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Generalized Edouard Numbers

Research Article

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Abstract: In this paper, we define and investigate the generalized Edouard sequences and we deal with, in detail, two special cases, namely, Edouard and Edouard-Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there are close relations between Edouard and Edouard-Lucas numbers and balancing, modified Lucas-balancing and Lucas-balancing numbers.

MSC: 11B37 • 11B39 • 11B83

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1. Introduction

Balancing sequence $\{B_n\}_{n\geq 0}$, modified Lucas-balancing sequence $\{H_n\}_{n\geq 0}$ and Lucas-balancing sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$B_n = 6B_{n-1} - B_{n-2}, \quad B_0 = 0, B_1 = 1, \tag{1}$$
$$H_n = 6H_{n-1} - H_{n-2}, \quad H_0 = 2, H_1 = 6, \tag{2}$$
$$C_n = 6C_{n-1} - C_{n-2}, \quad C_0 = 1, C_1 = 3. \tag{3}$$

The sequences $\{B_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$B_{-n} = 6B_{-(n-1)} - B_{-(n-2)},$$
$$H_{-n} = 6H_{-(n-1)} - H_{-(n-2)},$$
$$C_{-n} = 6C_{-(n-1)} - C_{-(n-2)},$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (1)-(3) hold for all integer n.

Balancing and Lucas-balancing sequences has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1–4, 6–22, 24**?**].

Now, we define two sequences related to balancing, modified Lucas-balancing sequence, Lucas-balancing numbers. Edouard and Edouard-Lucas numbers are defined as

$$E_n = 6E_{n-1} - E_{n-2} + 1, \quad \text{with } E_0 = 0, E_1 = 1, \quad n \geq 2,$$
$$K_n = 6K_{n-1} - K_{n-2} - 4, \quad \text{with } K_0 = 3, K_1 = 7, \quad n \geq 2,$$

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respectively. The first few values of Edouard and Edouard-Lucas numbers are

$$0,1,7,42,246,1435,8365,48756,\ldots$$

and

$$3,7,35,199,1155,6727,39203,228487,\ldots$$

respectively. The sequences $\{E_n\}$ and $\{K_n\}$ satisfy the following third order linear recurences:

$$\begin{aligned} E_n &= 7E_{n-1}-7E_{n-2}+E_{n-3}, \quad E_0=0,E_1=1,E_2=7,\\ K_n &= 7K_{n-1}-7K_{n-2}+K_{n-3}, \quad K_0=3,K_1=7,K_2=35.\end{aligned}$$

There are close relations between Edouard, Edouard-Lucas and balancing, modified Lucas-balancing, Lucas-balancing numbers. For example, they satisfy the following interrelations:

$$\begin{aligned} E_n &= \frac{1}{4}(B_{n+1}-B_n-1),\\ K_n &= H_n+1=2C_n+1\end{aligned}$$

and

$$\begin{aligned} 32E_n &= H_{n+1}+H_n-8=2C_{n+1}+2C_n-8,\\ K_n &= 2B_{n+1}-6B_n+1.\end{aligned}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Edouard, Edouard-Lucas numbers). First, we recall some properties of generalized Tribonacci numbers.

The generalized (r,s,t) sequence (or generalized Tribonacci sequence or generalized 3-step Fibonacci sequence)

$$\{W_n(W_0,W_1,W_2;r,s,t)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}, \quad W_0=a,W_1=b,W_2=c, \;\; n\geq 3 \tag{4}$$

where W_0,W_1,W_2 are arbitrary complex (or real) numbers and r,s,t are real numbers.

This sequence has been studied by many authors, see for example [26]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)}-\frac{r}{t}W_{-(n-2)}+\frac{1}{t}W_{-(n-3)}$$

for $n=1,2,3,\ldots$ when $t\neq 0$. Therefore, recurrence (4) holds for all integer n. As $\{W_n\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation is

$$x^3-rx^2-sx-t=0 \tag{5}$$

whose roots are

$$\begin{aligned} \alpha &= \frac{r}{3}+A+B,\\ \beta &= \frac{r}{3}+\omega A+\omega^2 B,\\ \gamma &= \frac{r}{3}+\omega^2 A+\omega B,\end{aligned}$$

where

$$\begin{aligned} A &= \left(\frac{r^3}{27}+\frac{rs}{6}+\frac{t}{2}+\sqrt{\Delta}\right)^{1/3}, B=\left(\frac{r^3}{27}+\frac{rs}{6}+\frac{t}{2}-\sqrt{\Delta}\right)^{1/3},\\ \Delta &= \Delta(r,s,t)=\frac{r^3t}{27}-\frac{r^2s^2}{108}+\frac{rst}{6}-\frac{s^3}{27}+\frac{t^2}{4}, \;\; \omega=\frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3).\end{aligned}$$

Using these roots and the recurrence relation, Binet's formula can be given as follows:

Theorem 1.1.
(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma$) Binet's formula of generalized Tribonacci numbers is

$$\begin{aligned} W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\ &= A_1\alpha^n + A_2\beta^n + A_3\gamma^n, \end{aligned} \tag{6}$$

where

$$p_1 = W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0,\ p_2 = W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0,\ p_3 = W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0,$$

and

$$\begin{aligned} A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)} = \frac{W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)}, \\ A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)} = \frac{W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)}, \\ A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)} = \frac{W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)}. \end{aligned}$$

2. Generalized Edouard Sequence

In this paper, we consider the case $r = 7, s = -7, t = 1$. A generalized Edouard sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 7W_{n-1} - 7W_{n-2} + W_{n-3} \tag{7}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 7W_{-(n-1)} - 7W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (7) holds for all integer n.

(6) can be used to obtain Binet's formula of generalized Edouard numbers. Binet's formula of generalized Edouard numbers can be given as

$$\begin{aligned} W_n &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{z_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{z_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{z_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} - \frac{z_3}{4} \end{aligned}$$

where

$$\begin{aligned} z_1 &= W_2 - (\beta+1)W_1 + \beta W_0, \\ z_2 &= W_2 - (\alpha+1)W_1 + \alpha W_0, \\ z_3 &= W_2 - 6W_1 + W_0. \end{aligned}$$

i.e.,

$$W_n = \frac{(W_2 - (\beta+1)W_1 + \beta W_0)\alpha^n}{(\alpha-\beta)(\alpha-1)} + \frac{(W_2 - (\alpha+1)W_1 + \alpha W_0)\beta^n}{(\beta-\alpha)(\beta-1)} - \frac{(W_2 - 6W_1 + W_0)}{4}.$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 7x^2 + 7x - 1 = \left(x^2 - 6x + 1\right)(x-1) = 0.$$

Moreover

$$\begin{aligned} \alpha &= 3 + 2\sqrt{2}, \\ \beta &= 3 - 2\sqrt{2}, \\ \gamma &= 1, \end{aligned}$$

Table 1. A few generalized Edouard numbers

n	W_n	W_{-n}
0	$W_0 =$	$W_0 = x_0 = W_0$
1	$W_1 =$	$W_{-1} = 7W_0 - 7W_1 + W_2$
2	$W_2 =$	$W_{-2} = 42W_0 - 48W_1 + 7W_2$
3	$W_3 = W_0 - 7W_1 + 7W_2$	$W_{-3} = 246W_0 - 287W_1 + 42W_2$
4	$W_4 = 7W_0 - 48W_1 + 42W_2$	$W_{-4} = 1435W_0 - 1680W_1 + 246W_2$
5	$W_5 = 42W_0 - 287W_1 + 246W_2$	$W_{-5} = 8365W_0 - 9799W_1 + 1435W_2$
6	$W_6 = 246W_0 - 1680W_1 + 1435W_2$	$W_{-6} = 48756W_0 - 57120W_1 + 8365W_2$
7	$W_7 = 1435W_0 - 9799W_1 + 8365W_2$	$W_{-7} = 284172W_0 - 332927W_1 + 48756W_2$
8	$W_8 = 8365W_0 - 57120W_1 + 48756W_2$	$W_{-8} = 1656277W_0 - 1940448W_1 + 284172W_2$
9	$W_9 = 48756W_0 - 332927W_1 + 284172W_2$	$W_{-9} = 9653491W_0 - 11309767W_1 + 1656277W_2$

Note that

$$\begin{aligned}\alpha+\beta+\gamma &= 7,\\ \alpha\beta+\alpha\gamma+\beta\gamma &= 7,\\ \alpha\beta\gamma &= 1,\end{aligned}$$

or

$$\alpha+\beta = 6,\ \alpha\beta = 1.$$

The first few generalized Edouard numbers with positive subscript and negative subscript are given in the following Table 1.

Now we define two special cases of the sequence $\{W_n\}$. Edouard sequence $\{E_n\}_{n\geq 0}$ and Edouard-Lucas sequence $\{K_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}, \quad E_0 = 0, E_1 = 1, E_2 = 7, \tag{8}$$
$$K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3}, \quad K_0 = 3, K_1 = 7, K_2 = 35. \tag{9}$$

The sequences $\{E_n\}_{n\geq 0}$ and $\{K_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}E_{-n} &= 7E_{-(n-1)} - 7E_{-(n-2)} + E_{-(n-3)},\\ K_{-n} &= 7K_{-(n-1)} - 7K_{-(n-2)} + K_{-(n-3)},\end{aligned}$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences (8)-(9) hold for all integer n.

E_n and K_n are the sequences A053142, A081555 in [23], respectively.

Next, we present the first few values of the Edouard and Edouard-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11
E_n	0	1	7	42	246	1435	8365	48756	284172	1656277	9653491	56264670
E_{-n}		0	1	7	42	246	1435	8365	48756	284172	1656277	9653491
K_n	3	7	35	199	1155	6727	39203	228487	1331715	7761799	45239075	263672647
K_{-n}	....	7	35	199	1155	6727	39203	228487	1331715	7761799	45239075	263672647

For all integers n, Edouard and Edouard-Lucas numbers can be expressed using Binet's formulas as

$$\begin{aligned}E_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)} - \frac{1}{4},\\ K_n &= \alpha^n + \beta^n + 1\end{aligned}$$

respectively.

Note that Binet's formulas of balancing, modified Lucas-balancing and Lucas-balancing numbers, respectively, are

$$\begin{aligned}B_n &= \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)},\\ H_n &= \alpha^n + \beta^n,\\ C_n &= \frac{\alpha^n+\beta^n}{2},\end{aligned}$$

and so

$$E_n = \frac{1}{4}(B_{n+1} - B_n - 1), \quad (10)$$

$$K_n = H_n + 1 = 2C_n + 1. \quad (11)$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 2.1.

Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Edouard sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 7W_0)x + (W_2 - 7W_1 + 7W_0)x^2}{1 - 7x + 7x^2 - x^3}.$$

Proof. Take $r = 7, s = -7, t = 1$ in Soykan [[26], Lemma 1.1]. □

The previous lemma gives the following results as particular examples.

Corollary 2.1.

Generated functions of Edouard and Edouard-Lucas numbers are

$$\sum_{n=0}^{\infty} E_n x^n = \frac{x}{1 - 7x + 7x^2 - x^3},$$

$$\sum_{n=0}^{\infty} K_n x^n = \frac{3 - 14x + 7x^2}{1 - 7x + 7x^2 - x^3},$$

respectively.

3. Simson Formulas

There is a well-known Simson Identity (formula) for balancing sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Edouard sequence $\{W_n\}_{n\geq 0}$.

Theorem 3.1 (Simson Formula of Generalized Edouard Numbers).

For all integers n, we have

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = (6W_1 - W_0 - W_2)(W_2^2 + 8W_1^2 + W_0^2 - 8W_1W_2 + 6W_0W_2 - 8W_0W_1).$$

Proof. Take $r = 7, s = -7, t = 1$ in Soykan [[25], Theorem 2.2]. □

The previous theorem gives the following results as particular examples.

Corollary 3.1.

For all integers n, Simson formula of Edouard and Edouard-Lucas numbers are given as

$$\begin{vmatrix} E_{n+2} & E_{n+1} & E_n \\ E_{n+1} & E_n & E_{n-1} \\ E_n & E_{n-1} & E_{n-2} \end{vmatrix} = -1,$$

$$\begin{vmatrix} K_{n+2} & K_{n+1} & K_n \\ K_{n+1} & K_n & K_{n-1} \\ K_n & K_{n-1} & K_{n-2} \end{vmatrix} = 512,$$

respectively.

4. Some Identities

In this section, we obtain some identities of Edouard and Edouard-Lucas numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{E_n\}$.

Lemma 4.1.
The following equalities are true:

(a) $W_n = (246W_0 - 287W_1 + 42W_2)E_{n+4} + (1961W_1 - 1680W_0 - 287W_2)E_{n+3} + (1435W_0 - 1680W_1 + 246W_2)E_{n+2}$.

(b) $W_n = (42W_0 - 48W_1 + 7W_2)E_{n+3} + (329W_1 - 287W_0 - 48W_2)E_{n+2} + (246W_0 - 287W_1 + 42W_2)E_{n+1}$.

(c) $W_n = (7W_0 - 7W_1 + W_2)E_{n+2} + (49W_1 - 48W_0 - 7W_2)E_{n+1} + (42W_0 - 48W_1 + 7W_2)E_n$.

(d) $W_n = W_0E_{n+1} + (W_1 - 7W_0)E_n + (7W_0 - 7W_1 + W_2)E_{n-1}$.

(e) $W_n = W_1E_n + (W_2 - 7W_1)E_{n-1} + W_0E_{n-2}$.

(f) $(W_0 - 6W_1 + W_2)(W_0^2 + 8W_1^2 + W_2^2 - 8W_0W_1 + 6W_0W_2 - 8W_1W_2)E_n = (7W_1^2 + W_2^2 - W_0W_1 - 7W_1W_2)W_{n+4} + (-48W_1^2 - 7W_2^2 + 7W_0W_1 - W_0W_2 + 49W_1W_2)W_{n+3} + (W_0^2 + 49W_1^2 + 7W_2^2 - 14W_0W_1 + 7W_0W_2 - 50W_1W_2)W_{n+2}$.

(g) $(W_0 - 6W_1 + W_2)(W_0^2 + 8W_1^2 + W_2^2 - 8W_0W_1 + 6W_0W_2 - 8W_1W_2)E_n = (W_1^2 - W_0W_2)W_{n+3} + (W_0^2 - 7W_0W_1 + 7W_0W_2 - W_1W_2)W_{n+2} + (7W_1^2 + W_2^2 - W_0W_1 - 7W_1W_2)W_{n+1}$.

(h) $(W_0 - 6W_1 + W_2)(W_0^2 + 8W_1^2 + W_2^2 - 8W_0W_1 + 6W_0W_2 - 8W_1W_2)E_n = (W_0^2 + 7W_1^2 - 7W_0W_1 - W_1W_2)W_{n+2} + (W_2^2 - W_0W_1 + 7W_0W_2 - 7W_1W_2)W_{n+1} + (W_1^2 - W_0W_2)W_n$.

(i) $(W_0 - 6W_1 + W_2)(W_0^2 + 8W_1^2 + W_2^2 - 8W_0W_1 + 6W_0W_2 - 8W_1W_2)E_n = (7W_0^2 + 49W_1^2 + W_2^2 - 50W_0W_1 + 7W_0W_2 - 14W_1W_2)W_{n+1} - (7W_0^2 + 48W_1^2 - 49W_0W_1 + W_0W_2 - 7W_1W_2)W_n + (W_0^2 + 7W_1^2 - 7W_0W_1 - W_1W_2)W_{n-1}$.

(j) $(W_0 - 6W_1 + W_2)(W_0^2 + 8W_1^2 + W_2^2 - 8W_0W_1 + 6W_0W_2 - 8W_1W_2)E_n = (42W_0^2 + 295W_1^2 + 7W_2^2 - 301W_0W_1 + 48W_0W_2 - 91W_1W_2)W_n - (48W_0^2 + 336W_1^2 + 7W_2^2 - 343W_0W_1 + 49W_0W_2 - 97W_1W_2)W_{n-1} + (7W_0^2 + 49W_1^2 + W_2^2 - 50W_0W_1 + 7W_0W_2 - 14W_1W_2)W_{n-2}$.

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$W_n = a \times E_{n+4} + b \times E_{n+3} + c \times E_{n+2}$$

and solving the system of equations

$$\begin{aligned} W_0 &= a \times E_4 + b \times E_3 + c \times E_2 \\ W_1 &= a \times E_5 + b \times E_4 + c \times E_3 \\ W_2 &= a \times E_6 + b \times E_5 + c \times E_4 \end{aligned}$$

we find that $a = 246W_0 - 287W_1 + 42W_2, b = 1961W_1 - 1680W_0 - 287W_2, c = 1435W_0 - 1680W_1 + 246W_2$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{W_n\}$ and $\{K_n\}$.

Lemma 4.2.
The following equalities are true:

(a) $64W_n = (581W_0 - 700W_1 + 103W_2)K_{n+4} - 4(991W_0 - 1190W_1 + 175W_2)K_{n+3} + (3367W_0 - 3964W_1 + 581W_2)K_{n+2}$.

(b) $64W_n = (103W_0 - 140W_1 + 21W_2)K_{n+3} - 4(175W_0 - 234W_1 + 35W_2)K_{n+2} + (581W_0 - 700W_1 + 103W_2)K_{n+1}$.

(c) $64W_n = (21W_0 - 44W_1 + 7W_2)K_{n+2} - 4(35W_0 - 70W_1 + 11W_2)K_{n+1} + (103W_0 - 140W_1 + 21W_2)K_n$.

(d) $64W_n = (7W_0 - 28W_1 + 5W_2)K_{n+1} - 4(11W_0 - 42W_1 + 7W_2)K_n + (21W_0 - 44W_1 + 7W_2)K_{n-1}$.

(e) $64W_n = (5W_0 - 28W_1 + 7W_2)K_n - 4(7W_0 - 38W_1 + 7W_2)K_{n-1} + (7W_0 - 28W_1 + 5W_2)K_{n-2}$.

(f) $(W_0 - 6W_1 + W_2)(W_0^2 + 8W_1^2 + W_2^2 - 8W_0W_1 + 6W_0W_2 - 8W_1W_2)K_n = (7W_0^2 + 248W_1^2 + 35W_2^2 - 84W_0W_1 + 46W_0W_2 - 252W_1W_2)W_{n+4} - 2(23W_0^2 + 840W_1^2 + 119W_2^2 - 280W_0W_1 + 154W_0W_2 - 856W_1W_2)W_{n+3} + (35W_0^2 + 1400W_1^2 + 199W_2^2 - 444W_0W_1 + 238W_0W_2 - 1428W_1W_2)W_{n+2}$.

(g) $(W_0-6W_1+W_2)(W_0^2+8W_1^2+W_2^2-8W_0W_1+6W_0W_2-8W_1W_2)K_n = (3W_0^2+56W_1^2+7W_2^2-28W_0W_1+14W_0W_2-52W_1W_2)W_{n+3}-2(7W_0^2+168W_1^2+23W_2^2-72W_0W_1+42W_0W_2-168W_1W_2)W_{n+2}+(7W_0^2+248W_1^2+35W_2^2-84W_0W_1+46W_0W_2-252W_1W_2)W_{n+1}$.

(h) $(W_0-6W_1+W_2)(W_0^2+8W_1^2+W_2^2-8W_0W_1+6W_0W_2-8W_1W_2)K_n = (7W_0^2+56W_1^2+3W_2^2-52W_0W_1+14W_0W_2-28W_1W_2)W_{n+2}-2(7W_0^2+72W_1^2+7W_2^2-56W_0W_1+26W_0W_2-56W_1W_2)W_{n+1}+(3W_0^2+56W_1^2+7W_2^2-28W_0W_1+14W_0W_2-52W_1W_2)W_n$.

(i) $(W_0-6W_1+W_2)(W_0^2+8W_1^2+W_2^2-8W_0W_1+6W_0W_2-8W_1W_2)K_n = (35W_0^2+248W_1^2+7W_2^2-252W_0W_1+46W_0W_2-84W_1W_2)W_{n+1}-2(23W_0^2+168W_1^2+7W_2^2-168W_0W_1+42W_0W_2-72W_1W_2)W_n+(7W_0^2+56W_1^2+3W_2^2-52W_0W_1+14W_0W_2-28W_1W_2)W_{n-1}$.

(j) $(W_0-6W_1+W_2)(W_0^2+8W_1^2+W_2^2-8W_0W_1+6W_0W_2-8W_1W_2)K_n = (199W_0^2+1400W_1^2+35W_2^2-1428W_0W_1+238W_0W_2-444W_1W_2)W_n-2(119W_0^2+840W_1^2+23W_2^2-856W_0W_1+154W_0W_2-280W_1W_2)W_{n-1}+(35W_0^2+248W_1^2+7W_2^2-252W_0W_1+46W_0W_2-84W_1W_2)W_{n-2}$.

Now, we give a few basic relations between $\{E_n\}$ and $\{K_n\}$.

Lemma 4.3.
The following equalities are true

$$
\begin{aligned}
64E_n &= 21K_{n+4}-140K_{n+3}+103K_{n+2},\\
64E_n &= 7K_{n+3}-44K_{n+2}+21K_{n+1},\\
64E_n &= 5K_{n+2}-28K_{n+1}+7K_n,\\
64E_n &= 7K_{n+1}-28K_n+5K_{n-1},\\
64E_n &= 21K_n-44K_{n-1}+7K_{n-2},
\end{aligned}
$$

and

$$
\begin{aligned}
K_n &= 199E_{n+4}-1358E_{n+3}+1155E_{n+2},\\
K_n &= 35E_{n+3}-238E_{n+2}+199E_{n+1},\\
K_n &= 7E_{n+2}-46E_{n+1}+35E_n,\\
K_n &= 3E_{n+1}-14E_n+7E_{n-1},\\
K_n &= 7E_n-14E_{n-1}+3E_{n-2}.
\end{aligned}
$$

5. Relations Between Special Numbers

In this section, we present identities on Edouard and Edouard-Lucas numbers and balancing, modified Lucas-balancing and Lucas-balancing numbers. We know that

$$
\begin{aligned}
E_n &= \frac{1}{4}(B_{n+1}-B_n-1),\\
K_n &= H_n+1=2C_n+1.
\end{aligned}
$$

We also note that from Lemma 4.3, we have

$$
\begin{aligned}
64E_n &= 5K_{n+2}-28K_{n+1}+7K_n.\\
K_n &= 7E_{n+2}-46E_{n+1}+35E_n.
\end{aligned}
$$

Using the above identities we see that

$$
\begin{aligned}
32E_n &= H_{n+1}+H_n-8=2C_{n+1}+2C_n-8,\\
K_n &= 2B_{n+1}-6B_n+1.
\end{aligned}
$$

Using the above identities (and Lemma 4.1) we obtain the following Binet's formula of generalized Edouard numbers in the following forms:

$$
\begin{aligned}
W_n &= (7W_0-7W_1+W_2)E_{n+2}+(49W_1-48W_0-7W_2)E_{n+1}+(42W_0-48W_1+7W_2)E_n\\
&= \frac{(W_2-6W_1+5W_0)B_{n+1}-(5W_2-34W_1+29W_0)B_n-(W_2-6W_1+W_0)}{4}.
\end{aligned}
$$

6. On the Recurrence Properties of Generalized Edouard Sequence

Taking $r = 7, s = -7, t = 1$ in Soykan [[27], Theorem 2], we obtain the following Proposition.

Proposition 6.1.
For $n \in \mathbb{Z}$, generalized Edouard numbers (the case $r = 7, s = -7, t = 1$) have the following identity:

$$W_{-n} = W_{2n} - K_n W_n + \frac{1}{2}(K_n^2 - K_{2n})W_0.$$

From the above Proposition and Corollary 6 in [27], we have the following corollary which gives the connection between the special cases of generalized Edouard sequence at the positive index and the negative index: for modified Edouard, Edouard-Lucas and Edouard numbers: take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 7$ and take $W_n = K_n$ with $K_0 = 3, K_1 = 7, K_2 = 35$, respectively. Note that in this case $K_n = K_n$.

Corollary 6.1.
For $n \in \mathbb{Z}$, we have the following recurrence relations:

(a) *Edouard sequence:*

$$E_{-n} = E_{2n} - E_n K_n.$$

(b) *Edouard-Lucas sequence:*

$$K_{-n} = \frac{1}{2}(K_n^2 - K_{2n}).$$

By using the identity $K_n = 7E_{n+2} - 46E_{n+1} + 35E_n$ (and Proposition 6.1 or Corollary 6.1), we get

$$E_{-n} = -35E_n^2 + E_{2n} + 46E_n E_{n+1} - 7E_n E_{n+2}.$$

Note also that since $E_n = \frac{1}{4}(B_{n+1} - B_n - 1)$ and $B_{-n} = -B_n$, we get

$$E_{-n} = \frac{1}{4}(B_n - B_{n-1} - 1)$$

and since $K_n = H_n + 1 = 2C_n + 1$ and $H_{-n} = H_n$, $C_{-n} = C_n$ we obtain

$$K_{-n} = H_n + 1 = 2C_n + 1.$$

7. Sums

The following Corollary gives sum formulas of balancing, modified Lucas-balancing and Lucas-balancing numbers.

Corollary 7.1.
For $n \geq 0$, balancing, modified Lucas-balancing and Lucas-balancing numbers have the following properties:

1.

- **(a)** $\sum_{k=0}^{n} B_k = \frac{1}{4}(5B_n - B_{n-1} - 1)$.
- **(b)** $\sum_{k=0}^{n} B_{2k} = \frac{1}{32}(33B_{2n} - B_{2n-2} - 6)$.
- **(c)** $\sum_{k=0}^{n} B_{2k+1} = \frac{1}{32}(33B_{2n+1} - B_{2n-1} - 2)$.

2.

- **(a)** $\sum_{k=0}^{n} H_k = \frac{1}{4}(5H_n - H_{n-1} + 4)$.
- **(b)** $\sum_{k=0}^{n} H_{2k} = \frac{1}{32}(33H_{2n} - H_{2n-2} + 32)$.
- **(c)** $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{32}(33H_{2n+1} - H_{2n-1})$.

3.

(a) $\sum_{k=0}^{n} C_k = \frac{1}{4}(5C_n - C_{n-1} + 2)$.

(b) $\sum_{k=0}^{n} C_{2k} = \frac{1}{32}(33C_{2n} - C_{2n-2} + 16)$.

(c) $\sum_{k=0}^{n} C_{2k+1} = \frac{1}{32}(33C_{2n+1} - C_{2n-1})$.

Proof. It is given in Soykan [[24], Corollary 6.6]. □

The following Corollary presents sum formulas of Edouard and Edouard-Lucas numbers.

Corollary 7.2.
For $n \geq 0$, Edouard and Edouard-Lucas numbers have the following properties:

1.

(a) $\sum_{k=0}^{n} E_k = \frac{1}{4}(B_{n+1} - (n+1))$.

(b) $\sum_{k=0}^{n} E_{2k} = \frac{1}{32}(7B_{2n+1} - B_{2n} - 8n - 7)$.

(c) $\sum_{k=0}^{n} E_{2k+1} = \frac{1}{32}(41B_{2n+1} - 7B_{2n} - 8n - 9)$.

2.

(a) $\sum_{k=0}^{n} K_k = \frac{1}{4}(5H_n - H_{n-1} + 4(n+2))$.

(b) $\sum_{k=0}^{n} K_{2k} = \frac{1}{32}(33H_{2n} - H_{2n-2} + 32(n+2))$.

(c) $\sum_{k=0}^{n} K_{2k+1} = \frac{1}{32}(33H_{2n+1} - H_{2n-1} + 32(n+1))$.

Proof. The proof follows from Corollary 7.1 and the identities (10) and (11), i.e.,

$$\begin{aligned} E_n &= \frac{1}{4}(B_{n+1} - B_n - 1), \\ K_n &= H_n + 1 = 2C_n + 1. \end{aligned}$$

□

8. Matrices Related With Generalized Edouard Numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{12}$$

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = -1$. From (7) we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix} \tag{13}$$

and from (12) (or using (13) and induction) we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}.$$

If we take $W = E$ in (13) we have

$$\begin{pmatrix} E_{n+2} \\ E_{n+1} \\ E_n \end{pmatrix} = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} E_{n+1} \\ E_n \\ E_{n-1} \end{pmatrix}.$$

We also define

$$B_n = \begin{pmatrix} E_{n+1} & -7E_n + E_{n-1} & E_n \\ E_n & -7E_{n-1} + E_{n-2} & E_{n-1} \\ E_{n-1} & -7E_{n-2} + E_{n-3} & E_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & -7W_n + W_{n-1} & W_n \\ W_n & -7W_{n-1} + W_{n-2} & W_{n-1} \\ W_{n-1} & -7W_{n-2} + W_{n-3} & W_{n-2} \end{pmatrix}$$

Theorem 8.1.
For all integers m, n, we have

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof. Take $r = 7, s = -7, t = 1$ in Soykan [[26], Theorem 5.1.]. □
Some properties of matrix A^n can be given as

$$A^n = 7A^{n-1} - 7A^{n-2} - A^{n-3}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = 1$$

for all integer m and n.

Corollary 8.1.
For all integers n, we have the following formulas for the Edouard and Edouard-Lucas numbers.

(a) *Edouard Numbers.*

$$A^n = \begin{pmatrix} 7 & -7 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} E_{n+1} & -7E_n + E_{n-1} & E_n \\ E_n & -7E_{n-1} + E_{n-2} & E_{n-1} \\ E_{n-1} & -7E_{n-2} + E_{n-3} & E_{n-2} \end{pmatrix}.$$

(b) *Edouard-Lucas Numbers.*

$$A^n = \frac{1}{64} \begin{pmatrix} 7K_{n+2} - 28K_{n+1} + 5K_n & -28K_{n+2} + 152K_{n+1} - 28K_n & 5K_{n+2} - 28K_{n+1} + 7K_n \\ 5K_{n+2} - 28K_{n+1} + 7K_n & -28K_{n+2} + 168K_{n+1} - 44K_n & 7K_{n+2} - 44K_{n+1} + 21K_n \\ 7K_{n+2} - 44K_{n+1} + 21K_n & -44K_{n+2} + 280K_{n+1} - 140K_n & 21K_{n+2} - 140K_{n+1} + 103K_n \end{pmatrix}.$$

Proof.

(a) It is given in Theorem 8.1 (a).

(b) Note that, from Lemma 4.3, we have

$$64E_n = 5K_{n+2} - 28K_{n+1} + 7K_n.$$

Using the last equation and (a), we get required result. □

Using the above last Corollary and the identities (10) and (11), i.e.,

$$\begin{aligned} E_n &= \frac{1}{4}(B_{n+1} - B_n - 1), \\ K_n &= H_n + 1 = 2C_n + 1, \end{aligned}$$

we obtain the following identities for balancing, modified Lucas-balancing and Lucas-balancing numbers.

Corollary 8.2.
For all integers n, we have the following formulas for balancing, modified Lucas-balancing and Lucas-balancing numbers.

(a) *balancing Numbers.*

$$A^n = \frac{1}{4}\begin{pmatrix} 5B_{n+1}-B_n-1 & -6B_{n+1}+2B_n+6 & B_{n+1}-B_n-1 \\ B_{n+1}-B_n-1 & -2B_{n+1}+6B_n+6 & B_{n+1}-5B_n-1 \\ B_{n+1}-5B_n-1 & -6B_{n+1}+34B_n+6 & 5B_{n+1}-29B_n-1 \end{pmatrix}.$$

(b) *modified Lucas-balancing Numbers.*

$$A^n = \frac{1}{32}\begin{pmatrix} 7H_{n+1}-H_n-8 & -8(H_{n+1}-6) & H_{n+1}+H_n-8 \\ H_{n+1}+H_n-8 & -8(H_n-6) & -H_{n+1}+7H_n-8 \\ -H_{n+1}+7H_n-8 & -8(-H_{n+1}+6H_n-6) & -7H_{n+1}+41H_n-8 \end{pmatrix}.$$

(c) *Lucas-balancing Numbers.*

$$A^n = \frac{1}{16}\begin{pmatrix} 7C_{n+1}-C_n-4 & -8(C_{n+1}-3) & C_{n+1}+C_n-4 \\ C_{n+1}+C_n-4 & -8(C_n-3) & -C_{n+1}+7C_n-4 \\ -C_{n+1}+7C_n-4 & -8(-C_{n+1}+6C_n-3) & -7C_{n+1}+41C_n-4 \end{pmatrix}.$$

Theorem 8.2.
For all integers m, n, we have

$$W_{n+m} = W_n E_{m+1} + (-7W_{n-1} + W_{n-2}) E_m + W_{n-1} E_{m-1} \tag{14}$$

Proof. Take $r = 7, s = -7, t = 1$ in Soykan [[26], Theorem 5.2.]. □
By Lemma 4.1, we know that

$$\begin{aligned} & (W_0 - 6W_1 + W_2)(W_0^2 + 8W_1^2 + W_2^2 - 8W_0W_1 + 6W_0W_2 - 8W_1W_2)E_m \\ = \ & (W_0^2 + 7W_1^2 - 7W_0W_1 - W_1W_2)W_{m+2} + (W_2^2 - W_0W_1 + 7W_0W_2 - 7W_1W_2)W_{m+1} + (W_1^2 - W_0W_2)W_m \end{aligned}$$

so (14) can be written in the following form

$$\begin{aligned} & (W_0 - 6W_1 + W_2)(W_0^2 + 8W_1^2 + W_2^2 - 8W_0W_1 + 6W_0W_2 - 8W_1W_2)W_{n+m} \\ = \ & W_n((W_0^2 + 7W_1^2 - 7W_0W_1 - W_1W_2)W_{m+3} + (W_2^2 - W_0W_1 + 7W_0W_2 - 7W_1W_2)W_{m+2} + (W_1^2 - W_0W_2)W_{m+1}) \\ & + (-7W_{n-1} + W_{n-2})((W_0^2 + 7W_1^2 - 7W_0W_1 - W_1W_2)W_{m+2} + (W_2^2 - W_0W_1 + 7W_0W_2 - 7W_1W_2)W_{m+1} + (W_1^2 - W_0W_2)W_m) \\ & + W_{n-1}((W_0^2 + 7W_1^2 - 7W_0W_1 - W_1W_2)W_{m+1} + (W_2^2 - W_0W_1 + 7W_0W_2 - 7W_1W_2)W_m + (W_1^2 - W_0W_2)W_{m-1}). \end{aligned}$$

Corollary 8.3.
For all integers m, n, we have

$$\begin{aligned} E_{n+m} &= E_n E_{m+1} + (-7E_{n-1} + E_{n-2}) E_m + E_{n-1} E_{m-1}, \\ K_{n+m} &= K_n E_{m+1} + (-7K_{n-1} + K_{n-2}) E_m + K_{n-1} E_{m-1}, \end{aligned}$$

and

$$512K_{m+n} = (40K_{m+3} - 224K_{m+2} + 56K_{m+1})K_n + (40K_{m+2} - 224K_{m+1} + 56K_m)(K_{n-2} - 7K_{n-1}) + (40K_{m+1} - 224K_m + 56K_{m-1})K_{n-1}$$

Taking $m = n$ in the last corollary we obtain the following identities:

$$\begin{aligned} E_{2n} &= E_n E_{n+1} + (-7E_{n-1} + E_{n-2}) E_n + E_{n-1}^2, \\ K_{2n} &= K_n E_{n+1} + (-7K_{n-1} + K_{n-2}) E_n + K_{n-1} E_{n-1}, \\ 512K_{2n} &= (40K_{n+3} - 224K_{n+2} + 56K_{n+1})K_n \\ & \quad + (40K_{n+2} - 224K_{n+1} + 56K_n)(K_{n-2} - 7K_{n-1}) + (40K_{n+1} - 224K_n + 56K_{n-1})K_{n-1}. \end{aligned}$$

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