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Some properties via $G^{\beta}-$open sets in $G-$metrizable topological space

Research Article

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Abstract: In this paper we introduce and investigate weak form of G-open sets in $G-$metrizable topological space, namely G^{β}-open sets, we introduce and investigate the concept of $G^{\beta}-$disconnected sets and $G^{\beta}-$connected in $G-$metrizable topological space, and we introduce and investigate the concept of $G^{\beta}-$compact sets in $G-$metrizable topological space.

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1. Introduction

The concept of metric space was introduced by Frechet in 1906, [6]. It has a very important basicrole in mathematics and its application. Many mathematical concepts that can be discussed in this space.The first attempt to generalize the ordinary distance function to a distance of three points was introduced by Gahler, [7, 8], in 1993. Ha, et al; [5], showed that a 2-metric is not a generalization of the usual notion of a metric. It was mentioned by Gahler, [7], that the notion of a 2-metric is an extension of an idea of ordinary metric and geometrically $d(x,y,z)$ represents the area of a triangle formed by the points x,y and z in X as its vertices. But this is not always true. A.Sharma, [1], showed that $d(x,y,z)=0$ for any three distinct points $x,y,z\in R$. Dhage in 1963 introduced a new class of generalized metrics called D-metrics, [8],. However, several errors for fundamental topological properties in a D-metric space were found by Mustafa and Sims, [9]. Due to these considerations, Mustafa and Sims , [10], proposed a more appropriate notion of a generalized metric space, called G-metric space, In 2021, [2], we introdcued the concept of $G^{\beta}-$open sets by the open balls in G-metric spaces

This paper is organized as follows.In Section 3 we introduce the concept of $G^{\beta}-$open sets in $G-$metrizable topological space, In Section 4 we introduce the concept of $G^{\beta}-$disconnected sets and $G^{\beta}-$connected. Furthermore, the relationship with the other known sets will be studied. In Section 5 we introduce the concept of $G^{\beta}-$compact sets.

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2. PRELIMINARIES

Definition 2.1.
[6] Let X be any nonempty set. A function $d : X \times X \to [0,\infty)$ is called a metric function on X if it satisfies the following three conditions for all $x, y, z \in X$:

1. *(positive property) $d(x, y) \geq 0$ with equality if and only if $x = y$;*
2. *(symmetric property) $d(x, y) = d(x, y)$;*
3. *(triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.*

A pair (X, d), where d is a metric on X is called a metric space.

Definition 2.2.
[9] Let X be a nonempty set and $\mathbb{R}$ be the set of real numbers. A function $G : X \times X \times X \to R$ is called a G-metric function on X if it satisfies the following:

1. *$G(x, x, y) > 0$ for all $x \neq y \in X$;*
2. *$G(x, y, z) = 0$ if and only if $x = y = z$;*
3. *$G(x, x, y) \leq G(x, y, z)$ for every $x, y, z \in X$ with $y \neq z$;*
4. *$G(x, y, z) = G(p(x, y, z))$ for every $x, y, z \in X$ and for any permutation p of x, y, z;*
5. *$G(x, y, z) \leq G(x, u, u) + G(u, y, z)$ for every $x, y, z, u \in X$.*

If G is a G-metric function on X, then the pair (X, G) is called a G-metric space.

Let (X, G) be a G-metric space, $x \in X$ and $A \subseteq X$. The open ball with center x and radius ϵ in metric space (X, G) is denoted by $B_G(x, \epsilon)$ and defined by

$$B_G(x, \epsilon) = \{y \in X | d(x, y, y) < \epsilon\}.$$

The closed ball with center x and radius ϵ in G-metric space (X, G) is denoted by $C_G(x, \epsilon)$and defined by

$$C_G(x, \epsilon) = \{y \in X | d(x, y, y) \leq \epsilon\}.$$

The set A is called an open set in G-metric space (X, G) if for every $x \in A$, there is $\epsilon > 0$ such that $B_G(x, \epsilon) \subseteq A$. The set A is called closed set in metric space (X, G) if $X - A$ is an open set in G-metric space (X, G).

Definition 2.3.
[10] Let (X, G) be a G-metric space, and let $\epsilon > 0$ be given, then a set $A \subseteq X$ is called an $\epsilon-$net of (X, G) if given any x in X there is at least one point a in A such that $x \in B_G(a, \epsilon)$ if the set A is finite then A is called a finite $\epsilon-$ net of (X, G) Note that if A is an ϵ-net then$X = \cup_{a \in A} B_G(a, \epsilon)$.

Theorem 2.1.
[10] For a G-metric space, (X, G) the following are equivalent:
1. (X, G) is a compact G- metric space;
2. $(X, \tau(G))$ is a compact topological space;
3. (X, d_G) is a compact metric space.

Definition 2.4.
[3] Let (X, G) be a $G-$metric space, the collection $\beta = \{B(x_0, \epsilon) : x \in X, \epsilon > 0\}$ of $G-$open balls induces a topology on X, called $G-$metric topology.
Thus the G-metric space X together with a topology τ generated by $G-$metric is called a $G-$metric topological space and τ is called G-metric topology on X.
A topological space X is called $G-$metrizable if there exists a $G-$metric G on X that induces a topology on X.
A $G-$metric space X is $G-$metrizable space together with the specific $G-$metric that induces the topology of X.

3. G^β-OPEN SETS

Definition 3.1.
Let $(X,\tau(G))$ be a $G-$metrizable topological spacee and $A \subseteq X$.A point $x \in X$ is called a G-point of A in $G-$metrizable topological space $(X,\tau(G))$ if there is $\delta > 0$ such that for every $y \in B_G(x,\delta)$,

$$B_G(y,\epsilon) \cap G \neq \emptyset \quad \forall \epsilon > 0.$$

$G^\beta(A)$ denotes the set of all G^β-points of A in $G-$metrizable topological space $(X,\tau(G))$

Example 3.1.
Let $(R,\tau(G))$ be $G-$metrizable topological space Diven by $G(x,y,z) = max\{|x-y|,|y-z|,|z-x|\}$ .Let $A = (0,2)$ and $B = Q$ be that set of rational numbers. Note that G^β (A)= (0,2) and $G^\beta(B) = R$.

Theorem 3.1.
Let $(X,\tau(G))$ be any $G-$metrizable topological space and $A, B \subseteq X$.Then

1. *$G^\beta(\phi) = \phi$ and $G^\beta(X) = X$;*
2. *if $A \subseteq B$ Then $G^\beta(A) \cup G^\beta(B)$;*
3. *$G^\beta(A \cap B) \subseteq G^\beta(A) \cap G^\beta(B)$;*
4. *$G^\beta(A) \cup G^\beta(B) \subseteq G^\beta(A \cup B)$.*

Proof. 1. It is clear from the definition ,we get that $G^\beta(\phi) = \phi$ and $G^\beta(X) = X$.

2. Let $A \subseteq B$ and $x \in G^\beta(A)$ Then is $\delta > 0$ such that for every $y \in B_G(y,\epsilon) \cap A \neq \emptyset$, for all Since $A \subseteq B$ Then $B_G(y,\epsilon) \cap B \neq \emptyset$, for all $\epsilon > 0$. That is,$x \in G^\beta(B)$ Then $G^\beta(A) \cup G^\beta(B)$.
3. Since $A \cap B \subseteq A$ Then by part (2) $G^\beta(A \cap B) \subseteq G^\beta(A)$, Similar $A \cap B \subseteq B$ Then $G^\beta(A \cap B) \subseteq G^\beta(B)$ Then $G^\beta(A \cap B) \subseteq G^\beta(A) \cap G^\beta(B)$.
4. Since $A \subseteq (A \cup B)$, Then by part (2) $G^\beta(A) \subseteq G^\beta(A \cup B)$, Similar $B \subseteq (A \cup B)$ Then $G^\beta(B) \subseteq G^\beta(A \cup B)$ Then $G^\beta(A) \cup G^\beta(B) \subseteq G^\beta(A \cup B)$.

□

Definition 3.2.
Let $(X,\tau(G))$ be a $G-$metrizable topological space. A subset $A \subseteq X$ is called a G^β-open set in $G-$metrizable topological space $(X,\tau(G))$ if for every $x \in A$,

$$B_G(x,\epsilon) \cap \; G^\beta(A) \neq \emptyset \quad \forall \epsilon > 0.$$

A subset$A \in X$ is called a G^β-closed set in G-metric space $(X,\tau(G))$ if $X - A$ is a G^β-open set in $G-$metrizable topological space $(X,\tau(G))$.

Example 3.2.
In Example (3.1), the stes A and B are G^β-open sets. Note that any finite sub sets of R are not G^β-open.

Theorem 3.2.
Every G-open set is a G^β-open set.

Proof. Let A be any G-open set in $G-$metrizable topological space $(X,\tau(G))$. Let $x \in G$ be arbitrary point. Then there is $\delta > 0$ such that $B_G(x,\varepsilon) \subseteq G$. For every $y \in B_G(x,\varepsilon)$, $y \in B_G(x,\varepsilon)(y)$ and $y \in A$ for every $\varepsilon > 0$. That is, $B_G(y,\varepsilon) \cap G \neq \emptyset$ for every $\varepsilon > 0$. Hence A is G^β-open set. □

The converse of above theorem need not be true.

Example 3.3.
In Example(3.1), the set of rational numbers Q is a G^β-open set but not G-open set in $(\mathbb{R},\tau(G))$.

The intersection of two G^{β}-open sets no need to be G^{β}-open set. In Example(3.1), the closed interval $A=[a,b]$ is a G^{β}-open set but not G-open set in $(\mathbb{R},\tau(G))$ and the set $IR\cup\{q\}$ is a G^{β}-open set in $(\mathbb{R},\tau(G))$, where IR is the set of irrational numbers and q is any rational number, but $Q\cap(IR\cup\{q\})=\{q\}$ is not G^{β}-open set.

The following theorem shows that the intersection of a G-open set and a G^{β}-open set is a G^{β}-open set.

Theorem 3.3.
The intersection of a G-open set and a G^{β}-open set is a G^{β}-open set.

Proof. Let A be G-open set and B be G^{β}-open set in $G-$metrizable topological space in $(X,\tau(G))$. Let $x\in A\cap B$ be arbitrary point. Then $x\in A$ and $x\in B$. Then there are $\delta_1>0$ and $\delta_2>0$ such that $B_G(x,\delta_1)\subseteq A$ and for every $y\in B_G(x,\delta_2)$, $B_G(y,\varepsilon)\cap B\neq\emptyset$ for every $\varepsilon>0$. Take $\delta=min\{\delta_1,\delta_2\}>0$. Then $B_G(x,\delta)\subseteq A$ and for every $y\in B_G(x,\delta)$, $B_G(y,\varepsilon))\cap B\neq\emptyset$ for every $\varepsilon>0$. Now for every $y\in B_G(x,\delta)$ and since A is G-open set, then there is $\varepsilon_y>0$ such that $B_G(y,\varepsilon_y))\subseteq A$ and $B_G(y,min\{\delta_1,\delta_2\})\cap B\neq\emptyset$. Since $B_G(y,min\{\delta_1,\delta_2\})\cap B\subseteq B_G(y,\varepsilon))\cap A\cap B$, then $B_G(y,\varepsilon)\cap(A\cap B)\neq\emptyset$ for every $\varepsilon>0$. That is $A\cap B$ is G-open set.Hnece $x\in G^{\beta}(A\cap B)$. Then $B_G(y,\varepsilon)\cap G^{\beta}(A\cap B)\neq\emptyset$ for all $\varepsilon>0$ .There for $A\cap B$ is G^{β}-open set. □

Theorem 3.4.
The union of any family of G^{β}-open sets is G^{β}-open set.

Proof. Let H_λ be a G^{β}-open in $G-$metrizable topological space $(X,\tau(G))$ for all $\lambda\in\Delta$. Let $x\in\cup_{\lambda\in\Delta}H_\lambda$ be an arbitrary point. Then there is at least $\lambda_0\in\Delta$ such that $x\in H_{\lambda_0}$. Since H_{λ_0} is a G^{β}-open set then $B_G(x,\varepsilon)\cap G^{\beta}(H_{\lambda_0})\neq\emptyset$ for all $\varepsilon>0$. Hence by Theorem (3.1) $G^{\beta}(H_{\lambda_0})\subseteq G^{\beta}(\cup_{\lambda\in\Delta})$ Hence $B_G(x,\varepsilon)\cap G^{\beta}(\cup_{\lambda\in\Delta}H_\lambda)\neq\emptyset$ for all $\varepsilon>0$. That is $\cup_{\lambda\in\Delta}H_\lambda$ is G^{β}-open set.

□

Definition 3.3.
Let $(X,\tau(G))$ be a $G-$metrizable topological space and $A\subseteq X$. The G-closure operator of A is denoted by $Cl^{\beta}_G(A)$ and defined by

$$Cl^{\beta}_G(A)=\cap\{H\subseteq X: A\subseteq H \text{ and } H \text{ is } G^{\beta}\text{-closed set}\}.$$

The G-interior functor of A is denoted by $Int^{\beta}_G(A)$ and defined by

$$Int^{\beta}_G(A)=\cup\{H\subseteq X: H\subseteq A \text{ and } H \text{ is } G^{\beta}\text{-open set}\}.$$

4. $G^{\beta}-$CONNECTED

Definition 4.1.
Let $(X,\tau(G))$ be a $G-$metrizable topological space and A,B be two nonempty subsets of X. The sets A and B are called a $G^{\beta}-$separated sets *if $Cl^{\beta}_G(A)\cap B=\emptyset$ and $A\cap Cl^{\beta}_G(B)=\emptyset$.*

Remark 4.1.
Let $(X,\tau(G))$ be a $G-$metrizable topological space. Then
1. Any $G^{\beta}-$-separated sets are disjoint sets, since $A\cap B\subseteq A\cap Cl^{\beta}_G(B)=\emptyset$.
2. Any two nonempty $G^{\beta}-$closed sets in X are $G^{\beta}-$separated if they are disjoint sets.

Definition 4.2.
A $G-$metrizable topological space $(X,\tau(G))$ is called a $G^{\beta}-$disconnected space *if it is the union of two $G^{\beta}-$separated sets. Otherwise A $G-$metrizable topological space $(X,\tau(G))$ is called* a $G^{\beta}-$connected space.

Theorem 4.1.
Any a $G-$metrizable topological space $(X,\tau(G))$ with a finite set X is a $G^{\beta}-$disconnected space if X has more that one point.

Proof. The proof of the theorem is clear since every set in X is $G^{\beta}-$clopen set. □

The proof of the following theorem is clear since $Cl_G^{\beta}(A) \subset Cl_G(A)$.

Theorem 4.2.
Every disconnected space is a $G^{\beta}-$disconnected space.

The converse of the above theorem need not be true.

Example 4.1.
In the $G-$metrizable topological space $(R, \tau(G))$, where $G = \{\emptyset, R\}$ and $M = \{R\}$, is G^{β}-disconnected space but it is a connected space, where $\tau(G) = \{\phi, R\}$.

Theorem 4.3.
A $G-$metrizable topological space $(X, \tau(G))$ is a $G^{\beta}-$disconnected space if and only if it is the union of two disjoint nonempty $G^{\beta}-$open sets.

Proof. Suppose that $(X, \tau(G))$ is a $G^{\beta}-$disconnected space. Then X is the union of two $G^{\beta}-$separated sets, that is, there are two nonempty subsets A and B of X such that

$$Cl_G^{\beta}(A) \cap B = \emptyset,\ A \cap Cl_G^{\beta}(B) = \emptyset \text{ and } A \cup B = X.$$

Take

$$M = X - Cl_G^{\beta}(A) \text{ and } H = X - Cl_G^{\beta}(B).$$

Then M and H are $G^{\beta}-$open sets. Since $B \neq \emptyset$ and $Cl_G^{\beta}(A) \cap B = \emptyset$, then $B \subseteq X - Cl_G^{\beta}(A)$, that is,

$$M = X - Cl_G^{\beta}(A) \neq \emptyset.$$

Similar $H \neq \emptyset$. Since

$$Cl_G^{\beta}(A) \cap B = \emptyset,\ A \cap Cl_G^{\beta}(B) = \emptyset \text{ and } A \cup B = X,$$

then

$$X - (M \cap H) = (X - M) \cup (X - H) = [Cl_G^{\beta}(A)] \cup [Cl_G^{\beta}(B)] = X.$$

That is, $M \cap H = \emptyset$.
Conversely, suppose that $(X, \tau(G))$ is the union of two disjoint nonempty $G^{\beta}-$open subsets, say M and H. Take

$$A = X - M \text{ and } B = X - H.$$

Then A and B are G^{β}-closed sets, that is, $Cl_G^{\beta}(A) = A$ and $Cl_G^{\beta}(B) = B$. Since $H \neq \emptyset$ and $H \cap M = \emptyset$, then $H \subseteq X - M = A$, that is, $A \neq \emptyset$. Similar $B \neq \emptyset$. Since $M \cap H = \emptyset$ and $M \cup H = X$, then

$$\begin{aligned} Cl_G^{\beta}(A) \cap B &= A \cap B = (X - M) \cap (X - H) \\ &= X - (M \cup H) = X - X = \emptyset. \end{aligned}$$

Similar, $A \cap Cl_G^{\beta}(B) = \emptyset$. Note that

$$A \cup B = (X - M) \cup (X - H) = X - (M \cap H) = X - \emptyset = X.$$

That is, $(X, \tau(G))$ is a $G^{\beta}-$disconnected space. □

Corollary 4.1.
A $G-$metrizable topological space $(X, \tau(G))$ is a $G^{\beta}-$disconnected space if and only if it is the union of two disjoint nonempty $G^{\beta}-$closed subsets.

Proof. Suppose that $(X,\tau(G))$ is a $G^\beta-$disconnected space. Then by Theorem (2.7), $(X,\tau(G))$ is the union of two disjoint nonempty $G^\beta-$open subsets, say M and H. Then $X-M$ and $X-H$ are $G^\beta-$closed subsets. Since $M\neq\emptyset$, $H\neq\emptyset$ and $X=M\cup H$ then $X-M\neq\emptyset$, $X-H\neq\emptyset$ and

$$(X-M)\cap(X-H)=X-(M\cup H)=X-X=\emptyset.$$

Since $M\cap H=\emptyset$ then

$$(X-M)\cup(X-H)=X-(M\cap H)=X-\emptyset=X.$$

Hence X is the union of two disjoint nonempty $G^\beta-$closed subsets.
Conversely, suppose that $(X,\tau(G))$ is the union of two disjoint nonempty $G^\beta-$closed subsets, say M and H. Take

$$A=X-M \text{ and } B=X-H.$$

Then A and B are $G^\beta-$open sets. Since $H\neq\emptyset$ and $H\cap M=\emptyset$, then $H\subseteq X-M=A$, that is, $A\neq\emptyset$. Similar $B\neq\emptyset$. Since $M\cap H=\emptyset$ and $M\cup H=X$, then

$$\begin{aligned} Cl_G^\beta(A)\cap B &= A\cap B=(X-M)\cap(X-H)\\ &= X-(M\cup H)\\ &= X-X=\emptyset. \end{aligned}$$

Similar, $A\cap Cl_G^\beta(B)=\emptyset$. Note that

$$A\cup B=(X-M)\cup(X-H)=X-(M\cap H)=X-\emptyset=X.$$

Then by Theorem (2.7), $(X,\tau(G))$ is a $G^\beta-$disconnected space. □

The $G-$open cover (resp. $G^\beta-$open cover) of a subset A of a $G-$metrizable topological space $(X,\tau(G))$ is a collection $\{\mathcal{G}_\lambda:\lambda\in I\}$ of $G-$open (resp. $G^\beta-$open) subsets of X such that $A\subseteq\cup_{\lambda\in I}\mathcal{G}_\lambda$, where I is an index set. In particular for X if $X=\cup_{\lambda\in I}\mathcal{G}_\lambda$.

5. $G^\beta-$COMPACT SPACES

Definition 5.1.
Let $(X,\tau(G))$ be a $G-$metrizable topological space and $A\subseteq X$. A is called a $G^\beta-$compact set in a $G-$metrizable topological space $(X,\tau(G))$ if for every $G^\beta-$open cover $\{\mathcal{G}_\lambda:\lambda\in I\}$ of A has finite $G^\beta-$open subcover $\{\mathcal{G}_{\lambda k}:k=1,2,...,n\}$ of A such that $A\subseteq\cup_{k=1}^n\{\mathcal{G}_{\lambda k}\}$. Similar, X is called a $G^\beta-$compact space if $X=\cup_{k=1}^n\{\mathcal{G}_{\lambda k}\}$.

Theorem 5.1.
Every $G^\beta-$compact set is compact set.

Proof. The proof of the theorem is clear since every open set is $G^\beta-$open set. □

The converse of the above theorem need not be true.

Example 5.1.
In the $G-$metrizable topological space $(R,\tau(G))$, where $G=\{\emptyset,R\}$ and $M=\{R\}$, is not $G^\beta-$compact space but it is a compact space, where $\tau(G)=\{\phi,R\}$.

Theorem 5.2.
Let $(X,\tau(G))$ be $G-$metrizable topological space and $A\subseteq X$. The set A is a $G^\beta-$compact set in a $G-$metrizable topological space $(X,\tau(G))$ if and only if for every $G-$open cover of A has finite $G-$open subcover.

Proof. Suppose that for every $G-$open cover $\{\{\mathcal{G}_\lambda:\lambda\in I\}$ of A has finite $G-$open subcover. Let $\{\{\mathcal{G}_\lambda:\lambda\in I\}$ be a $G^\beta-$open cover of A and $A\subseteq\cup_{\lambda\in I}\{\{\mathcal{G}_\lambda$. Then for each $x\in A$, there is $\lambda_x\in I$ such that $x\in\mathcal{G}_{\lambda_x}$. Since $\mathcal{G}_{\lambda_x}$ is $G^\beta-$open set, then there is $G-$open set U_{λ_x} containing x such that $U_{\lambda_x}-\mathcal{G}_{\lambda_x}$ is a finite set. Then the collection $\{U_{\lambda_x}:x\in A\}$ forms $G-$open cover of A. Then by the hypothesis, this collection has finite $G-$open subcover $\{U_{\lambda_{x_k}}:k=1,2,...,n\}$ of A such that $A\subseteq\cup_{k=1}^n U_{\lambda_{x_k}}$. Note that

$$A\subseteq\cup_{k=1}^n[(U_{\lambda_{x_k}}-\mathcal{G}_{\lambda_{x_k}})\cup\mathcal{G}_{\lambda_{x_k}}]=[\cup_{k=1}^n(U_{\lambda_{x_k}}-\mathcal{G}_{\lambda_{x_k}})]\cup[\cup_{k=1}^n\mathcal{G}_{\lambda_{x_k}}].$$

For each x_k, the set $U_{\lambda_{x_k}} - \mathscr{G}_{\lambda_{x_k}}$ is a finite set and there is a finite subset $I(x_k)$ of I such that

$$(U_{\lambda_{x_k}} - \mathscr{G}_{\lambda_{x_k}}) \subseteq \cup\{\mathscr{G}_\lambda : \lambda \in I(x_k)\}.$$

Hence

$$A \subseteq [\cup_{k=1}^{n}(\cup\{\mathscr{G}_\lambda : \lambda \in I(x_k)\}] \cup [\cup_{k=1}^{n}\mathscr{G}_{\lambda_{x_k}}].$$

That is, A is a $G^\beta-$compact set.
Conversely, it is clear since every $G-$open set is $G^\beta-$open set. □

Corollary 5.1.
$G-$metrizable topological space $(X,\tau(G))$ is a $G^\beta-$compact space if and only if for every $G-$open cover of X has finite $G-$open subcover.

Corollary 5.2.
Every strongly compact space is $G^\beta-$compact space.

Proof. Let $\{\mathscr{G}_\lambda : \lambda \in I\}$ be a $G-$open cover of A be a $G-$metrizable topological space and $\{\mathscr{G}_\lambda : \lambda \in I\}$ be a $G-$open cover of X and $A \subseteq \cup_{\lambda\in I}\mathscr{G}_\lambda$. Then by Theorem(1.4), $\{\mathscr{G}_\lambda : \lambda \in I\}$ is a open cover of X. Since (X,G) is strongly compact space, then $\{\mathscr{G}_\lambda : \lambda \in I\}$ has finite $G-$open subcover. Hence by Theorem (3.4), X is a $G^\beta-$compact space. □

Theorem 5.3.
Every $G^\beta-$closed subset of $G^\beta-$compact space is G^β-compact set.

Proof. Suppose that F is a $G^\beta-$closed subset of $G^\beta-$compact space $(X,\tau(G))$. Let $\{V_\lambda : \lambda \in I\}$ be any $G^\beta-$open cover of F, where I is an index set. Since F is a $G^\beta-$closed set in $(X,\tau(G))$ then $X - F$ is a $G^\beta-$open in $(X,\tau(G))$. Then

$$\{(X-F), V_\lambda : \lambda \in I\}$$

is $G^\beta-$open cover of $(X,\tau(G))$. Since $(X,\tau(G))$ is a $G^\beta-$compact space then there is a finite subcover

$$\{(X-F), V_{\lambda k} : k = 1,2,...,m\}$$

such that

$$X = (X-F) \cup [\cup_{k=1}^{m} V_{\lambda k}].$$

Hence $F \subseteq \cup_{k=1}^{m} V_{\lambda k}$. Hence F is a $G^\beta-$compact set in $(X,\tau(G))$. □

Theorem 5.4.
Let $(X,\tau(G))$ be a $G-$metrizable topological space, Every proper $G^\beta-$closed subset of X is $G^\beta-$compact set if and only if X is a $G^\beta-$compact space.

Proof. Suppose that every proper $G^\beta-$closed subset of X is $G^\beta-$compact set. Let $\{H_\lambda : \lambda \in I\}$ be any $G-$open cover of X. Choose $\lambda_0 \in I$ such that H_{λ_0} is a proper subset of X. Then $\{H_\lambda : \lambda \in I - \{\lambda_0\}\}$ is $G-$open cover of a $G^\beta-$closed set $X - H_{\lambda_0}$. By Theorem (3.7), $\{H_\lambda : \lambda \in I - \{\lambda_0\}\}$ is $G^\beta-$open cover of a $G^\beta-$closed set $X - H_{\lambda_0}$. Since $X - H_{\lambda_0}$ is $G^\beta-$compact set in X then there is a finite subcover

$$\{H_{\lambda k} : k = 1,2,...,m\}$$

such that

$$X - H_{\lambda_0} \subseteq \cup_{k=1}^{m} H_{\lambda k}$$

This implies,

$$X \subseteq H_{\lambda_0} \cup [\cup_{k=1}^{m} H_{\lambda k}].$$

That is, X is a $G^\beta-$compact space.
Conversely, by Theorem (3.7). □

6. $G^{\beta}-$SEPARATION AXIOMS

Definition 6.1.
A $G-$metrizable topological space is called:
1. $G^{\beta}-T_0$ space *if for two points $x \neq y \in X$, there is $G^{\beta}-$open set M in X such that $x \in M$ and $y \notin M$.*
2. $G^{\beta}-T_1$ space *if for two points $x \neq y \in X$, there are two $G^{\beta}-$open set M and U in X such that $x \in M$ and $y \notin M$, $y \in U$ and $x \notin U$.*
3.$G^{\beta}-T_2$ space or Hausdorff space *if for two points $x \neq y \in X$, there are two $G^{\beta}-$open set M and U in X such that $x \in M$ and $y \in U$ and $U \cap M = \emptyset$.*
4. $G^{\beta}-$regular space *if for each closed set F in $(X, \tau(G))$ and each $x \notin F$, there are two $G^{\beta}-$open sets M and U in $(X, \tau(G))$ such that $F \subseteq M$, $x \in U$ and $U \cap M = \emptyset$. A $G-$metrizable topological space $(X, \tau(G))$ is called* $G^{\beta}-T_3$ space *if it is $G^{\beta}-$regular space and $G^{\beta}-T_1$ space.*
5. $G^{\beta}-$normal space *if for each two disjoint closed sets F and M in $(X, \tau(G))$, there are two $G^{\beta}-$open sets M and U in $(X, \tau(G))$ such that $F \subseteq M$, $N \subseteq U$ and $U \cap M = \emptyset$. A $G-$metrizable topological space $(X, \tau(G))$ is called* $G^{\beta}-T_4$ space *if it is $G^{\beta}-$normal space and $G^{\beta}-T_1$ space.*

Theorem 6.1.
If $(X, \tau(G))$ is T_i space then the $G-$metrizable topological space $(X, \tau(G))$ is a $G^{\beta}-T_i$ space for all $i = 0, 1, 2, 3, 4$.

Proof. The proof of the theorem is clear since every open set is G^{β}-open set. □

The converse of the last theorem need not be true.

Example 6.1.
A $G-$metrizable topological space $(X, \tau(G))$ is a $G^{\beta}-T_i$ space but the space is not T_i space for all $i = 0, 1, 2$, where $X = \{a, b, c\}$, $\tau = \{\emptyset, X\}$ and $G = \{\{c\}, \{a, c\}, \{b, c\}, X\}$. Also a $G-$metrizable topological space $(X, \tau(G))$ is a $G^{\beta}-$regular and $G^{\beta}-$normal space but the space $(X, \tau(G))$ is not regular and not normal space.

Theorem 6.2.
Every $G^{\beta}-T_3$ space is a $G^{\beta}-T_2$ space.

Proof. Let be a $G^{\beta}-T_3$ space and $x \neq y \in X$ be any points in $(X, \tau(G))$. Since $(X, \tau(G))$ is a $G^{\beta}-T_1$ space then by Theorem (2.9), $\{x\}$ is a closed set in $(X, \tau(G))$ and $y \notin \{x\}$. Since $(X, \tau(G))$ is a $G^{\beta}-$regular space then there are two $G^{\beta}-$open sets M and U in $(X, \tau(G))$ such that $x \in \{x\} \subseteq M$, $y \in U$ and $U \cap M = \emptyset$. Hence $(X, \tau(G))$ is a $G^{\beta}-T_2$ space. □

Theorem 6.3.
Every $G^{\beta}-T_4$ space is a $G^{\beta}-T_3$ space.

Proof. Let be a $G^{\beta}-T_4$ space. Let F be any closed set in $(X, \tau(G))$ and $x \notin F$ be any point in $(X, \tau(G))$. Since $(X, \tau(G))$ is a $G^{\beta}-T_1$ space then by Theorem (2.9), $\{x\}$ is a closed set in $(X, \tau(G))$ and $F \cap \{x\} = \emptyset$. Since $(X, \tau(G))$ is a $G^{\beta}-$normal space then there are two $G^{\beta}-$open sets M and U in $(X, \tau(G))$ such that $x \in \{x\} \subseteq M$, $F \subseteq U$ and $U \cap M = \emptyset$. Hence $(X, \tau(G))$ is a $G^{\beta}-T_3$ space. □

Theorem 6.4.
A $G-$metrizable topological space $(X, \tau(G))$ is a $G^{\beta}-T_0$ space if and only if for two points $x \neq y \in X$, $Cl^{\beta}_G(\{x\}) \neq Cl^{\beta}_G(\{y\})$.

Proof. Suppose that is a $G^{\beta}-T_0$ space. Then for two points $x \neq y \in X$, there is $G^{\beta}-$open set M in such that $x \in M$, $y \notin M$. Since $X - M$ is a $G^{\beta}-$closed set and $y \in X - G$ then

$$Cl^{\beta}_G(\{y\}) \subseteq Cl^{\beta}_G(X - M) = X - M.$$

Since $x \in Cl^{\beta}_G(\{x\})$ and $x \notin X - G$ then $x \notin Cl^{\beta}_G(\{y\})$. That is, $Cl^{\beta}_G(\{x\}) \neq Cl^{\beta}_G(\{y\})$.
Conversely, Let $x \neq y \in X$ be any two points in X. Then by hypothesis, $Cl^{\beta}_G(\{x\}) \neq Cl^{\beta}_G(\{y\})$. Then there is at least one point $z \in X$ such that $z \in Cl^{\beta}_G(\{x\})$ and $z \notin Cl^{\beta}_G(\{y\})$. If $x \in Cl^{\beta}_G(\{y\})$ then $z \in Cl^{\beta}_G(\{x\}) \subseteq Cl^{\beta}_G(\{y\})$, and this is a contradiction. Hence $x \notin Cl^{\beta}_G(\{y\})$. Hence $M = X - Cl^{\beta}_G(\{y\})$ is a $G^{\beta}-$open set in $(X, \tau(G))$ such that $x \in M$ and $y \notin M$. Therefore is a $G^{\beta}-T_0$ space. □

Theorem 6.5.
A $G-$metrizable topological space $(X,\tau(G))$ is a $G^{\beta}-T_1$ space if and only if $\{x\}$ is $G^{\beta}-$closed set in $(X,\tau(G))$ for all $x\in X$.

Proof. Suppose that $(X,\tau(G))$ is a $G^{\beta}-T_1$ space. Let $x\in X$ be any point in X. Now we prove that $X-\{x\}$ is $G^{\beta}-$open set in $(X,\tau(G))$. Let $y\in X-\{x\}$. Then $x\neq y\in X$. Since $(X,\tau(G))$ is a $G^{\beta}-T_1$ space then there are two $G^{\beta}-$open sets M and U in $(X,\tau(G))$ such that $x\in M$, $y\notin M$, $y\in U$ and $x\notin U$. Then $y\in U\subseteq X-\{x\}$ Hence $X-\{x\}$ is $G^{\beta}-$open set in $(X,\tau(G))$. That is, $\{x\}$ is $G^{\beta}-$closed set in $(X,\tau(G))$.
Conversely, Let $x\neq y\in X$ be any two points in X. Then by hypothesis, $\{x\}$ and $\{y\}$ are $G^{\beta}-$closed set in $(X,\tau(G))$. Then $X-\{x\}$ and $X-\{y\}$ are $G^{\beta}-$open set in $(X,\tau(G))$, $x\in X-\{y\}$, $y\notin X-\{y\}$, $x\notin X-\{x\}$ and $y\in X-\{x\}$. Therefore $(X,\tau(G))$ is a $G^{\beta}-T_1$ space. □

Theorem 6.6.
A $G-$metrizable topological space $(X,\tau(G))$ is $G^{\beta}-T_2$ space if and only if for each $x\in X$ and for $x\neq y\in X$, there is a $G^{\beta}-$open set H in $(X,\tau(G))$ containing x such that $y\notin Cl_G^{\beta}(H)$.

Proof. Suppose that $(X,\tau(G))$ is $G^{\beta}-T_2$ space. Let $x\in X$ be any point in X and $x\neq y\in X$ be any point in X. Then there are two $G^{\beta}-$open sets M and U in $(X,\tau(G))$ such that $x\in M$, $y\in U$ and $U\cap M=\emptyset$. Take $H=M$ is a $G^{\beta}-$open set in $(X,\tau(G))$ containing x and so $y\notin H\subseteq Cl_G^{\beta}(H)$.
Conversely, Let $x\neq y\in X$ be any points in $(X,\tau(G))$. and By the hypothesis, there is a $G^{\beta}-$open set H in $(X,\tau(G))$ containing x such that $y\notin Cl_G^{\beta}(H)$. Then $X-Cl_G^{\beta}(H)$ is a $G^{\beta}-$open sets in $(X,\tau(G))$ containing y such that $x\in H$, $y\in X-Cl_G^{\beta}(H)$ and $H\cap X-Cl_G^{\beta}(H)=\emptyset$. Then $(X,\tau(G))$ is $G^{\beta}-T_2$ space. □

Theorem 6.7.
A $G-$metrizable topological space$(X,\tau(G))$ is $G^{\beta}-$regular space if and only if for each $x\in X$ and for each open set N in $(X,\tau(G))$ containing x, there is a $G^{\beta}-$open set H in $(X,\tau(G))$ containing x such that $Cl_G^{\beta}(H)\subseteq N$.

Proof. Suppose that $(X,\tau(G))$ is $G^{\beta}-$regular space. Let $x\in X$ be any point in X and N be any open set in $(X,\tau(G))$ containing x. Then $X-N$ is a closed set in $(X,\tau(G))$ and $x\notin X-N$. Since $(X,\tau(G))$ is $G^{\beta}-$regular space then there are two $G^{\beta}-$open sets M and U in $(X,\tau(G))$ such that $(X-N)\subseteq M$, $x\in U$ and $U\cap M=\emptyset$. Take $H=U$ is a $G^{\beta}-$open set in $(X,\tau(G))$ containing x. Then $H=U\subseteq(X-M)$, this implies,

$$Cl_G^{\beta}(H)\subseteq Cl_G^{\beta}(X-M)\subseteq(X-M)\subseteq N.$$

Conversely, Let F be any closed set in $(X,\tau(G))$ and $x\notin F$. Then $x\in X-F$ and $X-F$ is an open set in $(X,\tau(G))$ containing x. By the hypothesis, there is a $G^{\beta}-$open set H in $(X,\tau(G))$ containing x such that $Cl_G^{\beta}(H)\subseteq(X-F)$. Then $F\subseteq[X-Cl_G^{\beta}(H)]$ and $X-Cl_G^{\beta}(H)$ is a $G^{\beta}-$open set in $(X,\tau(G))$. Since H is a $G^{\beta}-$open set in $(X,\tau(G))$ containing x and $H\cap[X-Cl_G^{\beta}(H)]=\emptyset$, then $(X,\tau(G))$ is $G^{\beta}-$regular space. □

Theorem 6.8.
A $G-$metrizable topological space $(X,\tau(G))$ is $G^{\beta}-$normal space if and only if for each closed set F in $(X,\tau(G))$ and for each open set M in $(X,\tau(G))$ containing F, there is a $G^{\beta}-$open set V in $(X,\tau(G))$ containing F such that $Cl_G^{\beta}(V)\subseteq M$.

Proof. Suppose that $(X,\tau(G))$ is $G^{\beta}-$normal space. Let F be any closed set in $(X,\tau(G))$ and M be any open set in $(X,\tau(G))$ containing F. Then $X-M$ is a closed set in $(X,\tau(G))$ and $F\cap(X-M)=\emptyset$. Since $(X,\tau(G))$ is $G^{\beta}-$normal space then there are two $G^{\beta}-$open sets H and U in $(X,\tau(G))$ such that $(X-M)\subseteq U$, $F\subseteq H$ and $U\cap H=\emptyset$. Take $V=H$ is a $G^{\beta}-$open set in $(X,\tau(G))$ containing F. Then $V=H\subseteq(X-U)$, this implies,

$$Cl_G^{\beta}(V)\subseteq Cl_G^{\beta}(X-U)\subseteq(X-U)\subseteq M.$$

Conversely, Let F and H be any two closed sets in $(X,\tau(G))$ such that $F\cap H=\emptyset$. Then $H\subseteq(X-F)$ and $X-F$ is an open set in $(X,\tau(G))$ containing closed set H. By the hypothesis, there is a $G^{\beta}-$open set V in $(X,\tau(G))$ containing H such that $Cl_G^{\beta}(V)\subseteq(X-F)$. Then $F\subseteq X-Cl_G^{\beta}(V)$ and $X-Cl_G^{\beta}(V)$ is a $G^{\beta}-$open set in $(X,\tau(G)$. Since V is a $G^{\beta}-$open set in $(X,\tau(G))$ containing H and $V\cap[X-Cl_G^{\beta}(V)]=\emptyset$, then $(X,\tau(G))$ is $G^{\beta}-$normal space. □

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