On the Sum of the Squares of Generalized Mersenne Numbers: the Sum Formula \( \sum_{k=0}^{n} x^k W^2_{mk+j} \)

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**Abstract:** In this paper, closed forms of the sum formulas \( \sum_{k=0}^{n} x^k W^2_{mk+j} \) for generalized Mersenne numbers are presented. As special cases, we give sum formulas of Mersenne and Mersenne-Lucas numbers.

**MSC:** 11B37 • 11B39 • 11B83

**Keywords:** Mersenne numbers • Mersenne-Lucas numbers • sum formulas

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1. **Introduction**

A Mersenne number, denoted by \( M_n \), is a number of the form \( M_n = 2^n - 1 \). The Mersenne sequence \( \{M_n\}_{n \geq 0} \) can also be defined recursively by

\[ M_n = 3M_{n-1} - 2M_{n-2} \]

with initial conditions \( M_0 = 0, M_1 = 1 \). A Mersenne-Lucas number, denoted by \( H_n \), is a number of the form \( H_n = 2^n + 1 \). The Mersenne-Lucas sequence \( \{H_n\}_{n \geq 0} \) can also be defined, by the second-order recurrence relation,

\[ H_n = 3H_{n-1} - 2H_{n-2} \]

with initial conditions \( H_0 = 2, H_1 = 3 \).

A generalized Mersenne sequence \( \{W_n\}_{n \geq 0} = \{W_n(W_0, W_1)\}_{n \geq 0} \) is defined by the second-order recurrence relation

\[ W_n = 3W_{n-1} - 2W_{n-2} \quad (1) \]

with the initial values \( W_0 = c_0, W_1 = c_1 \) not all being zero.

The sequence \( \{W_n\}_{n \geq 0} \) can be extended to negative subscripts by defining

\[ W_{-n} = \frac{3}{2} W_{-(n-1)} - \frac{1}{2} W_{-(n-2)} \]

for \( n = 1, 2, 3, \ldots \). Therefore, recurrence (1) holds for all integer \( n \). For more information on generalized Mersenne numbers, see Soykan [47].

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The following theorem presents sum formulas of generalized Mersenne numbers.

**Theorem 2.1.**

Let $x$ be a real (or complex) number. For all integers $m$ and $j$, for generalized Mersenne numbers we have the following sum formulas:

(a) If $(1 + 2^m x^2 - x H_2m)(2^m x - 1) \neq 0$ then

$$
\sum_{k=0}^{n} x^k W_{mk+j}^2 \Omega_1 = \frac{\Omega_1}{(1 + 2^m x^2 - x H_2m)(2^m x - 1)}
$$

where

$$
\Omega_1 = (2^m x - 1)(2^{m+1} x - H_2m) x^n W_{mn+j}^2 + 2^{m+1} \frac{x^n + 1}{x^n - 1} x W_{mn+j}^2 + 2^m x W_{mn+j}^2 - 2^m x W_{mn+j}^2(2^m x - 1) x W_{mn+j}^2
$$

(b) If $(1 + 2^m x^2 - x H_2m)(2^m x - 1) = u(x - a)(x - b)(x - c) = 0$ for some $u, a, b, c \in \mathbb{C}$ with $u \neq 0$ and $a \neq b \neq c$, i.e., $x = a$ or $x = b$ or $x = c$, then

$$
\sum_{k=0}^{n} x^k W_{mk+j}^2 \Omega_2 = \frac{\Omega_2}{(3 x^2 + 2 \frac{H_2m + 2^m x}{x + 2 H_2m + 2^m})}
$$

where

$$
\Omega_2 = (2^m x - H_2m) x^n W_{mn+j}^2 + 2^m(2^m (n + 2) x - (n + 1) H_2m) x^n W_{mn+j}^2 W_{mn+j}^2 + 2^m(2^m (n + 2) x - (n + 1)) x^n W_{mn+j}^2 W_{mn+j}^2 + 2^m x W_{mn+j}^2 - 2^m x W_{mn+j}^2 + 2^m x W_{mn+j}^2 + 2^m x W_{mn+j}^2 + 2^m x W_{mn+j}^2 - 3 W_1 W_0(2^m (n + 1) x^n - 1)(H_2m - 2^m + 1)
$$

(c) If $(1 + 2^m x^2 - x H_2m)(2^m x - 1) = u(x - a)^2(x - c) = 0$ for some $u, a, c \in \mathbb{C}$ with $u \neq 0$ and $a \neq c$ then if $x = c$ then

$$
\sum_{k=0}^{n} x^k W_{mk+j}^2 \Omega_3 = \frac{\Omega_3}{(3 x^2 + 2 \frac{H_2m + 2^m x}{x + 2 H_2m + 2^m})}
$$

where

$$
\Omega_3 = (2^m x - H_2m) x^n W_{mn+j}^2 + 2^m(2^m (n + 2) x - (n + 1) H_2m) x^n W_{mn+j}^2 W_{mn+j}^2 + 2^m(2^m (n + 2) x - (n + 1)) x^n W_{mn+j}^2 W_{mn+j}^2 + 2^m x W_{mn+j}^2 - 2^m x W_{mn+j}^2 + 2^m x W_{mn+j}^2 + 2^m x W_{mn+j}^2 + 2^m x W_{mn+j}^2 - 3 W_1 W_0(2^m (n + 1) x^n - 1)(H_2m - 2^m + 1)
$$

and if $x = a$ then

$$
\sum_{k=0}^{n} x^k W_{mk+j}^2 \Omega_4 = \frac{\Omega_4}{(2^m + 1)(3 x^2 + 2 \frac{H_2m + 2^m x}{x + 2 H_2m + 2^m})}
$$

where

$$
\Omega_4 = (2^m (n + 3)(n + 2) x^n - x^2 - x^2 (n + 2)(n + 1)(H_2m + 2^m) + n(n + 1)(H_2m + 2^m) x^n W_{mn+j}^2 + 2^m (n + 1)(2^m + 1) x^n W_{mn+j}^2 - n x^{n+1} W_{mn+j}^2 - 2^{m+1} W_{mn+j}^2 + 2^m x W_{mn+j}^2 - 2^m x W_{mn+j}^2 - 3 W_1 W_0(2^m (n + 1) x^n - 1)(H_2m - 2^m + 1) x^{n-1}
$$
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(d) If $(1 + 2^{2m}x^2 - xH_{2m})(2^n x - 1) = u(x - a)^3 = 0$ for some $u, a \in C$ with $u \neq 0$, i.e., $x = a$, then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_5}{6 \times 2^{m}}$$

where

$$\Omega_5 = (n + 1)(2^{3m}(n + 3)(n + 2)x^2 - n2^{m}(n + 2)(H_{2m} + 2^m)x + n(n - 1) H_{2m})x^{n-2} W_{mn+j}^2 + n2^{m}(2^m + 4x)(n + 1)$$

$$((n + 2)x^2 + 1 - n)x^{x-2} W_{mn-m+j}^2 + (n + 1)(n + 1)2^{mn+1}(H_{2m} - 2^{m+1})(W_0 + 2W_0^2 - 2W_1 W_0)x^{n-2}$$

**Proof.** Take $r = 3, s = -2$ and $H_n = H_n$ in Soykan [[46], Theorem 2.1]. □

Note that (2) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j}^2 = \frac{\Omega_6}{(1 + 2^{2m}x^2 - xH_{2m})(2^{m}x - 1)}$$

where

$$\Omega_6 = (2^{m}x - 1)(2^{2m}x - H_{2m})x^{n+1} W_{mn+j}^2 + 2^{m}(2^{m}x - 1)x^{n+1} W_{mn-m+j}^2 - (2^{m}x - 1)(2^{m}x - H_{2m})x W_j^2 - 2^{m}(2^{m}x - 1)W_{j-m-x}^2 + 2^{m+1}(W_0^2 + 2W_0^2 - 3W_1 W_0)(2^{m}x - 1)(H_{2m} - 2^{m+1})x$$

As special cases of $m$ and $j$ in the last Theorem, we obtain the following proposition.

**Proposition 2.1.**

For generalized Mersenne numbers, we have the following sum formulas for $n \geq 0$:

(a) $(m = 1, j = 0)$

If $(2x - 1)(4x - 1)(x - 1) \neq 0$, i.e., $x \neq \frac{1}{4}, x \neq \frac{1}{2}, x \neq 1$, then

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Delta}{(2x - 1)(4x - 1)(x - 1)}$$

where

$$\Delta = (2x - 1)(4x - 5)x^{n+1} W_n^2 + 4(2x - 1)x^{n+1} W_{n-1}^2 + (2x - 1)W_0^2 - (2x - 1)(3W_0 - W_1)^2 + 2(2W_0^2 + 2W_0^2 - 3W_1 W_0)(2^{n}x^n - 1)x$$

and

if $(2x - 1)(4x - 1)(x - 1) = 0$, i.e., $x = \frac{1}{2}$ or $x = \frac{1}{2}$ or $x = 1$ then

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Psi}{(24x^2 - 28x + 7)}$$

where

$$\Psi = (n(2x - 1)(4x - 5) + 24x^2 - 28x + 5)x^{n+1} W_n^2 + 4(2x - 1)n + 4x - 1)x^{n+1} W_{n-1}^2 + 2W_0^2 - (4x - 1)(3W_0 - W_1)^2 + 2(W_0^2 + 2W_0^2 - 3W_1 W_0)(2^{n}x^n - 1)$$

(b) $(m = 2, j = 0)$

If $(4x - 1)(16x - 1)(x - 1) \neq 0$, i.e., $x \neq \frac{1}{16}, x \neq \frac{1}{4}, x \neq 1$, then

$$\sum_{k=0}^{n} x^k W_{2k}^2 = \frac{\Delta}{(4x - 1)(16x - 1)(x - 1)}$$

where

$$\Delta = (4x - 1)(16x - 17)x^{n+1} W_{2n}^2 + 16(4x - 1)x^{n+1} W_{2n-2}^2 + (4x - 1)W_0^2 - (4x - 1)(7W_0 - 3W_1)^2 + 18(W_0^2 + 2W_0^2 - 3W_1 W_0)(2^{n}x^n - 1)x$$

and

if $(4x - 1)(16x - 1)(x - 1) = 0$, i.e., $x = \frac{1}{16}$ or $x = \frac{1}{4}$ or $x = 1$ then

$$\sum_{k=0}^{n} x^k W_{2k}^2 = \frac{\Psi}{3(64x^2 - 56x + 7)}$$

where

$$\Psi = (n(4x - 1)(16x - 17) + 192x^2 - 168x + 17)x^{n} W_{2n}^2 + 16((4x - 1)n + 8x - 1)x^{n} W_{2n-2}^2 + 4W_0^2 - (8x - 1)(7W_0 - 3W_1)^2 + 18(W_0^2 + 2W_0^2 - 3W_1 W_0)(2^{n}(n + 1)x^n - 1)$$
(c) \( m = 2, j = 1 \)
\[
\sum_{k=0}^{n} x^k W_{2k+1} = \frac{\Delta}{(4x-1)(16x-1)(x-1)}
\]
where
\[
\Delta = (4x-1)(16x-17)x^{n+1}W_{2n+1}^2 + 16(4x-1)x^{n+1}W_{2n-1}^2 + (4x-1)W_1^2 - 4(4x-1)x(3W_0 - W_1)^2 + 36(W_1^2 + 2W_0^2 - 3W_1W_0)(2^{2n}x^n - 1)x
\]
and
\[
\sum_{k=0}^{n} x^k W_{2k+1} = \frac{\Psi}{3(64x^2 - 56x + 7)}
\]
where
\[
\Psi = (n(4x-1)(16x-17) + 192x^2 - 168x + 17)x^nW_{2n+1}^2 + 16((4x-1)n + 8x - 1).x^mW_{2n-1}^2 + 4W_1^2 - 4(8x-1)(3W_0 - W_1)^2 + 36(W_1^2 + 2W_0^2 - 3W_1W_0)(2^{2n}(n+1)x^n - 1)x
\]

(d) \( m = -1, j = 0 \)
\[
\sum_{k=0}^{n} x^k W_k = \frac{\Delta}{(x-2)(x-1)(x-4)}
\]
where
\[
\Delta = (x-2)x^{n+1}W_{-n+1}^2 + (x-2)(x-5)x^{n+1}W_{-n+1}^2 + 4(x-2)W_0^2 - x(x-2)W_1^2 + 4(W_1^2 + 2W_0^2 - 3W_1W_0)(2^{-n}x^n - 1)x
\]
and
\[
\sum_{k=0}^{n} x^k W_k = \frac{\Psi}{3x^2 - 14x + 14}
\]
where
\[
\Psi = (n(x-2) + 2(x-1)).x^nW_{-n+1} - (n(x-2)(x-5) + 3x^2 - 14x + 10).x^nW_{-n+1}^2 + 4W_0^2 - 2(x-1)W_1^2 + 4(W_1^2 + 2W_0^2 - 3W_1W_0)(2^{-n}(n+1)x^n - 1)
\]

(e) \( m = -2, j = 0 \)
\[
\sum_{k=0}^{n} x^k W_{-2k} = \frac{\Delta}{(x-4)(x-1)(x-16)}
\]
where
\[
\Delta = (x-4)x^{n+1}W_{2n+2}^2 + (x-4)(x-17)x^{n+1}W_{2n}^2 + 16(x-4)W_0^2 - x(x-4)(2W_0 - 3W_1)^2 + 72(W_1^2 + 2W_0^2 - 3W_1W_0)(2^{-2n}x^n - 1)x
\]
and
\[
\sum_{k=0}^{n} x^k W_{-2k} = \frac{\Psi}{3(x^2 - 14x + 28)}
\]
where
\[
\Psi = (n(x-4) + 2(x-2)).x^nW_{2n+2} - (n(x-4)(x-17) + 3x^2 - 42x + 68)x^nW_{2n} + 16W_0^2 - 2(x-2)(2W_0 - 3W_1)^2 + 72(W_1^2 + 2W_0^2 - 3W_1W_0)(2^{-2n}(n+1)x^n - 1)
\]
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(f) $(m = -2, j = 1)$

If $(x - 4)(x - 1)(x - 16) \neq 0$, i.e., $x \neq 1, x \neq 4, x \neq 16$, then

$$\sum_{k=0}^{n} x^k W_{-2k+1}^2 = \frac{\Delta}{(x - 4)(x - 1)(x - 16)}$$

where

$$\Delta = (x - 4)x^{n+1}W_{2n+3}^2 + (x - 4)(x - 17)x^{n+1}W_{2n+1}^2 + 16(x - 4)W_1^2 - x(x - 4)(6W_0 - 7W_1)^2 + 144(W_1^2 + 2W_0^2 - 3W_1W_0)(2^{-2n}x^n - 1)x$$

and

If $(x - 4)(x - 1)(x - 16) = 0$, i.e., $x = 1$ or $x = 4$ or $x = 16$ then

$$\sum_{k=0}^{n} x^k W_{-2k+1}^2 = \frac{\Psi}{3(x^2 - 14x + 28)}$$

where

$$\Psi = (n(x - 4) + 2(x - 2))x^n W_{2n+3}^2 + (n(x - 4)(x - 17) + 3x^2 - 42x + 68)x^n W_{2n+1}^2 + 16W_1^2 - 2(x - 2)(6W_0 - 7W_1)^2 + 144(W_1^2 + 2W_0^2 - 3W_1W_0)(2^{-2n}(n + 1)x^n - 1)$$

From the above proposition, we have the following corollary which gives sum formulas of Mersenne numbers (take $W_n = M_n$ with $M_0 = 0, M_1 = 1$).

**Corollary 2.1.**

For $n \geq 0$, Mersenne numbers have the following properties:

(a) $(m = 1, j = 0)$

If $(2x - 1)(4x - 1)(x - 1) \neq 0$, i.e., $x \neq \frac{1}{4}, x \neq \frac{1}{2}, x \neq 1$, then

$$\sum_{k=0}^{n} x^k M_k^2 = \frac{(2x - 1)(4x - 5)x^{n+1}M_n^2 + 4(2x - 1)x^{n+1}M_{n-1}^2 + x(2^{n+1}x^n - 2x - 1)}{(2x - 1)(4x - 1)(x - 1)}$$

and

If $(2x - 1)(4x - 1)(x - 1) = 0$, i.e., $x = \frac{1}{4}$ or $x = \frac{1}{2}$ or $x = 1$ then

$$\sum_{k=0}^{n} x^k M_k^2 = \frac{\Theta_1}{(24x^2 - 28x + 7)}$$

where

$$\Theta_1 = (n(2x - 1)(4x - 5) + 24x^2 - 28x + 5)x^n M_n^2 + 4((2x - 1)n + 4x - 1)x^n M_{n-1}^2 + (n + 1)2^{n+1}x^n - 4x - 1.$$

(b) $(m = 2, j = 0)$

If $(4x - 1)(16x - 1)(x - 1) \neq 0$, i.e., $x \neq \frac{1}{16}, x \neq \frac{1}{4}, x \neq 1$, then

$$\sum_{k=0}^{n} x^k M_{2k}^2 = \frac{(4x - 1)(16x - 17)x^{n+1}M_{2n}^2 + 16(4x - 1)x^{n+1}M_{2n-2}^2 + 9(2^{2n+1}x^n - 4x - 1)x}{(4x - 1)(16x - 1)(x - 1)}$$

and

If $(4x - 1)(16x - 1)(x - 1) = 0$, i.e., $x = \frac{1}{16}$ or $x = \frac{1}{4}$ or $x = 1$ then

$$\sum_{k=0}^{n} x^k M_{2k}^2 = \frac{\Theta_2}{3(64x^2 - 56x + 7)}$$

where

$$\Theta_2 = (n(4x - 1)(16x - 17) + 192x^2 - 168x + 17)x^n M_{2n}^2 + 16((4x - 1)n + 8x - 1)x^n M_{2n-2}^2 + 9((n + 1)2^{2n+1}x^n - 8x - 1).$$
(c) \( m = 2, j = 1 \)

If \((4x - 1)(16x - 1)(x - 1) \neq 0\), i.e., \(x \neq \frac{1}{4}, x \neq \frac{1}{16}, x \neq 1\), then

\[
\sum_{k=0}^{n} x^k M_{2k+1}^2 = \frac{(4x - 1)(16x - 17)x^{n+1}M_{2n+1}^2 + 16(4x - 1)x^n M_{2n-1}^2 + (9 \times 2^{2n+2}x^n + 16x^2 - 28x - 1)}{(4x - 1)(16x - 1)(x - 1)}
\]

and

if \((4x - 1)(16x - 1)(x - 1) = 0\), i.e., \(x = \frac{1}{16}\) or \(x = \frac{1}{4}\) or \(x = 1\)

\[
\sum_{k=0}^{n} x^k M_{2k+1}^2 = \frac{\Theta_3}{3(64x^2 - 56x + 7)}
\]

where

\[
\Theta_3 = (n(4x - 1)(16x - 17) + 192x^2 - 168x + 17)x^n M_{2n+1}^2 + 16((4x - 1)n + 8x - 1)x^n M_{2n-1}^2 + 4(9(n + 1)2^{2n}x^n - 8x - 7).
\]

(d) \( m = -1, j = 0 \)

If \((x - 2)(x - 1)(x - 4) \neq 0\), i.e., \(x \neq 1, x \neq 2, x \neq 4\), then

\[
\sum_{k=0}^{n} x^k M_{-k}^2 = \frac{(x - 2)x^{n+1}M_{-n+1}^2 + (x - 2)(x - 5)x^n M_{-n}^2 + x(2^{-n+2}x^n - x - 2)}{(x - 2)(x - 1)(x - 4)}
\]

and

if \((x - 2)(x - 1)(x - 4) = 0\), i.e., \(x = 1\) or \(x = 2\) or \(x = 4\)

\[
\sum_{k=0}^{n} x^k M_{-k}^2 = \frac{\Theta_4}{3x^2 - 14x + 14}
\]

where

\[
\Theta_4 = (n(x - 2) + 2(x - 1))x^n M_{-n+1}^2 + (n(x - 2)(x - 5) + 3x^2 - 14x + 10)x^n M_{-n}^2 + 2((n + 1)2^{-n+1}x^n - (x + 1)).
\]

(e) \( m = -2, j = 0 \)

If \((x - 4)(x - 1)(x - 16) \neq 0\), i.e., \(x \neq 1, x \neq 4, x \neq 16\), then

\[
\sum_{k=0}^{n} x^k M_{-2k}^2 = \frac{(x - 4)x^{n+1}M_{-2n+2}^2 + (x - 4)(x - 17)x^{n+1}M_{-2n}^2 + 9(2^{-2n+3}x^n - x - 4)x}{(x - 4)(x - 1)(x - 16)}
\]

and

if \((x - 4)(x - 1)(x - 16) = 0\), i.e., \(x = 1\) or \(x = 4\) or \(x = 16\)

\[
\sum_{k=0}^{n} x^k M_{-2k}^2 = \frac{\Theta_5}{3(x^2 - 14x + 28)}
\]

where

\[
\Theta_5 = (n(x - 4) + 2(x - 2))x^n M_{-2n+2}^2 + (n(x - 4)(x - 17) + 3x^2 - 42x + 68)x^n M_{-2n}^2 + 18(2^{-2n+2}(n + 1)x^n - x - 2).
\]

(f) \( m = -2, j = 1 \)

If \((x - 4)(x - 1)(x - 16) \neq 0\), i.e., \(x \neq 1, x \neq 4, x \neq 16\), then

\[
\sum_{k=0}^{n} x^k M_{-2k+1}^2 = \frac{(x - 4)x^{n+1}M_{-2n+3}^2 + (x - 4)(x - 17)x^{n+1}M_{-2n+1}^2 + (9 \times 2^{-2n+4}x^{n+1} - 49x^2 + 68x - 64)}{(x - 4)(x - 1)(x - 16)}
\]

and

if \((x - 4)(x - 1)(x - 16) = 0\), i.e., \(x = 1\) or \(x = 4\) or \(x = 16\)

\[
\sum_{k=0}^{n} x^k M_{-2k+1}^2 = \frac{\Theta_6}{3(x^2 - 14x + 28)}
\]

where

\[
\Theta_6 = (n(x - 4) + 2(x - 2))x^n M_{-2n+3}^2 + (n(x - 4)(x - 17) + 3x^2 - 42x + 68)x^n M_{-2n+1}^2 + 2(9 \times 2^{-2n+3}(n + 1)x^n - 49x + 34).
\]
Taking $W_n = H_n$ with $H_0 = 2, H_1 = 3$ in the last proposition, we have the following corollary which presents sum formulas of Mersenne-Lucas numbers.

**Corollary 2.2.**

For $n \geq 0$, Mersenne-Lucas numbers have the following properties:

(a) $(m = 1, j = 0)$  
If $(2x - 1)(4x - 1)(x - 1) \neq 0$, i.e., $x \neq \frac{1}{2}, x \neq \frac{1}{4}, x \neq 1$, then  
$$
\sum_{k=0}^{n} x^k H_k^2 = \frac{(2x - 1)(4x - 5)x^{n+1}H_n^2 + 4(2x - 1)x^{n+1}H_{n-1}^2 - (2^{n+1}x^{n+1} + 18x^2 - 19x + 4)}{(2x - 1)(4x - 1)(x - 1)}
$$
and if $(2x - 1)(4x - 1)(x - 1) = 0$, i.e., $x = \frac{1}{4}$ or $x = \frac{1}{2}$ or $x = 1$ then  
$$
\sum_{k=0}^{n} x^k H_k^2 = \frac{\Theta_7}{(24x^2 - 28x + 7)}
$$
where  
$$
\Theta_7 = (n(2x - 1)(4x - 5) + 24x^2 - 28x + 5)x^nH_n^2 + 4((2x - 1)n + 4x - 1)x^nH_{n-1}^2 - ((n + 1)2^{n+1}x^n + 36x - 19).
$$

(b) $(m = 2, j = 0)$  
If $(4x - 1)(16x - 1)(x - 1) \neq 0$, i.e., $x \neq \frac{1}{16}, x \neq \frac{1}{4}, x \neq 1$, then  
$$
\sum_{k=0}^{n} x^k H_k^2 = \frac{(4x - 1)(16x - 17)x^{n+1}H_{2n}^2 + 16(4x - 1)x^{n+1}H_{2n-2}^2 - (9 \times 2^{n+1}x^{n+1} + 100x^2 - 59x + 4)}{(4x - 1)(16x - 1)(x - 1)}
$$
and if $(4x - 1)(16x - 1)(x - 1) = 0$, i.e., $x = \frac{1}{16}$ or $x = \frac{1}{4}$ or $x = 1$ then  
$$
\sum_{k=0}^{n} x^k H_k^2 = \frac{\Theta_8}{3(64x^2 - 56x + 7)}
$$
where  
$$
\Theta_8 = (n(4x - 1)(16x - 17) + 192x^2 - 168x + 17)x^nH_{2n}^2 + 16((4x - 1)n + 8x - 1)x^nH_{2n-2}^2 - (9(n + 1)2^{n+1}x^n + 200x - 59).
$$

(c) $(m = 2, j = 1)$  
If $(4x - 1)(16x - 1)(x - 1) \neq 0$, i.e., $x \neq \frac{1}{16}, x \neq \frac{1}{4}, x \neq 1$, then  
$$
\sum_{k=0}^{n} x^k H_k^2 = \frac{(4x - 1)(16x - 17)x^{n+1}H_{2n+1}^2 + 16(4x - 1)x^{n+1}H_{2n-1}^2 - 9(2^{n+2}x^{n+1} + 16x^2 - 12x + 1)}{(4x - 1)(16x - 1)(x - 1)}
$$
and if $(4x - 1)(16x - 1)(x - 1) = 0$, i.e., $x = \frac{1}{16}$ or $x = \frac{1}{4}$ or $x = 1$ then  
$$
\sum_{k=0}^{n} x^k H_k^2 = \frac{\Theta_9}{3(64x^2 - 56x + 7)}
$$
where  
$$
\Theta_9 = (n(4x - 1)(16x - 17) + 192x^2 - 168x + 17)x^nH_{2n+1}^2 + 16((4x - 1)n + 8x - 1)x^nH_{2n-1}^2 - 36((n + 1)2^{2n}x^n + 8x - 3).
$$

(d) $(m = -1, j = 0)$  
If $(x - 2)(x - 1)(x - 4) \neq 0$, i.e., $x \neq 1, x \neq 2, x \neq 4$, then  
$$
\sum_{k=0}^{n} x^k H_k^2 = \frac{(x - 2)x^{n+1}H_{n+1}^2 + (x - 2)(x - 5)x^{n+1}H_n^2 - (2^{n+2}x^{n+1} + 9x^2 - 38x + 32)}{(x - 2)(x - 1)(x - 4)}
$$
and if $(x - 2)(x - 1)(x - 4) = 0$, i.e., $x = 1$ or $x = 2$ or $x = 4$ then  
$$
\sum_{k=0}^{n} x^k H_k^2 = \frac{\Theta_{10}}{3x^2 - 14x + 14}
$$
where  
$$
\Theta_{10} = (n(x - 2) + 2(x - 1))x^nH_{n+1}^2 + (n(x - 2)(x - 5) + 3x^2 - 14x + 10)x^nH_n^2 - 2((n + 1)2^{n+1}x^n + 9x - 19).
(e) \(m = -2, j = 0\)

If \((x - 4)(x - 1)(x - 16) \neq 0\), i.e., \(x \neq 1, x \neq 4, x \neq 16\), then

\[
\sum_{k=0}^{n} x^k H_{2k}^2 = \frac{(x - 4)x^{n+1}H_{2n+2}^2 + (x - 4)(x - 17)x^{n+1}H_{2n}^2 - (9x^{n+1}2^{-2n+3} + 25x^2 - 236x + 256)}{(x - 4)(x - 1)(x - 16)}
\]

and

if \((x - 4)(x - 1)(x - 16) = 0\), i.e., \(x = 1 \) or \(x = 4 \) or \(x = 16\) then

\[
\sum_{k=0}^{n} x^k H_{2k}^2 = \frac{\Theta_{11}}{3(x^2 - 14x + 28)}
\]

where

\[
\Theta_{11} = (n(x - 4) + 2(x - 2))x^n H_{2n+2}^2 + (n(x - 4)(x - 17) + 3x^2 - 42x + 68)x^n H_{2n}^2 - 2(9(n + 1)2^{-2n+2}x^n + 25x - 118).
\]

(f) \(m = -2, j = 1\)

If \((x - 4)(x - 1)(x - 16) \neq 0\), i.e., \(x \neq 1, x \neq 4, x \neq 16\), then

\[
\sum_{k=0}^{n} x^k H_{2k+1}^2 = \frac{(x - 4)x^{n+1}H_{2n+3}^2 + (x - 4)(x - 17)x^{n+1}H_{2n}^2 - 9(2^{-2n+4}x^{n+1} + 9x^2 + 64 - 68x)}{(x - 4)(x - 1)(x - 16)}
\]

and

if \((x - 4)(x - 1)(x - 16) = 0\), i.e., \(x = 1 \) or \(x = 4 \) or \(x = 16\) then

\[
\sum_{k=0}^{n} x^k H_{2k+1}^2 = \frac{\Theta_{12}}{3(x^2 - 14x + 28)}
\]

where

\[
\Theta_{12} = (n(x - 4) + 2(x - 2))x^n H_{2n+3}^2 + (n(x - 4)(x - 17) + 3x^2 - 42x + 68)x^n H_{2n+1}^2 - 18((n + 1)2^{-2n+3}x^n + 9x - 34).
\]

Taking \(x = 1\) in the last two corollaries we get the following corollary.

**Corollary 2.3.**

For \(n \geq 0\), Mersenne numbers, and Mersenne-Lucas numbers have the following properties:

1.

(a) \(\sum_{k=0}^{n} M_k^2 = \frac{1}{3}(-n)M_n^2 + 4(n + 3)M_{n-1}^2 + (n + 1)2^{n+1} - 5).\)

(b) \(\sum_{k=0}^{n} M_k^2 = \frac{1}{3}(-3n - 41)M_{2n}^2 + 16(3n + 7)M_{2n-2}^2 + 9(n + 1)2^{2n+1} - 81).\)

(c) \(\sum_{k=0}^{n} M_k^2 = \frac{1}{3n}(-3n - 41)M_{2n+1}^2 + 16(3n + 7)M_{2n-1}^2 + 9(n + 1)2^{2n+2} - 60).\)

(d) \(\sum_{k=0}^{n} M_k^2 = \frac{1}{3}(-nM_{n+1}^2 + 4(n - 1)M_n^2 + (n + 1)2^{n+2} - 4).\)

(e) \(\sum_{k=0}^{n} M_{2k}^2 = \frac{1}{35}(-3n + 2)M_{2n+2}^2 + (48n + 29)M_{2n}^2 + 9(n + 1)2^{-2n+3} - 54).\)

(f) \(\sum_{k=0}^{n} M_{2k}^2 = \frac{1}{35}(-3n + 2)M_{2n+3}^2 + (48n + 29)M_{2n+1}^2 + 9(n + 1)2^{-2n+4} - 30).\)

2.

(a) \(\sum_{k=0}^{n} H_k^2 = \frac{1}{3}(-n - 1)H_n^2 + 4(n + 3)H_{n-1}^2 - (n + 1)2^{n+1} - 17).\)

(b) \(\sum_{k=0}^{n} H_k^2 = \frac{1}{45}(-3n - 41)H_{2n}^2 + 16(3n + 7)H_{2n-2}^2 - 9(n + 1)2^{2n+1} - 141).\)

(c) \(\sum_{k=0}^{n} H_k^2 = \frac{1}{45}(-3n - 41)H_{2n+1}^2 + 16(3n + 7)H_{2n-1}^2 - 9(n + 1)2^{2n+2} - 180).\)

(d) \(\sum_{k=0}^{n} H_k^2 = \frac{1}{3}(-nH_{n+1}^2 + 4(n - 1)H_n^2 - (n + 1)2^{-n+2} + 20).\)

(e) \(\sum_{k=0}^{n} H_k^2 = \frac{1}{45}(-3n+2)H_{2n+2}^2 + (48n+29)H_{2n}^2 - 9(n + 1)2^{-2n+3} + 186).\)

(f) \(\sum_{k=0}^{n} H_k^2 = \frac{1}{45}(-3n+2)H_{2n+3}^2 + (48n+29)H_{2n+1}^2 - 9(n + 1)2^{-2n+4} + 450).\)
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