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On the Sum of the Squares of Generalized Mersenne Numbers: the Sum Formula $\sum_{k=0}^{n} x^k W_{mk+j}^2$

Research Article

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Abstract: In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} x^k W_{mk+j}^2$ for generalized Mersenne numbers are presented. As special cases, we give sum formulas of Mersenne and Mersenne-Lucas numbers.

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Keywords: Mersenne numbers • Mersenne-Lucas numbers • sum formulas

1. Introduction

A Mersenne number, denoted by M_n, is a number of the form $M_n = 2^n - 1$. The Mersenne sequence $\{M_n\}_{n\geq 0}$ can also be defined recursively by

$$M_n = 3M_{n-1} - 2M_{n-2}$$

with initial conditions $M_0 = 0, M_1 = 1$. A Mersenne-Lucas number, denoted by H_n, is a number of the form $H_n = 2^n + 1$. The Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ can also be defined, by the second-order recurrence relation,

$$H_n = 3H_{n-1} - 2H_{n-2}$$

with initial conditions $H_0 = 2, H_1 = 3$.

A generalized Mersenne sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 3W_{n-1} - 2W_{n-2} \quad (1)$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = \frac{3}{2}W_{-(n-1)} - \frac{1}{2}W_{-(n-2)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (1) holds for all integer n. For more information on generalized Mersenne numbers, see Soykan [47].

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$\{M_n\}_{n\geq 0}$ is the sequence $A000225$ in the OEIS [?], whereas $\{H_n\}_{n\geq 0}$ is the id-number $A000051$ in OEIS. Note that Mersen-Lucas numbers are also called as Fermat numbers. In fact, there are two definitions of the Fermat numbers. The less common is a number of the form 2^n+1, the first few of which are $2,3,5,9,17,33,...$ (OEIS $A000051$). The much more commonly encountered Fermat numbers are a special case, given by the binomial number of the form $F_n = 2^{2^n}+1$. The first few for $n = 0,1,2,...$ are $3,5,17,257,65537,4294967297,...$ (OEIS $A000215$).

Mersenne sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to this sequence, see for example, [1],[2],[3],[4],[7],[9],[10],[11],[15],[16],[17],[20],[21],[22],[26],[27],[29],[49],[50].

In this work, we derive expressions for sums of second powers of generalized Mersenne numbers. We present some works on sum formulas of powers of the numbers in the following Table 1.

Table 1. A few special study on sum formulas of second, third and arbitrary powers.

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[5],[6],[14],[18],[19],[36],[37],[39],[42],[46]]	[13],[31],[33],[34],[40],[41],[45],[48],[51]]	[8],[12],[23]
Generalized Tribonacci	[25],[32],[38],[43]		
Generalized Tetranacci	[24],[28],[35],[44]		

2. The Sum Formula $\sum_{k=0}^{n} x^k W_{mk+j}^2$

The following theorem presents sum formulas of generalized Mersenne numbers.

Theorem 2.1.
Let x be a real (or complex) number. For all integers m and j, for generalized Mersenne numbers we have the following sum formulas:

(a) *If $(1+2^{2m}x^2 - xH_{2m})(2^m x - 1) \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_1}{(1+2^{2m}x^2 - xH_{2m})(2^m x-1)} \tag{2}$$

where

$$\Omega_1 = (2^m x-1)(2^{2m}x - H_{2m})x^{n+1}W_{mn+j}^2 + 2^{2m}(2^m x-1)x^{n+1}W_{mn-m+j}^2 + (2^m x-1)W_j^2 - 2^{2m}(2^m x-1)xW_{j-m}^2 + 2^{j+1}(W_1^2+2W_0^2-3W_1W_0)(2^{mn}x^n-1)(H_{2m}-2^{m+1})x.$$

(b) *If $(1+2^{2m}x^2 - xH_{2m})(2^m x-1) = u(x-a)(x-b)(x-c) = 0$ for some $u,a,b,c \in \mathbb{C}$ with $u \neq 0$ and $a \neq b \neq c$, i.e., $x = a$ or $x = b$ or $x = c$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_2}{(3\times 2^{3m}x^2 - 2\times 2^m(H_{2m}+2^m)x + H_{2m}+2^m)}$$

where

$$\Omega_2 = (2^m(2^{2m}x - H_{2m})x^{n+1} + (2^m x-1)(2^{2m}(n+2)x - (n+1)H_{2m})x^n)W_{mn+j}^2 + 2^{2m}(2^m(n+2)x-(n+1))x^n W_{mn-m+j}^2 + 2^m W_j^2 - 2^{2m}(2^{m+1}x-1)W_{j-m}^2 + 2^{j+1}(W_1^2+2W_0^2-3W_1W_0)(2^{mn}(n+1)x^n-1)(H_{2m}-2^{m+1})$$

(c) *If $(1+2^{2m}x^2 - xH_{2m})(2^m x-1) = u(x-a)^2(x-c) = 0$ for some $u,a,c \in \mathbb{C}$ with $u \neq 0$ and $a \neq c$ then if $x = c$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_3}{(3\times 2^{3m}x^2 - 2\times 2^m(H_{2m}+2^m)x + H_{2m}+2^m)}$$

where

$$\Omega_3 = (2^m(2^{2m}x - H_{2m})x^{n+1} + (2^m x-1)(2^{2m}(n+2)x - (n+1)H_{2m})x^n)W_{mn+j}^2 + 2^{2m}(2^m(n+2)x-(n+1))x^n W_{mn-m+j}^2 + 2^m W_j^2 - 2^{2m}(2^{m+1}x-1)W_{j-m}^2 + 2^{j+1}(W_1^2+2W_0^2-3W_1W_0)(2^{mn}(n+1)x^n-1)(H_{2m}-2^{m+1})$$

and if $x = a$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_4}{2^{m+1}(3\times 2^{2m}x - 2^m - H_{2m})}$$

where

$$\Omega_4 = (2^{3m}(n+3)(n+2)x^2 - x\times 2^m(n+2)(n+1)(H_{2m}+2^m) + n(n+1)H_{2m})x^{n-1}W_{mn+j}^2 + 2^{2m}(n+1)((2+n)2^m x^n - nx^{n-1})W_{mn-m+j}^2 - 2^{3m+1}W_{j-m}^2 + 2n(n+1)2^{mn+j}(W_1^2+2W_0^2-3W_1W_0)\left(H_{2m}-2^{m+1}\right)x^{n-1}$$

(d) *If* $(1+2^{2m}x^2 - xH_{2m})(2^m x - 1) = u(x-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $x = a$, *then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_5}{6 \times 2^{3m}}$$

where

$\Omega_5 = (n+1)(2^{3m}(n+3)(n+2)x^2 - n2^m(n+2)(H_{2m}+2^m)x + n(n-1)H_{2m})x^{n-2}W_{mn+j}^2 + n2^{2m}(r^2+4s)(n+1)((n+2)2^m x + 1 - n)x^{n-2}W_{mn-m+j}^2 + 2(n-1)n(n+1)2^{mn+j}(H_{2m} - 2^{m+1})(W_1^2 + 2W_0^2 - 3W_1W_0)x^{n-2}$

Proof. Take $r = 3, s = -2$ and $H_n = H_n$ in Soykan [[46], Theorem 2.1.]. □

Note that (2) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j}^2 = \frac{\Omega_6}{(1+2^{2m}x^2 - xH_{2m})(2^m x - 1)}$$

where

$\Omega_6 = (2^m x - 1)(2^{2m}x - H_{2m})x^{n+1}W_{mn+j}^2 + 2^{2m}(2^m x - 1)x^{n+1}W_{mn-m+j}^2 - (2^m x - 1)(2^{2m}x - H_{2m})xW_j^2 - 2^{2m}(2^m x - 1)W_{j-m}^2 x + 2^{j+1}(W_1^2 + 2W_0^2 - 3W_1W_0)(2^{mn}x^n - 1)(H_{2m} - 2^{m+1})x.$

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 2.1.

For generalized Mersenne numbers, we have the following sum formulas for $n \geq 0$:

(a) $(m = 1, j = 0)$

If $(2x-1)(4x-1)(x-1) \neq 0$, *i.e.,* $x \neq \frac{1}{4}, x \neq \frac{1}{2}, x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Delta}{(2x-1)(4x-1)(x-1)}$$

where

$\Delta = (2x-1)(4x-5)x^{n+1}W_n^2 + 4(2x-1)x^{n+1}W_{n-1}^2 + (2x-1)W_0^2 - (2x-1)x(3W_0 - W_1)^2 + 2(W_1^2 + 2W_0^2 - 3W_1W_0)(2^n x^n - 1)x$

and

if $(2x-1)(4x-1)(x-1) = 0$, *i.e.,* $x = \frac{1}{4}$ *or* $x = \frac{1}{2}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Psi}{(24x^2 - 28x + 7)}$$

where

$\Psi = (n(2x-1)(4x-5) + 24x^2 - 28x + 5)x^n W_n^2 + 4((2x-1)n + 4x - 1)x^n W_{n-1}^2 + 2W_0^2 - (4x-1)(3W_0 - W_1)^2 + 2(W_1^2 + 2W_0^2 - 3W_1W_0)(2^n(n+1)x^n - 1)$

(b) $(m = 2, j = 0)$

If $(4x-1)(16x-1)(x-1) \neq 0$, *i.e.,* $x \neq \frac{1}{16}, x \neq \frac{1}{4}, x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k W_{2k}^2 = \frac{\Delta}{(4x-1)(16x-1)(x-1)}$$

where

$\Delta = (4x-1)(16x-17)x^{n+1}W_{2n}^2 + 16(4x-1)x^{n+1}W_{2n-2}^2 + (4x-1)W_0^2 - (4x-1)x(7W_0 - 3W_1)^2 + 18(W_1^2 + 2W_0^2 - 3W_1W_0)(2^{2n}x^n - 1)x$

and

if $(4x-1)(16x-1)(x-1) = 0$, *i.e.,* $x = \frac{1}{16}$ *or* $x = \frac{1}{4}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k W_{2k}^2 = \frac{\Psi}{3\left(64x^2 - 56x + 7\right)}$$

where

$\Psi = (n(4x-1)(16x-17) + 192x^2 - 168x + 17)x^n W_{2n}^2 + 16((4x-1)n + 8x - 1)x^n W_{2n-2}^2 + 4W_0^2 - (8x-1)(7W_0 - 3W_1)^2 + 18(W_1^2 + 2W_0^2 - 3W_1W_0)(2^{2n}(n+1)x^n - 1)$

(c) $(m=2,\ j=1)$

If $(4x-1)(16x-1)(x-1)\neq 0$, *i.e.*, $x\neq \frac{1}{16}$, $x\neq \frac{1}{4}$, $x\neq 1$, *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^2 = \frac{\Delta}{(4x-1)(16x-1)(x-1)}$$

where

$\Delta = (4x-1)(16x-17)x^{n+1}W_{2n+1}^2 + 16(4x-1)x^{n+1}W_{2n-1}^2 + (4x-1)W_1^2 - 4(4x-1)x(3W_0-W_1)^2 + 36(W_1^2+2W_0^2-3W_1W_0)(2^{2n}x^n-1)x$

and

if $(4x-1)(16x-1)(x-1)=0$, *i.e.*, $x=\frac{1}{16}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^2 = \frac{\Psi}{3\left(64x^2-56x+7\right)}$$

where

$\Psi = (n(4x-1)(16x-17)+192x^2-168x+17)x^nW_{2n+1}^2 + 16((4x-1)n+8x-1)x^nW_{2n-1}^2 + 4W_1^2 - 4(8x-1)(3W_0-W_1)^2 + 36(W_1^2+2W_0^2-3W_1W_0)(2^{2n}(n+1)x^n-1)$

(d) $(m=-1,\ j=0)$

If $(x-2)(x-1)(x-4)\neq 0$, *i.e.*, $x\neq 1$, $x\neq 2$, $x\neq 4$, *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Delta}{(x-2)(x-1)(x-4)}$$

where

$\Delta = (x-2)x^{n+1}W_{-n+1}^2 + (x-2)(x-5)x^{n+1}W_{-n}^2 + 4(x-2)W_0^2 - x(x-2)W_1^2 + 4(W_1^2+2W_0^2-3W_1W_0)(2^{-n}x^n-1)x$

and

if $(x-2)(x-1)(x-4)=0$, *i.e.*, $x=1$ *or* $x=2$ *or* $x=4$ *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Psi}{\left(3x^2-14x+14\right)}$$

where

$\Psi = (n(x-2)+2(x-1))x^nW_{-n+1}^2 + (n(x-2)(x-5)+3x^2-14x+10)x^nW_{-n}^2 + 4W_0^2 - 2(x-1)W_1^2 + 4(W_1^2+2W_0^2-3W_1W_0)(2^{-n}(n+1)x^n-1)$

(e) $(m=-2,\ j=0)$

If $(x-4)(x-1)(x-16)\neq 0$, *i.e.*, $x\neq 1$, $x\neq 4$, $x\neq 16$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^2 = \frac{\Delta}{(x-4)(x-1)(x-16)}$$

where

$\Delta = (x-4)x^{n+1}W_{-2n+2}^2 + (x-4)(x-17)x^{n+1}W_{-2n}^2 + 16(x-4)W_0^2 - x(x-4)(2W_0-3W_1)^2 + 72(W_1^2+2W_0^2-3W_1W_0)(2^{-2n}x^n-1)x$

and

if $(x-4)(x-1)(x-16)=0$, *i.e.*, $x=1$ *or* $x=4$ *or* $x=16$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^2 = \frac{\Psi}{3\left(x^2-14x+28\right)}$$

where

$\Psi = (n(x-4)+2(x-2))x^nW_{-2n+2}^2 + (n(x-4)(x-17)+3x^2-42x+68)x^nW_{-2n}^2 + 16W_0^2 - 2(x-2)(2W_0-3W_1)^2 + 72(W_1^2+2W_0^2-3W_1W_0)(2^{-2n}(n+1)x^n-1)$

(f) $(m=-2,\ j=1)$

If $(x-4)(x-1)(x-16)\neq 0$, *i.e.,* $x\neq 1,\ x\neq 4,\ x\neq 16$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^2 = \frac{\Delta}{(x-4)(x-1)(x-16)}$$

where

$\Delta = (x-4)x^{n+1}W_{-2n+3}^2 + (x-4)(x-17)x^{n+1}W_{-2n+1}^2 + 16(x-4)W_1^2 - x(x-4)(6W_0-7W_1)^2 + 144(W_1^2+2W_0^2-3W_1W_0)(2^{-2n}x^n-1)x$

and

if $(x-4)(x-1)(x-16)=0$, *i.e.,* $x=1$ *or* $x=4$ *or* $x=16$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^2 = \frac{\Psi}{3\left(x^2-14x+28\right)}$$

where

$\Psi = (n(x-4)+2(x-2))x^nW_{-2n+3}^2 + (n(x-4)(x-17)+3x^2-42x+68)x^nW_{-2n+1}^2 + 16W_1^2 - 2(x-2)(6W_0-7W_1)^2 + 144(W_1^2+2W_0^2-3W_1W_0)(2^{-2n}(n+1)x^n-1)$

From the above proposition, we have the following corollary which gives sum formulas of Mersenne numbers (take $W_n=M_n$ with $M_0=0, M_1=1$).

Corollary 2.1.
For $n\geq 0$, *Mersenne numbers have the following properties:*

(a) $(m=1,\ j=0)$

If $(2x-1)(4x-1)(x-1)\neq 0$, *i.e.,* $x\neq \frac{1}{4},\ x\neq \frac{1}{2},\ x\neq 1$, *then*

$$\sum_{k=0}^{n} x^k M_k^2 = \frac{(2x-1)(4x-5)x^{n+1}M_n^2 + 4(2x-1)x^{n+1}M_{n-1}^2 + x(2^{n+1}x^n-2x-1)}{(2x-1)(4x-1)(x-1)}$$

and

if $(2x-1)(4x-1)(x-1)=0$, *i.e.,* $x=\frac{1}{4}$ *or* $x=\frac{1}{2}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k M_k^2 = \frac{\Theta_1}{(24x^2-28x+7)}$$

where

$\Theta_1 = (n(2x-1)(4x-5)+24x^2-28x+5)x^nM_n^2 + 4((2x-1)n+4x-1)x^nM_{n-1}^2 + (n+1)2^{n+1}x^n - 4x - 1.$

(b) $(m=2,\ j=0)$

If $(4x-1)(16x-1)(x-1)\neq 0$, *i.e.,* $x\neq \frac{1}{16},\ x\neq \frac{1}{4},\ x\neq 1$, *then*

$$\sum_{k=0}^{n} x^k M_{2k}^2 = \frac{(4x-1)(16x-17)x^{n+1}M_{2n}^2 + 16(4x-1)x^{n+1}M_{2n-2}^2 + 9(2^{2n+1}x^n-4x-1)x}{(4x-1)(16x-1)(x-1)}$$

and

if $(4x-1)(16x-1)(x-1)=0$, *i.e.,* $x=\frac{1}{16}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k M_{2k}^2 = \frac{\Theta_2}{3\left(64x^2-56x+7\right)}$$

where

$\Theta_2 = (n(4x-1)(16x-17)+192x^2-168x+17)x^nM_{2n}^2 + 16((4x-1)n+8x-1)x^nM_{2n-2}^2 + 9((n+1)2^{2n+1}x^n-8x-1).$

(c) $(m=2,\ j=1)$

If $(4x-1)(16x-1)(x-1)\neq 0$, *i.e.,* $x\neq \frac{1}{16}$, $x\neq \frac{1}{4}$, $x\neq 1$, *then*

$$\sum_{k=0}^{n} x^k M_{2k+1}^2 = \frac{(4x-1)(16x-17)x^{n+1}M_{2n+1}^2 + 16(4x-1)x^{n+1}M_{2n-1}^2 + (9\times 2^{2n+2}x^{n+1} - 16x^2 - 28x - 1)}{(4x-1)(16x-1)(x-1)}$$

and

if $(4x-1)(16x-1)(x-1)=0$, *i.e.,* $x=\frac{1}{16}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k M_{2k+1}^2 = \frac{\Theta_3}{3\left(64x^2-56x+7\right)}$$

where

$\Theta_3 = (n(4x-1)(16x-17)+192x^2-168x+17)x^n M_{2n+1}^2 + 16((4x-1)n+8x-1)x^n M_{2n-1}^2 + 4(9(n+1)2^{2n}x^n - 8x - 7).$

(d) $(m=-1,\ j=0)$

If $(x-2)(x-1)(x-4)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 2$, $x\neq 4$, *then*

$$\sum_{k=0}^{n} x^k M_{-k}^2 = \frac{(x-2)x^{n+1}M_{-n+1}^2 + (x-2)(x-5)x^{n+1}M_{-n}^2 + x(2^{-n+2}x^n - x - 2)}{(x-2)(x-1)(x-4)}$$

and

if $(x-2)(x-1)(x-4)=0$, *i.e.,* $x=1$ *or* $x=2$ *or* $x=4$ *then*

$$\sum_{k=0}^{n} x^k M_{-k}^2 = \frac{\Theta_4}{\left(3x^2-14x+14\right)}$$

where

$\Theta_4 = (n(x-2)+2(x-1))x^n M_{-n+1}^2 + (n(x-2)(x-5)+3x^2-14x+10)x^n M_{-n}^2 + 2((n+1)2^{-n+1}x^n - (x+1)).$

(e) $(m=-2,\ j=0)$

If $(x-4)(x-1)(x-16)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 4$, $x\neq 16$, *then*

$$\sum_{k=0}^{n} x^k M_{-2k}^2 = \frac{(x-4)x^{n+1}M_{-2n+2}^2 + (x-4)(x-17)x^{n+1}M_{-2n}^2 + 9(2^{-2n+3}x^n - x - 4)x}{(x-4)(x-1)(x-16)}$$

and

if $(x-4)(x-1)(x-16)=0$, *i.e.,* $x=1$ *or* $x=4$ *or* $x=16$ *then*

$$\sum_{k=0}^{n} x^k M_{-2k}^2 = \frac{\Theta_5}{3\left(x^2-14x+28\right)}$$

where

$\Theta_5 = (n(x-4)+2(x-2))x^n M_{-2n+2}^2 + (n(x-4)(x-17)+3x^2-42x+68)x^n M_{-2n}^2 + 18(2^{-2n+2}(n+1)x^n - x - 2).$

(f) $(m=-2,\ j=1)$

If $(x-4)(x-1)(x-16)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 4$, $x\neq 16$, *then*

$$\sum_{k=0}^{n} x^k M_{-2k+1}^2 = \frac{(x-4)x^{n+1}M_{-2n+3}^2 + (x-4)(x-17)x^{n+1}M_{-2n+1}^2 + (9\times 2^{-2n+4}x^{n+1} - 49x^2 + 68x - 64)}{(x-4)(x-1)(x-16)}$$

and

if $(x-4)(x-1)(x-16)=0$, *i.e.,* $x=1$ *or* $x=4$ *or* $x=16$ *then*

$$\sum_{k=0}^{n} x^k M_{-2k+1}^2 = \frac{\Theta_6}{3\left(x^2-14x+28\right)}$$

where

$\Theta_6 = (n(x-4)+2(x-2))x^n M_{-2n+3}^2 + (n(x-4)(x-17)+3x^2-42x+68)x^n M_{-2n+1}^2 + 2(9\times 2^{-2n+3}(n+1)x^n - 49x + 34).$

Taking $W_n = H_n$ with $H_0 = 2, H_1 = 3$ in the last proposition, we have the following corollary which presents sum formulas of Mersenne-Lucas numbers.

Corollary 2.2.
For $n \geq 0$, Mersenne-Lucas numbers have the following properties:

(a) $(m = 1,\ j = 0)$

If $(2x-1)(4x-1)(x-1) \neq 0$, *i.e.,* $x \neq \frac{1}{4}, x \neq \frac{1}{2}, x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k H_k^2 = \frac{(2x-1)(4x-5)x^{n+1}H_n^2 + 4(2x-1)x^{n+1}H_{n-1}^2 - (2^{n+1}x^{n+1} + 18x^2 - 19x + 4)}{(2x-1)(4x-1)(x-1)}$$

and

if $(2x-1)(4x-1)(x-1) = 0$, *i.e.,* $x = \frac{1}{4}$ *or* $x = \frac{1}{2}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k H_k^2 = \frac{\Theta_7}{(24x^2 - 28x + 7)}$$

where

$\Theta_7 = (n(2x-1)(4x-5) + 24x^2 - 28x + 5)x^n H_n^2 + 4((2x-1)n + 4x - 1)x^n H_{n-1}^2 - ((n+1)2^{n+1}x^n + 36x - 19).$

(b) $(m = 2,\ j = 0)$

If $(4x-1)(16x-1)(x-1) \neq 0$, *i.e.,* $x \neq \frac{1}{16}, x \neq \frac{1}{4}, x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k H_{2k}^2 = \frac{(4x-1)(16x-17)x^{n+1}H_{2n}^2 + 16(4x-1)x^{n+1}H_{2n-2}^2 - (9 \times 2^{2n+1}x^{n+1} + 100x^2 - 59x + 4)}{(4x-1)(16x-1)(x-1)}$$

and

if $(4x-1)(16x-1)(x-1) = 0$, *i.e.,* $x = \frac{1}{16}$ *or* $x = \frac{1}{4}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k H_{2k}^2 = \frac{\Theta_8}{3\left(64x^2 - 56x + 7\right)}$$

where

$\Theta_8 = (n(4x-1)(16x-17) + 192x^2 - 168x + 17)x^n H_{2n}^2 + 16((4x-1)n + 8x - 1)x^n H_{2n-2}^2 - (9(n+1)2^{2n+1}x^n + 200x - 59).$

(c) $(m = 2,\ j = 1)$

If $(4x-1)(16x-1)(x-1) \neq 0$, *i.e.,* $x \neq \frac{1}{16}, x \neq \frac{1}{4}, x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k H_{2k+1}^2 = \frac{(4x-1)(16x-17)x^{n+1}H_{2n+1}^2 + 16(4x-1)x^{n+1}H_{2n-1}^2 - 9(2^{2n+2}x^{n+1} + 16x^2 - 12x + 1)}{(4x-1)(16x-1)(x-1)}$$

and

if $(4x-1)(16x-1)(x-1) = 0$, *i.e.,* $x = \frac{1}{16}$ *or* $x = \frac{1}{4}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k H_{2k+1}^2 = \frac{\Theta_9}{3\left(64x^2 - 56x + 7\right)}$$

where

$\Theta_9 = (n(4x-1)(16x-17) + 192x^2 - 168x + 17)x^n H_{2n+1}^2 + 16((4x-1)n + 8x - 1)x^n H_{2n-1}^2 - 36((n+1)2^{2n}x^n + 8x - 3).$

(d) $(m = -1,\ j = 0)$

If $(x-2)(x-1)(x-4) \neq 0$, *i.e.,* $x \neq 1, x \neq 2, x \neq 4$, *then*

$$\sum_{k=0}^{n} x^k H_{-k}^2 = \frac{(x-2)x^{n+1}H_{-n+1}^2 + (x-2)(x-5)x^{n+1}H_{-n}^2 - (2^{-n+2}x^{n+1} + 9x^2 - 38x + 32)}{(x-2)(x-1)(x-4)}$$

and

if $(x-2)(x-1)(x-4) = 0$, *i.e.,* $x = 1$ *or* $x = 2$ *or* $x = 4$ *then*

$$\sum_{k=0}^{n} x^k H_{-k}^2 = \frac{\Theta_{10}}{\left(3x^2 - 14x + 14\right)}$$

where

$\Theta_{10} = (n(x-2) + 2(x-1))x^n H_{-n+1}^2 + (n(x-2)(x-5) + 3x^2 - 14x + 10)x^n H_{-n}^2 - 2((n+1)2^{-n+1}x^n + 9x - 19).$

(e) $(m=-2,\ j=0)$

If $(x-4)(x-1)(x-16)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 4$, $x\neq 16$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k}^2 = \frac{(x-4)x^{n+1}H_{-2n+2}^2+(x-4)(x-17)x^{n+1}H_{-2n}^2-(9x^{n+1}2^{-2n+3}+25x^2-236x+256)}{(x-4)(x-1)(x-16)}$$

and

if $(x-4)(x-1)(x-16)=0$, *i.e.,* $x=1$ *or* $x=4$ *or* $x=16$ *then*

$$\sum_{k=0}^{n} x^k H_{-2k}^2 = \frac{\Theta_{11}}{3\left(x^2-14x+28\right)}$$

where

$$\Theta_{11}=(n(x-4)+2(x-2))x^nH_{-2n+2}^2+(n(x-4)(x-17)+3x^2-42x+68)x^nH_{-2n}^2-2(9(n+1)2^{-2n+2}x^n+25x-118).$$

(f) $(m=-2,\ j=1)$

If $(x-4)(x-1)(x-16)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 4$, $x\neq 16$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1}^2 = \frac{(x-4)x^{n+1}H_{-2n+3}^2+(x-4)(x-17)x^{n+1}H_{-2n+1}^2-9(2^{-2n+4}x^{n+1}+9x^2+64-68x)}{(x-4)(x-1)(x-16)}$$

and

if $(x-4)(x-1)(x-16)=0$, *i.e.,* $x=1$ *or* $x=4$ *or* $x=16$ *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1}^2 = \frac{\Theta_{12}}{3\left(x^2-14x+28\right)}$$

where

$$\Theta_{12}=(n(x-4)+2(x-2))x^nH_{-2n+3}^2+(n(x-4)(x-17)+3x^2-42x+68)x^nH_{-2n+1}^2-18((n+1)2^{-2n+3}x^n+9x-34).$$

Taking $x=1$ in the last two corollaries we get the following corollary.

Corollary 2.3.
For $n\geq 0$, *Mersenne numbers, and Mersenne-Lucas numbers have the following properties:*

1.

- **(a)** $\sum_{k=0}^{n} M_k^2=\frac{1}{3}(-(n-1)M_n^2+4(n+3)M_{n-1}^2+(n+1)2^{n+1}-5)$.
- **(b)** $\sum_{k=0}^{n} M_{2k}^2=\frac{1}{45}(-(3n-41)M_{2n}^2+16(3n+7)M_{2n-2}^2+9(n+1)2^{2n+1}-81)$.
- **(c)** $\sum_{k=0}^{n} M_{2k+1}^2=\frac{1}{45}(-(3n-41)M_{2n+1}^2+16(3n+7)M_{2n-1}^2+9(n+1)2^{2n+2}-60)$.
- **(d)** $\sum_{k=0}^{n} M_{-k}^2=\frac{1}{3}(-nM_{-n+1}^2+(4n-1)M_{-n}^2+(n+1)2^{-n+2}-4)$.
- **(e)** $\sum_{k=0}^{n} M_{-2k}^2=\frac{1}{45}(-(3n+2)M_{-2n+2}^2+(48n+29)M_{-2n}^2+9(n+1)2^{-2n+3}-54)$.
- **(f)** $\sum_{k=0}^{n} M_{-2k+1}^2=\frac{1}{45}(-(3n+2)M_{-2n+3}^2+(48n+29)M_{-2n+1}^2+9(n+1)2^{-2n+4}-30)$.

2.

- **(a)** $\sum_{k=0}^{n} H_k^2=\frac{1}{3}(-(n-1)H_n^2+4(n+3)H_{n-1}^2-(n+1)2^{n+1}-17)$.
- **(b)** $\sum_{k=0}^{n} H_{2k}^2=\frac{1}{45}(-(3n-41)H_{2n}^2+16(3n+7)H_{2n-2}^2-9(n+1)2^{2n+1}-141)$.
- **(c)** $\sum_{k=0}^{n} H_{2k+1}^2=\frac{1}{45}(-(3n-41)H_{2n+1}^2+16(3n+7)H_{2n-1}^2-9(n+1)2^{2n+2}-180)$.
- **(d)** $\sum_{k=0}^{n} H_{-k}^2=\frac{1}{3}(-nH_{-n+1}^2+(4n-1)H_{-n}^2-(n+1)2^{-n+2}+20)$.
- **(e)** $\sum_{k=0}^{n} H_{-2k}^2=\frac{1}{45}(-(3n+2)H_{-2n+2}^2+(48n+29)H_{-2n}^2-9(n+1)2^{-2n+3}+186)$.
- **(f)** $\sum_{k=0}^{n} H_{-2k+1}^2=\frac{1}{45}(-(3n+2)H_{-2n+3}^2+(48n+29)H_{-2n+1}^2-9(n+1)2^{-2n+4}+450)$.

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