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Sums and Partial Sums of Horadam Sequences: The Sum Formulas $\sum_{k=0}^{n} x^k W_k$ and $\sum_{k=n}^{n+m} x^k W_k$ via Generating Functions

Research Article

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Abstract: In this paper, we present sums and partial sums of Horadam sequences via generating functions which extends a recent result of Prodinger [6].

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Keywords: Horadam numbers • Horadam sequence • Sum • Partial sum • Fibonacci numbers • Lucas numbers

1. Introduction

The generalized Fibonacci sequence (or generalized (r,s)-sequence or Horadam sequence or 2-step Fibonacci sequence) $\{W_n(W_0,W_1;r,s)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined (by Horadam [2]) as follows:

$$W_n = rW_{n-1} + sW_{n-2}, \quad W_0 = a, W_1 = b, \quad n \geq 2 \tag{1}$$

where W_0, W_1 are arbitrary complex (or real) numbers and r, s are real numbers, see also Horadam [1, 3] and [4].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1,2,3,...$ when $s \neq 0$. Therefore, recurrence (1) holds for all integer n.

For more information on Horadam numbers, see for example, [8] and [9]. For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1 and Table 2) are used for the special cases of r, s and initial values.

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

Binet's formula of generalized Fibonacci sequence can be calculated using its characteristic equation (the quadratic equation) which is given as

$$x^2 - rx - s = 0. \tag{2}$$

The roots of characteristic equation are

$$\alpha = \frac{r+\sqrt{r^2+4s}}{2}, \quad \beta = \frac{r-\sqrt{r^2+4s}}{2}, \tag{3}$$

Binet's formula can be given as follows:

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Table 1. A few special case of generalized Fibonacci sequence.

No	Sequences (Numbers)	Notation
1	Generalized Fibonacci	$\{W_n\} = \{W_n(W_0, W_1; 1, 1)\}$
2	Generalized Pell	$\{W_n\} = \{W_n(W_0, W_1; 2, 1)\}$
3	Generalized Jacobsthal	$\{W_n\} = \{W_n(W_0, W_1; 1, 2)\}$
4	Generalized 2-primes	$\{W_n\} = \{W_n(W_0, W_1; 2, 3)\}$
5	Generalized Mersenne	$\{W_n\} = \{W_n(W_0, W_1; 3, -2)\}$
6	Generalized p-Mersenne	$\{W_n\} = \{W_n(W_0, W_1; p, -(p-1)\}$
7	Generalized balancing	$\{W_n\} = \{W_n(W_0, W_1; 6, -1)\}$
8	Generalized Oresme	$\{W_n\} = \{W_n(W_0, W_1; 1, -\frac{1}{4})\}$
9	Generalized p-Oresme	$\{W_n\} = \{W_n(W_0, W_1; 1, -\frac{1}{p^2}\}$

Table 2. Notation tables of a few special case of generalized (r,s) (generalized Fibonacci) sequence.

No	Name of sequence	Notation: $W_n(W_0, W_1; r, s)$	OEIS [7]
1	Fibonacci	$F_n = W_n(0, 1; 1, 1)$	A000045
2	Lucas	$L_n = W_n(2, 1; 1, 1)$	A000032
3	Pell	$P_n = W_n(0, 1; 2, 1)$	A000129
4	Pell-Lucas	$Q_n = W_n(2, 2; 2, 1)$	A002203
5	Jacobsthal	$J_n = W_n(0, 1; 1, 2)$	A001045
6	Jacobsthal-Lucas	$j_n = W_n(2, 1; 1, 2)$	A014551
7	2-primes	$G_n = W_n(1, 2; 2, 3)$	A015518
8	Lucas 2-primes	$H_n = W_n(2, 2; 2, 3)$	A102345
9	Modified 2-primes	$E_n = W_n(1, 1; 2, 3)$	A046717
10	Mersenne	$M_n = W_n(0, 1; 3, -2)$	A000225
11	Mersenne-Lucas	$H_n = W_n(2, 3; 3, -2)$	A000051
12	Balancing	$B_n = W_n(0, 1; 6, -1)$	A001109
13	modified Lucas-balancing	$H_n = W_n(2, 6; 6, -1)$	A003499
14	Lucas-balancing	$C_n = W_n(1, 3; 6, -1)$	A001541
15	Modified Oresme	$G_n = W_n(0, 1; 1, -\frac{1}{4})$	
16	Oresme-Lucas	$H_n = W_n(2, 1; 1, -\frac{1}{4})$	
17	Oresme	$O_n = W_n(0, \frac{1}{2}; 1, -\frac{1}{4})$	

Theorem 1.1.
The general term of the generalized Fibonacci sequence W_n can be presented by the following Binet formula:

$$W_n = \begin{cases} \dfrac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \dfrac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ (nW_1 - \alpha(n-1)W_0)\alpha^{n-1} & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases}. \tag{4}$$

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 1.1.
Suppose that $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Fibonacci sequence $\{W_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} W_n x^n$ is given by

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x}{1 - rx - sx^2}. \tag{5}$$

Now we define two special cases of the sequence $\{W_n\}$. (r,s) sequence $\{G_n(0,1;r,s)\}_{n\geq 0}$ and Lucas (r,s) sequence $\{H_n(2,r;r,s)\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = rG_{n+1} + sG_n, \quad G_0 = 0, G_1 = 1, \tag{6}$$

$$H_{n+2} = rH_{n+1} + sH_n, \quad H_0 = 2, H_1 = r. \tag{7}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{r}{s}G_{-(n-1)}+\frac{1}{s}G_{-(n-2)}, \\ H_{-n} &= -\frac{r}{s}H_{-(n-1)}+\frac{1}{s}H_{-(n-2)}, \end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (6) and (7) hold for all integer n.

Some special cases of (r,s) sequence $\{G_n(0,1;r,s)\}_{n\geq 0}$ and Lucas (r,s) sequence $\{H_n(2,r;r,s)\}_{n\geq 0}$ are as follows:

1. $G_n(0,1;1,1)=F_n$, Fibonacci sequence,
2. $H_n(2,1;1,1)=L_n$, Lucas sequence,
3. $G_n(0,1;2,1)=P_n$, Pell sequence,
4. $H_n(2,2;2,1)=Q_n$, Pell-Lucas sequence,
5. $G_n(0,1;1,2)=J_n$, Jacobsthal sequence,
6. $H_n(2,1;1,2)=j_n$, Jacobsthal-Lucas sequence.
7. $G_n(1,2;2,3)=G_n$, 2-primes sequence,
8. $H_n(2,2;2,3)=H_n$, Lucas 2-primes sequence,
9. $G_n(0,1;3,-2)=M_n$, Mersenne sequence,
10. $H_n(2,3;3,-2)=H_n$, Mersenne-Lucas sequence,
11. $G_n(0,1;6,-1)=B_n$, balancing sequence,
12. $H_n(2,6;6,-1)=H_n$, modified Lucas-balancing sequence,
13. $G_n(0,1;1,-\frac{1}{4})=G_n$, modified Oresme sequence,
14. $H_n(2,1;1,-\frac{1}{4})=H_n$, Oresme-Lucas-Lucas sequence.

Binet's formulas of (r,s) and Lucas (r,s) numbers are given as

$$G_n=\begin{cases} \dfrac{\alpha^n-\beta^n}{\alpha-\beta} & , \text{ if } \alpha\neq\beta \text{ (Distinct Roots Case)} \\ n\alpha^{n-1} & , \text{ if } \alpha=\beta \text{ (Single Root Case)} \end{cases}$$

and

$$H_n=\begin{cases} \alpha^n+\beta^n & , \text{ if } \alpha\neq\beta \text{ (Distinct Roots Case)} \\ 2\alpha^n & , \text{ if } \alpha=\beta \text{ (Single Root Case)} \end{cases}.$$

Lemma 1.1 gives the following results as particular examples (generating functions of (r,s) and Lucas (r,s) numbers).

Corollary 1.1.
Generating functions of (r,s) and Lucas (r,s) numbers are

$$\begin{aligned} \sum_{n=0}^{\infty} G_n x^n &= \frac{x}{1-rx-sx^2}, \\ \sum_{n=0}^{\infty} H_n x^n &= \frac{2-rx}{1-rx-sx^2}, \end{aligned}$$

respectively.

Note that if

$$K_n=\sum_{k=0}^{n} W_k$$

then

$$\sum_{k=n}^{n+m} W_k=\sum_{k=0}^{n+m} W_k-\sum_{k=0}^{n-1} W_k=K_{n+m}-K_{n-1}.$$

In a quite recent preprint, Prodinger [6] proved the following Theorem via generating functions which gives complete answer the sum problem addressed in [5].

Theorem 1.2.
(Prodinger) For $n \geq 0$, *we have*

$$K_n = \frac{W_0 + W_1 - W_0 r}{1-r-s} - \frac{(2W_0 s + W_1 r + 2W_1 s - W_0 r s)}{2(1-r-s)} G_n - \frac{W_1 + W_0 s}{2(1-r-s)} H_n$$

and

$$K_{n+m} - K_{n-1} = -\frac{(2W_0 s + W_1 r + 2W_1 s - W_0 r s)}{2(1-r-s)} (G_{m+n} - G_{n-1}) - \frac{(W_1 + W_0 s)}{2(1-r-s)} (H_{m+n} - H_{n-1}).$$

In the next sections, we extend the results of Prodinger. Note that Prodinger does not consider the case $1-r-s=0$ which we also deal with this case as well. For example, for the generalized Mersen numbers (the case $r=3$ and $s=-2$), we get $1-r-s=0$.

2. Extension of Prodinger's Theorem 1.2: The Sum Formulas $\sum_{k=0}^{n} x^k W_k$ and $\sum_{k=n}^{n+m} x^k W_k$ via Generating Functions

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} x^n W_n z^n$ of the sequence $\{x^n W_n\}$.

Lemma 2.1.
Suppose that $f_{x^n W_n}(z) = \sum\limits_{n=0}^{\infty} x^n W_n z^n$ *is the ordinary generating function of the sequence* $\{x^n W_n\}_{n\geq 0}$. *Then,* $\sum\limits_{n=0}^{\infty} x^n W_n z^n$ *is given by*

$$\sum_{n=0}^{\infty} x^n W_n z^n = \frac{W_0 + x(W_1 - r W_0) z}{1 - r x z - s x^2 z^2}. \tag{8}$$

Proof. Note that

$$x^n W_n = x^n (r W_{n-1} + s W_{n-2}).$$

Using the definition of generalized Fibonacci numbers, and substracting $rxz \sum_{n=0}^{\infty} x^n W_n z^n$ and $sx^2 z^2 \sum_{n=0}^{\infty} x^n W_n z^n$ from $\sum_{n=0}^{\infty} x^n W_n z^n$ we obtain

$$\begin{aligned}
(1 - rxz - sx^2 z^2) \sum_{n=0}^{\infty} x^n W_n z^n &= \sum_{n=0}^{\infty} x^n W_n z^n - rxz \sum_{n=0}^{\infty} x^n W_n z^n - sx^2 z^2 \sum_{n=0}^{\infty} x^n W_n z^n \\
&= \sum_{n=0}^{\infty} x^n W_n z^n - r \sum_{n=0}^{\infty} x^{n+1} W_n z^{n+1} - s \sum_{n=0}^{\infty} x^{n+2} W_n z^{n+2} \\
&= \sum_{n=0}^{\infty} x^n W_n z^n - r \sum_{n=1}^{\infty} x^n W_{n-1} z^n - s \sum_{n=2}^{\infty} x^n W_{n-2} z^n \\
&= (W_0 + x W_1 z) - r x W_0 z + \sum_{n=2}^{\infty} x^n (W_n - r W_{n-1} - s W_{n-2}) z^n \\
&= W_0 + x(W_1 - r W_0) z.
\end{aligned}$$

Rearranging above equation, we obtain (8). □

Lemma 2.1 gives the following results as particular examples.

Corollary 2.1.
Generating functions $\sum\limits_{n=0}^{\infty} x^n G_n z^n$ *and* $\sum\limits_{n=0}^{\infty} x^n H_n z^n$ *are*

$$\begin{aligned}
\sum_{n=0}^{\infty} x^n G_n z^n &= \frac{xz}{1 - rxz - sx^2 z^2}, \\
\sum_{n=0}^{\infty} x^n H_n z^n &= \frac{2 - rxz}{1 - rxz - sx^2 z^2},
\end{aligned}$$

respectively.

Let

$$S_n = \sum_{k=0}^{n} x^k W_k.$$

The following theorem presents some sum formulas of generalized Fibonacci (Horadam) numbers with positive subscripts.

Theorem 2.1.
Let x be a nonzero complex (or real) number.

(a) *If $1-rx-sx^2 \neq 0$ then*

$$\begin{aligned} S_n &= \frac{W_0+x(W_1-rW_0)}{1-rx-sx^2} - \frac{2W_1sx^2+2W_0sx+W_1rx-W_0rsx^2}{2(1-rx-sx^2)}x^nG_n - \frac{W_1x+W_0sx^2}{2(1-rx-sx^2)}x^nH_n \\ &= \frac{\Theta(x)}{2(1-rx-sx^2)} \end{aligned} \qquad (9)$$

where

$\Theta(x) = 2(W_0+(W_1-rW_0)x) - (s(2W_1-rW_0)x+(rW_1+2sW_0))x^{n+1}G_n - (W_1+sxW_0)x^{n+1}H_n.$

(b) *If $1-rx-sx^2 = u(x-a)(x-b) = 0$ for some $u,a,b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $x=a$ or $x=b$, then*

$$S_n = \frac{\Theta_1(x)}{-2(r+2sx)}$$

where

$\Theta_1(x) = 2(W_1-rW_0)+(-(r+nr+4sx+2nsx)W_1+s(-2n+2rx+nrx-2)W_0)x^nG_n - x^n((n+1)W_1+sx(n+2)W_0)H_n.$

(c) *If $1-rx-sx^2 = u(x-c)^2 = 0$ for some $u,c \in \mathbb{C}$ with $u \neq 0$, i.e., $x=c$, then*

$$S_n = \frac{\Theta_2(x)}{4s}$$

where

$\Theta_2(x) = (n+1)((nr+4sx+2nsx)W_1+s(2n-2rx-nrx)W_0)x^{n-1}G_n + (n+1)(nx^{n-1}W_1+sx^n(n+2)W_0)H_n.$

Proof.

(a) Note that using generating functions, we get

$$\begin{aligned} S(z) &= \sum_{n=0}^{\infty} S_n z^n = \frac{1}{1-z}\frac{W_0+x(W_1-rW_0)z}{1-rxz-sx^2z^2} \\ &= \frac{W_0+x(W_1-rW_0)}{1-rx-sx^2}\frac{1}{1-z} - \frac{2W_1sx^2+2W_0sx+W_1rx-W_0rsx^2}{2(1-rx-sx^2)}\frac{xz}{1-rxz-sx^2z^2} \\ &\quad -\frac{W_1x+W_0sx^2}{2(1-rx-sx^2)}\frac{2-rxz}{1-rxz-sx^2z^2} \\ &= \frac{W_0+x(W_1-rW_0)}{1-rx-sx^2}\sum_{n=0}^{\infty} z^n - \frac{2W_1sx^2+2W_0sx+W_1rx-W_0rsx^2}{2(1-rx-sx^2)}\sum_{n=0}^{\infty} x^nG_nz^n \\ &\quad -\frac{W_1x+W_0sx^2}{2(1-rx-sx^2)}\sum_{n=0}^{\infty} x^nH_nz^n \\ &= \sum_{n=0}^{\infty}\left(\frac{W_0+x(W_1-rW_0)}{1-rx-sx^2} - \frac{2W_1sx^2+2W_0sx+W_1rx-W_0rsx^2}{2(1-rx-sx^2)}x^nG_n - \frac{W_1x+W_0sx^2}{2(1-rx-sx^2)}x^nH_n\right)z^n. \end{aligned}$$

Comparing on both sides leads to (9).

(b) We use (9). For $x=a$ and $x=b$, the right hand side of the above sum formula (9) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

$$\sum_{k=0}^{n} a^k W_k = \left.\frac{\frac{d}{dx}\Theta(x)}{\frac{d}{dx}(2(1-rx-sx^2))}\right|_{x=a}$$

and similarly

$$\sum_{k=0}^{n} b^k W_k = \left.\frac{\frac{d}{dx}\Theta(x)}{\frac{d}{dx}(2(1-rx-sx^2))}\right|_{x=b}.$$

(c) We use (9). For $x = c$, the right hand side of the above sum formula (9) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (c) by using

$$\sum_{k=0}^{n} c^k W_k = \left.\frac{\frac{d^2}{dx^2}\Theta(x)}{\frac{d^2}{dx^2}(2(1-rx-sx^2))}\right|_{x=c}. \square$$

We now concentrate on finding expressions for the partial sums

$$\sum_{k=n}^{n+m} x^k W_k = \sum_{k=0}^{n+m} x^k W_k - \sum_{k=0}^{n-1} x^k W_k = S_{n+m} - S_{n-1}.$$

Corollary 2.2.
Let x be a non-zero complex (or real) number.

(a) *If* $1 - rx - sx^2 \neq 0$ *then*

$$S_{n+m} - S_{n-1} = \frac{\Gamma_1}{2(1-rx-sx^2)}$$

where

$\Gamma_1 = -(s(2W_1 - rW_0)x + (rW_1 + 2sW_0))(x^{m+1}G_{n+m} - G_{n-1})x^n - (W_1 + sxW_0)(x^{m+1}H_{n+m} - H_{n-1})x^n.$

(b) *If* $1 - rx - sx^2 = u(x-a)(x-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $x = a$ *or* $x = b$, *then*

$$S_{n+m} - S_{n-1} = \frac{\Gamma_2}{-2(r+2sx)}$$

where

$\Gamma_2 = (-(r+(n+m)r+4sx+2(n+m)sx)W_1 + s(-2(n+m)+2rx+(n+m)rx-2)W_0)x^{n+m}G_{n+m} - (-(r+(n-1)r+4sx+2(n-1)sx)W_1 + s(-2(n-1)+2rx+(n-1)rx-2)W_0)x^{n-1}G_{n-1} - (((n+m)+1)W_1 + sx((n+m)+2)W_0)x^{n+m}H_{n+m} + (((n-1)+1)W_1 + sx((n-1)+2)W_0)x^{n-1}H_{n-1}.$

(c) *If* $1 - rx - sx^2 = u(x-c)^2 = 0$ *for some* $u, c \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $x = c$, *then*

$$S_{n+m} - S_{n-1} = \frac{\Gamma_3}{4s}$$

where

$\Gamma_3 = ((n+m)+1)(((n+m)r + 4sx + 2(n+m)sx)W_1 + s(2(n+m) - 2rx - (n+m)rx)W_0)x^{n+m-1}G_{n+m} - ((n-1)+1)(((n-1)r + 4sx + 2(n-1)sx)W_1 + s(2(n-1) - 2rx - (n-1)rx)W_0)x^{n-2}G_{n-1} + ((n+m)+1)((n+m)x^{n+m-1}W_1 + sx^{n+m}((n+m)+2)W_0)H_{n+m} - ((n-1)+1)((n-1)x^{n-2}W_1 + sx^{n-1}((n-1)+2)W_0)H_{n-1}.$

3. The Sum Formulas $\sum_{k=0}^{n} kx^k W_k$ and $\sum_{k=n}^{n+m} kx^k W_k$

Let

$$Y_n = \sum_{k=0}^{n} kx^k W_k.$$

The following theorem presents some sum formulas of generalized Fibonacci (Horadam) numbers with positive subscripts.

Theorem 3.1.
Let x be a non-zero complex (or real) number. Then

(a) *If* $1 - rx - sx^2 \neq 0$ *then*

$$\begin{aligned} Y_n &= \frac{x(1-rx-sx^2)\Theta'(x) + x(r+2sx)\Theta(x)}{2(1-rx-sx^2)^2} \\ &= \frac{\Delta_1}{2(1-rx-sx^2)^2} \end{aligned} \tag{10}$$

where $\Theta(x)$ *is as in Theorem 2.1 (a) and* $\Theta'(x)$ *denotes the derivative of* $\Theta(x)$ *with respect to x, and*

$\Delta_1 = 2x((sx^2+1)W_1 - sx(rx-2)W_0) + ((2ns^2x^3 + rsx^2 + 3nrsx^2 + nr^2x - 2nsx - 4sx - r - nr)W_1 - s(nrsx^3 + 2sx^2 + r^2x^2 - 2nsx^2 + nr^2x^2 - 2rx - 3nrx + 2n + 2)W_0)x^{n+1}G_n + ((nsx^2 - sx^2 + nrx - n - 1)W_1 + sx(nsx^2 + nrx - n + rx - 2)W_0)x^{n+1}H_n.$

(b) *If* $1-rx-sx^2 = u(x-a)(x-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$*, i.e.,* $x = a$ *or* $x = b$*, then*

$$\begin{aligned} Y_n &= \frac{4s\Theta(x)+2sx\Theta^{'}(x)-\left(4sx^2+3rx-2\right)\Theta^{''}(x)+x(1-rx-sx^2)\Theta^{'''}(x)}{4\left(-2s+6s^2x^2+r^2+6rsx\right)} \\ &= \frac{\Delta_2}{4\left(-2s+6s^2x^2+r^2+6rsx\right)} \end{aligned}$$

where $\Theta^{''}(x)$ *and* $\Theta^{'''}(x)$ *denote the second and third derivatives of* $\Theta(x)$*, respectively, with respect to* x*, and*

$\Delta_2 = 12sxW_1-4s(3rx-2)W_0+((-nr(n+1)^2+x((n+2)(n+1)(nr^2-2ns-4s)+sx(n+3)(2n^2sx+3n^2r+8nsx+7nr+2r)))W_1+s(-2n(n+1)^2+x(r(3n+2)(n+2)(n+1)-x(n+3)(n^2rsx+3nr^2-2n^2s+n^2r^2+4nrsx-2ns+2r^2+4s)))W_0)x^{n-1}G_n+((-n(n+1)^2+x(n+2)(n^2r+n^2sx+2nsx+nr-3sx))W_1+sx(-(n+1)(n+2)^2+x(n+3)(n^2sx+n^2r+4nsx+2r+3nr))W_0)x^{n-1}H_n.$

(c) *If* $1-rx-sx^2 = u(x-c)^2 = 0$ *for some* $u, c \in \mathbb{C}$ *with* $u \neq 0$*, i.e.,* $x = c$*, then*

$$\begin{aligned} Y_n &= \frac{-4\left(2r+5sx\right)\Theta^{'''}(x)-(10sx^2+7rx-4)\Theta^{''''}(x)+x((1-rx-sx^2)\Theta^{'''''}(x)}{48s^2} \\ &= \frac{\Delta_3}{12s^2} \end{aligned}$$

where $\Theta^{''''}(x)$ *and* $\Theta^{'''''}(x)$ *denote the fourth and fifth derivatives of* $\Theta(x)$*, respectively, with respect to* x*, and*

$\Delta_3 = x^{-2}(((nx^n(2r+5sx)(n+1)(-r+nr+4sx+2nsx)+546x^{12}(630s^2x^3+187r^2x+760rsx^2-450sx-154r))W_1+s(nx^n(n+1)(2n-2rx-nrx-2)(2r+5sx)-546x^{12}(315rsx^3-440sx^2+270r^2x^2-599rx+308))W_0)G_n+((nx^n(n-1)(n+1)(2r+5sx)+6006x^{12}(20sx^2+17rx-14))W_1+sx(nx^n(n+2)(n+1)(2r+5sx)+24570x^{12}(7sx^2+6rx-5))W_0)H_n).$

Proof.

(a) We know from Theorem 2.1 that

$$S_n = \sum_{k=0}^{n} x^k W_k = \frac{\Theta(x)}{2(1-rx-sx^2)}$$

where

$$\Theta(x) = 2(W_0+(W_1-rW_0)x)-(s(2W_1-rW_0)x+(rW_1+2sW_0))x^{n+1}G_n-(W_1+sxW_0)x^{n+1}H_n.$$

By taking the derivative of the both sides of the above formulas with respect to x, we get

$$\sum_{k=0}^{n} kx^{k-1}W_k = \frac{(1-rx-sx^2)\Theta^{'}(x)+(r+2sx)\Theta(x)}{2(1-rx-sx^2)^2}$$

i.e.

$$\begin{aligned} Y_n &= \sum_{k=0}^{n} kx^k W_k = \frac{x(1-rx-sx^2)\Theta^{'}(x)+x(r+2sx)\Theta(x)}{2(1-rx-sx^2)^2} \\ &= \frac{\Delta_1}{2(1-rx-sx^2)^2}. \end{aligned}$$

(b) We use (a). For $x = a$ and $x = b$, the right hand side of the above sum formula 10) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (b) by using

$$\begin{aligned} Y_n &= \sum_{k=0}^{n} ka^k W_k = \left.\frac{\frac{d^2}{dx^2}(x(1-rx-sx^2)\Theta^{'}(x)+x(r+2sx)\Theta(x))}{\frac{d^2}{dx^2}(2(1-rx-sx^2)^2)}\right|_{x=a} \\ &= \frac{4s\Theta(a)+2sa\Theta^{'}(a)-\left(4sa^2+3ra-2\right)\Theta^{''}(a)+a(1-ra-sa^2)\Theta^{'''}(a)}{4\left(-2s+6s^2a^2+r^2+6rsa\right)} \\ &= \frac{\Delta_2}{4\left(-2s+6s^2a^2+r^2+6rsa\right)} \end{aligned}$$

and similarly

$$\begin{aligned} Y_n &= \sum_{k=0}^{n} ka^k W_k \\ &= \frac{4s\Theta(b)+2sb\Theta^{'}(b)-\left(4sb^2+3rb-2\right)\Theta^{''}(b)+b(1-rb-sb^2)\Theta^{'''}(b)}{4\left(-2s+6s^2b^2+r^2+6rsb\right)}. \end{aligned}$$

(c) We use (a). For $x = c$, the right hand side of the above sum formula 10) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (c) by using

$$\begin{aligned} Y_n &= \sum_{k=0}^{n} ka^k W_k = \left. \frac{\frac{d^4}{dx^4}(x(1-rx-sx^2)\Theta'(x)+x(r+2sx)\Theta(x))}{\frac{d^4}{dx^4}(2(1-rx-sx^2)^2)} \right|_{x=c} \\ &= \frac{-4(2r+5sx)\Theta'''(x)-(10sx^2+7rx-4)\Theta''''(x)+x((1-rx-sx^2)\Theta'''''(x)}{48s^2} \\ &= \frac{\Delta_3}{12s^2}. \square \end{aligned}$$

We now concentrate on finding expressions for the partial sums

$$\sum_{k=n}^{n+m} kx^k W_k = \sum_{k=0}^{n+m} kx^k W_k - \sum_{k=0}^{n-1} kx^k W_k = Y_{n+m} - Y_{n-1}.$$

Corollary 3.1.
Let x be a non-zero complex (or real) number. If $1 - rx - sx^2 \neq 0$ then

$$Y_{n+m} - Y_{n-1} = \frac{\Psi}{2(1-rx-sx^2)^2}$$

where
$\Psi = 2x((sx^2+1)W_1 - sx(rx-2)W_0) + ((2(n+m)s^2x^3 + rsx^2 + 3(n+m)rsx^2 + (n+m)r^2x - 2(n+m)sx - 4sx - r - (n+m)r)W_1 - s((n+m)rsx^3 + 2sx^2 + r^2x^2 - 2(n+m)sx^2 + (n+m)r^2x^2 - 2rx - 3(n+m)rx + 2(n+m) + 2)W_0)x^{n+m+1} G_{n+m} - (2x((sx^2+1)W_1 - sx(rx-2)W_0) + ((2(n-1)s^2x^3 + rsx^2 + 3(n-1)rsx^2 + (n-1)r^2x - 2(n-1)sx - 4sx - r - (n-1)r)W_1 - s((n-1)rsx^3 + 2sx^2 + r^2x^2 - 2(n-1)sx^2 + (n-1)r^2x^2 - 2rx - 3(n-1)rx + 2(n-1) + 2)W_0)x^{n-1+1} G_{n-1}) + (((n+m)sx^2 - sx^2 + (n+m)rx - (n+m) - 1)W_1 + sx((n+m)sx^2 + (n+m)rx - (n+m) + rx - 2)W_0)x^{n+m+1} H_{n+m} - ((((n-1)sx^2 - sx^2 + (n-1)rx - (n-1) - 1)W_1 + sx((n-1)sx^2 + (n-1)rx - (n-1) + rx - 2)W_0)x^{n-1+1} H_{n-1}).$

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