0$, *we have the following formulas:*

(a) $\sum_{k=0}^{n} \widehat{R}_k^2 = \frac{1}{21}(11\widehat{R}_{n+4}^2 + 186\widehat{R}_{n+3}^2 + 266\widehat{R}_{n+2}^2 - 10\widehat{R}_{n+1}^2 - 90\widehat{R}_{n+3}\widehat{R}_{n+4} + 98\widehat{R}_{n+2}\widehat{R}_{n+4} - 20\widehat{R}_{n+1}\widehat{R}_{n+4} - 420\widehat{R}_{n+2}\widehat{R}_{n+3} + 78\widehat{R}_{n+1}\widehat{R}_{n+3} - 112\widehat{R}_{n+1}\widehat{R}_{n+2} + 157)$.

(b) $\sum_{k=0}^{n} \widehat{R}_{k+1}\widehat{R}_k = \frac{1}{21}(10\widehat{R}_{n+4}^2 + 150\widehat{R}_{n+3}^2 + 238\widehat{R}_{n+2}^2 + 10\widehat{R}_{n+1}^2 - 78\widehat{R}_{n+3}\widehat{R}_{n+4} + 91\widehat{R}_{n+2}\widehat{R}_{n+4} - 22\widehat{R}_{n+1}\widehat{R}_{n+4} - 357\widehat{R}_{n+2}\widehat{R}_{n+3} + 69\widehat{R}_{n+1}\widehat{R}_{n+3} - 119\widehat{R}_{n+1}\widehat{R}_{n+2} + 116)$.

(c) $\sum_{k=0}^{n} \widehat{R}_{k+2}\widehat{R}_k = \frac{1}{21}(11\widehat{R}_{n+4}^2 + 81\widehat{R}_{n+3}^2 + 161\widehat{R}_{n+2}^2 + 11\widehat{R}_{n+1}^2 - 69\widehat{R}_{n+3}\widehat{R}_{n+4} + 98\widehat{R}_{n+2}\widehat{R}_{n+4} - 231\widehat{R}_{n+2}\widehat{R}_{n+3} - 41\widehat{R}_{n+1}\widehat{R}_{n+4} + 57\widehat{R}_{n+1}\widehat{R}_{n+3} - 91\widehat{R}_{n+1}\widehat{R}_{n+2} + 367)$.

(d) $\sum_{k=0}^{n} \widehat{R}_{k+3}\widehat{R}_k = \frac{1}{21}(31\widehat{R}_{n+4}^2 + 171\widehat{R}_{n+3}^2 + 259\widehat{R}_{n+2}^2 + 31\widehat{R}_{n+1}^2 - 183\widehat{R}_{n+3}\widehat{R}_{n+4} + 259\widehat{R}_{n+2}\widehat{R}_{n+4} - 148\widehat{R}_{n+1}\widehat{R}_{n+4} - 462\widehat{R}_{n+2}\widehat{R}_{n+3} + 237\widehat{R}_{n+1}\widehat{R}_{n+3} - 224\widehat{R}_{n+1}\widehat{R}_{n+2} + 1229)$.

8. Matrices Related with Binomial Transform of Generalized Tetranacci Numbers

We define the square matrix A of order 4 as:

$$A=\begin{pmatrix} 5 & -8 & 6 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. From (1) we have

$$\begin{pmatrix} b_{n+3} \\ b_{n+2} \\ b_{n+1} \\ b_n \end{pmatrix}=\begin{pmatrix} 5 & -8 & 6 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} b_{n+2} \\ b_{n+1} \\ b_n \\ b_{n-1} \end{pmatrix}. \tag{17}$$

and from (6) (or using (17) and induction) we have

$$\begin{pmatrix} b_{n+3} \\ b_{n+2} \\ b_{n+1} \\ b_n \end{pmatrix}=\begin{pmatrix} 5 & -8 & 6 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n\begin{pmatrix} b_3 \\ b_2 \\ b_1 \\ b_0 \end{pmatrix}.$$

If we take $b_n=\widehat{M}_n$ in (17) we have

$$\begin{pmatrix} \widehat{M}_{n+3} \\ \widehat{M}_{n+2} \\ \widehat{M}_{n+1} \\ \widehat{M}_n \end{pmatrix}=\begin{pmatrix} 5 & -8 & 6 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \widehat{M}_{n+2} \\ \widehat{M}_{n+1} \\ \widehat{M}_n \\ \widehat{M}_{n-1} \end{pmatrix}. \tag{18}$$

We also, for $n\geq 0$, define

$$B_n=\begin{pmatrix} \sum_{k=0}^{n+1}\sum_{l=k}^{n+1}\widehat{M}_k & E_1 & 6\sum_{k=0}^{n}\sum_{l=k}^{n}\widehat{M}_k-\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\widehat{M}_k & -\sum_{k=0}^{n}\sum_{l=k}^{n}\widehat{M}_k \\ \sum_{k=0}^{n}\sum_{l=k}^{n}\widehat{M}_k & E_2 & 6\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\widehat{M}_k-\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\widehat{M}_k & -\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\widehat{M}_k \\ \sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\widehat{M}_k & E_3 & 6\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\widehat{M}_k-\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\widehat{M}_k & -\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\widehat{M}_k \\ \sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\widehat{M}_k & E_4 & 6\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\widehat{M}_k-\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\widehat{M}_k & -\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\widehat{M}_k \end{pmatrix}$$

and

$$C_n=\begin{pmatrix} b_{n+1} & -8b_n+6b_{n-1}-b_{n-2} & 6b_n-b_{n-1} & -b_n \\ b_n & -8b_{n-1}+6b_{n-2}-b_{n-3} & 6b_{n-1}-b_{n-2} & -b_{n-1} \\ b_{n-1} & -8b_{n-2}+6b_{n-3}-b_{n-4} & 6b_{n-2}-b_{n-3} & -b_{n-2} \\ b_{n-2} & -8b_{n-3}+6b_{n-4}-b_{n-5} & 6b_{n-3}-b_{n-4} & -b_{n-3} \end{pmatrix}$$

where

$$\begin{matrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{matrix} = \begin{matrix} -8\sum_{k=0}^{n}\sum_{l=k}^{n}\widehat{M}_k+6\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\widehat{M}_k-\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\widehat{M}_k \\ -8\sum_{k=0}^{n-1}\sum_{l=k}^{n-1}\widehat{M}_k+6\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\widehat{M}_k-\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\widehat{M}_k \\ -8\sum_{k=0}^{n-2}\sum_{l=k}^{n-2}\widehat{M}_k+6\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\widehat{M}_k-\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\widehat{M}_k \\ -8\sum_{k=0}^{n-3}\sum_{l=k}^{n-3}\widehat{M}_k+6\sum_{k=0}^{n-4}\sum_{l=k}^{n-4}\widehat{M}_k-\sum_{k=0}^{n-5}\sum_{l=k}^{n-5}\widehat{M}_k \end{matrix}.$$

By convention, we assume that

$$\sum_{k=0}^{-5}\sum_{l=k}^{-5}\widehat{M}_k=-28,\ \sum_{k=0}^{-4}\sum_{l=k}^{-4}\widehat{M}_k=-6,\ \sum_{k=0}^{-3}\sum_{l=k}^{-3}\widehat{M}_k=-1,\ \sum_{k=0}^{-2}\sum_{l=k}^{-2}\widehat{M}_k=0,\ \sum_{k=0}^{-1}\sum_{l=k}^{-1}\widehat{M}_k=0.$$

Theorem 8.1.
For all integers $m,n\geq 0$, we have

(a) $B_n=A^n$.

(b) $C_1A^n=A^nC_1$.

(c) $C_{n+m}=C_nB_m=B_mC_n$.

Proof.

(a) Proof can be done by mathematical induction on n.

(b) After matrix multiplication, (b) follows.

(c) We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} 5 & -8 & 6 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_n & -8b_{n-1}+6b_{n-2}-b_{n-3} & 6b_{n-1}-b_{n-2} & -b_{n-1} \\ b_{n-1} & -8b_{n-2}+6b_{n-3}-b_{n-4} & 6b_{n-2}-b_{n-3} & -b_{n-2} \\ b_{n-2} & -8b_{n-3}+6b_{n-4}-b_{n-5} & 6b_{n-3}-b_{n-4} & -b_{n-3} \\ b_{n-3} & -8b_{n-4}+6b_{n-5}-b_{n-6} & 6b_{n-4}-b_{n-5} & -b_{n-4} \end{pmatrix} \\ &= \begin{pmatrix} b_{n+1} & -8b_n+6b_{n-1}-b_{n-2} & 6b_n-b_{n-1} & -b_n \\ b_n & -8b_{n-1}+6b_{n-2}-b_{n-3} & 6b_{n-1}-b_{n-2} & -b_{n-1} \\ b_{n-1} & -8b_{n-2}+6b_{n-3}-b_{n-4} & 6b_{n-2}-b_{n-3} & -b_{n-2} \\ b_{n-2} & -8b_{n-3}+6b_{n-4}-b_{n-5} & 6b_{n-3}-b_{n-4} & -b_{n-3} \end{pmatrix} = C_n. \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction, we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n.$$

□

Theorem 8.2.
For $m, n \geq 0$, we have

$$\begin{aligned} b_{n+m} &= b_n \sum_{k=0}^{m+1}\sum_{l=k}^{m+1} \widehat{M}_k + b_{n-1}\left(-8\sum_{k=0}^{m}\sum_{l=k}^{m} \widehat{M}_k + 6\sum_{k=0}^{m-1}\sum_{l=k}^{m-1} \widehat{M}_k - \sum_{k=0}^{m-2}\sum_{l=k}^{m-2} \widehat{M}_k\right) \\ &\quad + b_{n-2}\left(6\sum_{k=0}^{m}\sum_{l=k}^{m} \widehat{M}_k - \sum_{k=0}^{m-1}\sum_{l=k}^{m-1} \widehat{M}_k\right) - b_{n-3}\sum_{k=0}^{m}\sum_{l=k}^{m} \widehat{M}_k \end{aligned} \tag{19}$$

Proof. From the equation $C_{n+m} = C_nB_m = B_mC_n$, we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation, we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and C_nB_m. This completes the proof. □

Corollary 8.1.
For $m, n \geq 0$, we have

$$\begin{aligned} \widehat{M}_{n+m} &= \widehat{M}_n \sum_{k=0}^{m+1}\sum_{l=k}^{m+1} \widehat{M}_k + \widehat{M}_{n-1}\left(-8\sum_{k=0}^{m}\sum_{l=k}^{m} \widehat{M}_k + 6\sum_{k=0}^{m-1}\sum_{l=k}^{m-1} \widehat{M}_k - \sum_{k=0}^{m-2}\sum_{l=k}^{m-2} \widehat{M}_k\right) \\ &\quad + \widehat{M}_{n-2}\left(6\sum_{k=0}^{m}\sum_{l=k}^{m} \widehat{M}_k - \sum_{k=0}^{m-1}\sum_{l=k}^{m-1} \widehat{M}_k\right) - \widehat{M}_{n-3}\sum_{k=0}^{m}\sum_{l=k}^{m} \widehat{M}_k, \\ \widehat{R}_{n+m} &= \widehat{R}_n \sum_{k=0}^{m+1}\sum_{l=k}^{m+1} \widehat{M}_k + \widehat{R}_{n-1}\left(-8\sum_{k=0}^{m}\sum_{l=k}^{m} \widehat{M}_k + 6\sum_{k=0}^{m-1}\sum_{l=k}^{m-1} \widehat{M}_k - \sum_{k=0}^{m-2}\sum_{l=k}^{m-2} \widehat{M}_k\right) \\ &\quad + \widehat{R}_{n-2}\left(6\sum_{k=0}^{m}\sum_{l=k}^{m} \widehat{M}_k - \sum_{k=0}^{m-1}\sum_{l=k}^{m-1} \widehat{M}_k\right) - \widehat{R}_{n-3}\sum_{k=0}^{m}\sum_{l=k}^{m} \widehat{M}_k. \end{aligned}$$

Remark 8.1.
Note that Theorem 8.2 can be simplified by using the formula

$$\sum_{k=0}^{n} \widehat{M}_k = \widehat{M}_{n+4} - 4\widehat{M}_{n+3} + 4\widehat{M}_{n+2} - 2\widehat{M}_{n+1}$$

which is given in Corollary 7.1 and the other formulas such as

$$\sum_{l=k}^{m+1} \widehat{M}_k = \widehat{M}_k \sum_{l=k}^{m+1} 1 = \widehat{M}_k((m+1)-k+1) = (m-k+2)\widehat{M}_k.$$

and

$$\sum_{l=k}^{m} \widehat{M}_k = (m-k+1)\widehat{M}_k.$$

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