Int. J. Adv. Appl. Math. and Mech. 9(4) (2022) 1 – 11 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

Thermal damages of living tissues due to hyperthermic perfusion

Research Article

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Received 27 March 2022; accepted (in revised version) 28 May 2022

Abstract: This problem deals the hyperthermic perfusion in the elastic blood vessel to study the thermal damages of living tissues due to a moving outer thermal resource. The inner and outer boundaries of the blood vessels are assumed to be stress free and thermal loads are given to inner and outer boundaries. The governing equations are to be solved in the Laplace transformation domain using state-space approach. The inversion of the Laplace transformation is to be computed numerically using a method based on Fourier series expansion technique. The influences of the moving heat source on the temperature and stress of living tissues are precisely investigated and presented graphically. Significant effect of moving heat source is reported.

MSC: 74F05 • 74J02

Keywords: Bioheat equation • Blood vessels • Hyperthermic perfusion • Laplace transform; Fourier series expansion technique • Moving heat source

1. Introduction

Bioheat transfer is the study of the transport of thermal energy in living systems. Because biochemical processes are temperature dependent, heat transfer plays a major role in living systems. Also, because the mass transport of blood through tissue causes a consequent thermal energy transfer, bioheat transfer methods are applicable for diagnostic and therapeutic applications involving either mass or heat transfer.

Since the thermal behavior of biological tissues depended on some complicated phenomena such as blood circulation and metabolic heat generation, some governing equations have been derived by several authors. In 1948, Pennes [1] studied the distribution of temperature in the forearm.

Recently, the estimation of transient temperatures in biological tissues has been under the focus of researchers. The so called thermal therapy has been lately considered one of the best existing alternatives for modern clinical treatments. Sur et al. [2] studied influence of moving heat source on skin tissue in the context of two-temperature memory-dependent heat transport law. Vastly, the methods of thermal treatments have been used for modern clinical treatments such as hyperthermia.

Hyperthermia (also called thermal therapy or thermotherapy) is a type of cancer treatment in which body tissue is exposed to high temperatures up to 113 °F. Van der Zee [3] showed that high temperatures can damage and kill

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cancer cells, usually with minimal injury to normal tissues. Hildebrandt et al. [4] proved that hyperthermia may shrink tumors by killing cancer cells and damaging proteins and structures within cells. Van der Zee [3] and Wust et al. [5] reported that hyperthermia is almost always used with other forms of cancer therapy, such as radiation therapy and chemotherapy. Waterman et al. [6] studied the mechanism of heat removal during local hyperthermia. Lang et al. [7] described the impact of nonlinear heat transfer on temperature control in regional hyperthermia. Mondal et al. [8] analyzed the thermal damage within a functionally graded spherical skin tissue to study hyperthermia cancer treatment in the context of bioheat model of heat conduction.

Biot [9] developed the coupled theory of thermo-elasticity to deal with defeat of the uncoupled theory that mechanical cause has no effect on the temperature field. In this theory, the heat equation has a parabolic form which predicts an infinite speed for the propagation of mechanical wave. The theory of generalized thermoelasticity with one relaxation time was introduced by Lord and Shulman [10] where heat transfer equation is as follows

$$(1+\tau_0\frac{\partial}{\partial t})(\rho C_E\dot{\theta}+\beta_1 T_0\frac{\partial^2 u}{\partial x\partial t})=K\nabla^2\theta+\rho Q$$

where θ is the thermodynamic temperature, K is the thermal conductivity, $\rho>0$ is the mass density, Q is the heat source acting per unit mass per second, C_E is the specific heat at constant strain, the term $\beta_1 T_0\frac{\partial^2 u}{\partial x\partial t}$ is due to the coupling between temperature and strain field, τ_0 is called relaxation time, which is the time required to maintain steady state heat conduction in an element of volume of an elastic body when a sudden temperature gradient is imposed on that volume element.

Perfusion is measured as the rate at which blood is delivered to tissue or volume of blood per unit time (blood flow) per unit tissue mass. So it can be assumed that perfusion follows second law of thermodynamics and hence entropy needs to grow with time in this case. Hyperthermic perfusion is a delivery technique of heat from blood to living tissue considering concentration and temperature gradient. Stehlin [11] studied hyperthermic perfusion with chemotherapy for cancer of the extremities. Sur and Kanoria [12] analyzed elasto-thermodiffusive response using three phase lag model in an elastic solid under hydrostatic pressure.

Heat transfer in biological systems is usually modelled by the Pennes' bioheat equation [1] based on the classical Fourier's law. But heat pulses obtained by the classical bio-heat conduction equation propagate at infinite speed. Much attention has been devoted to modifying the classical heat conduction equation to ensure finite speed pulse propagation. In mathematical terms, the governing partial differential equation is transformed from parabolic to hyperbolic type. A general form of bioheat transfer model in living tissues based on Lord Shulman model is as follows

$$(1+\tau_0\frac{\partial}{\partial t})(\rho C_E\dot{\theta}-c\omega_b\rho_b(\theta+T_0)-(Q_{met}+Q_{ext}))=K\nabla^2\theta$$

where T_0 is the temperature of the tissue at natural state, C_E is the specific heat of the tissue, c is the specific heat of the blood, ρ_b is the blood mass density, ω_b is the perfusion rate of blood (volume blood per unit mass per unit time), Q_{met} is the metabolic thermal generations in the skin tissue, Q_{ext} is the moving line heat source and K is the thermal conductivity of the living tissue. Sur et al. [13] studied influence of moving heat source on skin tissue in the context of two-temperature Caputo-Fabrizio heat transport law. Mondal et al. [14] analyzed the thermal damage within the skin tissue to study hyperthermia cancer treatment using bioheat model of heat conduction. Sur et al. [15] also studied the variations of temperature profile and thermal damages within a spherical living tissue subjected to a thermal therapy, whose outer surface is thermally insulated. Sur and Kanoria [16] analyzed thermoelastic response in a functionally graded infinite space subjected to a Mode-I crack. Purkait et al. [17] studied thermoelastic interaction in a two-dimensional infinite space due to memory- dependent heat transfer.

In the last few years the mechanism of blood vessels has been an interesting topic for scientists and clinicians because blood vessels are elastic tubular formations that transport blood throughout the entire organism. The basic structural elements of blood vessels are as follows: endothelial cells, collagen fibers, elastin fibers, smooth muscular cells, and the substance which joins all the elements. Due to the mechanical property of blood vessels, hyperthermic perfusion in blood vessels demand an appropriate heat transfer equation. Such an equation must take into consideration three factors (i) elastic property of blood vessels (ii) finite bioheat transfer (iii) blood perfusion.

In the present analysis, the hyperthermic perfusion through elastic blood vessels using bioheat model of heat conduction due to an external heat source have been reported to analyze the thermal damage of living tissue in study of cancer treatment. The problem is solved using Laplace transform technique and the state-space approach. The analytical expressions for the thermoelastic stresses, temperature, concentration and chemical potential are obtained in the Laplace transform domain. The inversion of the Laplace transform is computed numerically by using a method on Fourier expansion technique [18]. The results have been demonstrated in the graphical forms to study the effect of velocity of moving heat source on stress and temperature.

2. Formulation of the problem

Let us consider the semi-infinite tissue of elastic blood vessels those are near to the skin and are at initial temperature 37°C after being heated by a moving heat source. In the context of thermoelasticity, the generalized form of heat transport equation in biological tissue is expressed as

$$(1+\tau_0\frac{\partial}{\partial t})(\rho C_E\dot{\theta}+\beta_1 T_0\frac{\partial^2 u}{\partial x\partial t}-c\omega_b\rho_b(\theta+T_0)-(Q_{met}+Q_{ext}))=K\nabla^2\theta \tag{1}$$

where $\beta_1=(3\lambda+2\mu)\alpha_t$, λ,μ are Lamé's constants, α_t is the coefficient of linear thermal expansion.
The equation of motion in absence of body force is given by [16]

$$(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}-\beta_1\frac{\partial\theta}{\partial x}-\beta_2\frac{\partial C}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2} \tag{2}$$

where $\beta_2=(3\lambda+2\mu)\alpha_c$, α_c is the coefficient of linear perfusion expansion and C is the concentration of blood.
The constitutive equation is given by [19]

$$\sigma_{xx}=(\lambda+2\mu)\frac{\partial u}{\partial x}-\beta_1\theta-\beta_2 C \tag{3}$$

For convenience we introduce the following non-dimensional variables,
$u'=c_1\eta u$, $x'=c_1\eta x$, $\theta'=\frac{\beta_1\theta}{\lambda+2\mu}$, $C'=\frac{\beta_2 C}{\lambda+2\mu}$ $t'=c_1^2\eta t$, $\tau_0'=c_1^2\eta\tau_0$

where

$$c_1^2=\frac{\lambda+2\mu}{\rho},\eta=\frac{\rho C_E}{K},\omega_b'\rho_b'=\frac{\beta_2}{c_1^2\eta(\lambda+2\mu)}\omega_b\rho_b,\frac{\sigma_{xx}}{\lambda+2\mu}=\sigma'$$

Therefore , the governing equations, given by Eqs. (1)-(3) can be expressed in the following forms (where the primes are suppressed for simplicity) as

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial\theta}{\partial x}=\frac{\partial^2 u}{\partial t^2} \tag{4}$$

$$(1+\tau_0\frac{\partial}{\partial t})[\dot{\theta}+\epsilon\dot{e}+\omega_b\rho_b(p_1\theta+\epsilon\alpha_1)-(Q_{met}+Q_{ext})]=\nabla^2\theta \tag{5}$$

$$\dot{\sigma}=\frac{\partial^2 u}{\partial x\partial t}-\dot{\theta}+\omega_b\rho_b \tag{6}$$

where $\epsilon=\frac{\beta_1^2 T_0}{(\lambda+2\mu)\rho C_E}$, $\alpha_1=\frac{c\rho c_1^2}{\beta_1\beta_2}$, $p_1=\frac{cc_1^2}{\beta_2 C_E}$, $Q_{ext}'=\frac{\beta_1}{c_1^4\rho\eta^2 K}Q_{ext}$,
$Q_{met}'=\frac{\beta_1}{c_1^4\rho\eta^2 K}Q_{met}$, $\eta=\frac{\rho C_E}{K}$, $c_1^2=\frac{\lambda+2\mu}{\rho}$
and $\dot{C}=-\omega_b\rho_b$ (where $\omega_b\rho_b$ is constant)
The form of the external heat source Q_{ext} being a movable thermal resource is

$$Q_{ext}(x,t)=Q_0\delta(x-vt) \tag{7}$$

where Q_0 is a constant dimensionless quantity, v is the velocity and $\delta(x)$ is the delta function.
The initial and boundary conditions for the problem are taken as

$$u=\theta=\sigma=0 \text{ at } t=0 \tag{8}$$

$$\frac{\partial u}{\partial t}=\frac{\partial\theta}{\partial t}=\frac{\partial\sigma}{\partial t}=0 \text{ at } t=0 \tag{9}$$

The problem is to solve Eqs. (4)-(6) and subject to the following boundary conditions:
(i) Thermal boundary condition

$$\theta(x,t)|_{x=0}=\theta_0 \tag{10}$$

$$\theta(x,t)|_{x=L}=\theta_L \tag{11}$$

(ii) Mechanical boundary condition
Since the boundary is taken to be stress free we obtained

$$\sigma(x,t)|_{x=0}=\sigma_0=0 \tag{12}$$

$$\sigma(x,t)|_{x=L}=\sigma_L=0 \tag{13}$$

3. Method of Solution

Taking the Laplace transforms defined by the relation

$$\overline{f}(x,s) = L[f(x,t)] = \int_0^{\infty} e^{-st} f(x,t)\,dt, Re(s) > 0$$

on both sides of the Eqs. (4)-(6), we obtain:

$$\frac{d^2\overline{u}}{dx^2} - \frac{d\overline{\theta}}{dx} = s^2\overline{u}(x,s) \tag{14}$$

$$\frac{d^2\overline{\theta}}{dx^2} = \overline{\theta}(s+\omega_b\rho_b p_1 + \tau_0 s^2 + \tau_0\omega_b\rho_b p_1 s) + (\epsilon s + \tau_0\epsilon s^2)\frac{d\overline{u}}{dx} + \frac{\omega_b\rho_b\epsilon\alpha_1}{s} - \frac{Q_0}{v}e^{-\frac{xs}{v}} - \frac{Q_{met}}{s} \tag{15}$$

$$s\overline{\sigma} = s\frac{d\overline{u}}{dx} - s\overline{\theta} + \omega_b\rho_b\frac{1}{s} \tag{16}$$

Eliminating $\bar{u}$ from Eqs. (14)-(15) using Eq. (16), we obtain

$$\frac{d^2\overline{\theta}}{dx^2} = L_1\overline{\theta} + L_2\overline{\sigma} + T_1 \tag{17}$$

where

$$L_1 = s + \omega_b\rho_b p_1 + \tau_0 s^2 + \tau_0\omega_b\rho_b p_1 s + \epsilon s + \tau_0\epsilon s^2$$

$$L_2 = \epsilon s + \tau_0\epsilon s^2$$

$$T_1 = \frac{\omega_b\rho_b\epsilon\alpha_1}{s} - (1+\tau_0 s)\frac{\epsilon\omega_b\rho_b}{s} - \frac{Q_0}{v}e^{-\frac{xs}{v}} - \frac{Q_{met}}{s}$$

and

$$\frac{d^2\overline{\sigma}}{dx^2} = M_1\overline{\theta} + M_2\overline{\sigma} + T_2 \tag{18}$$

where

$$M_1 = s^2, M_2 = s^2, T_2 = -\omega_b\rho_b$$

Choosing as state variables the temperature $\bar{\theta}$ and the stress component $\bar{\sigma}$ in the x-direction, Eqs. (17) and (18) can be written in the matrix form as:

$$\frac{d^2\bar{v}(x,s)}{dx^2} = A(s)\overline{v}(x,s) + T(x) \tag{19}$$

where

$$\bar{v}(x,s) = \begin{bmatrix} \bar{\theta}(x,s) \\ \bar{\sigma}(x,s) \end{bmatrix}, \quad A(s) = \begin{bmatrix} L_1 & L_2 \\ M_1 & M_2 \end{bmatrix} \quad \text{and} \quad T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}.$$

The formal solution of Eq. (19) can be written in the form

$$\therefore\ \bar{v}(x,s) = \left[\bar{v}(0,s) + \frac{T_0}{A(s)} - \frac{\bar{v}(0,s)e^{\sqrt{A(s)}L} - \bar{v}(L,s) + \frac{1}{A(s)}(T_0 e^{\sqrt{A(s)}L} - T_L)}{e^{\sqrt{A(s)}L} - e^{-\sqrt{A(s)}L}}\right] e^{\sqrt{A(s)}x}$$
$$+ \left[\frac{\bar{v}(0,s)e^{\sqrt{A(s)}L} - \bar{v}(L,s) + \frac{1}{A(s)}(T_0 e^{\sqrt{A(s)}L} - T_L)}{e^{\sqrt{A(s)}L} - e^{-\sqrt{A(s)}L}}\right] e^{-\sqrt{A(s)}x} - \frac{T(x)}{A(s)} \tag{20}$$

where

$$\bar{v}(0,s) = \begin{bmatrix} \bar{\theta}(0,s) \\ \bar{\sigma}(0,s) \end{bmatrix} = \begin{bmatrix} \bar{\theta}_0 \\ \bar{\sigma}_0 \end{bmatrix}$$

$$\bar{v}(L,s) = \begin{bmatrix} \bar{\theta}(L,s) \\ \bar{\sigma}(L,s) \end{bmatrix} = \begin{bmatrix} \bar{\theta}_L \\ \bar{\sigma}_L \end{bmatrix}$$

$$T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}$$

then $T_0 = T(0) = \begin{bmatrix} T_1(0) \\ T_2(0) \end{bmatrix} = \begin{bmatrix} T_{10} \\ T_{20} \end{bmatrix}$, $T_L = T(L) = \begin{bmatrix} T_1(L) \\ T_2(L) \end{bmatrix} = \begin{bmatrix} T_{1L} \\ T_{2L} \end{bmatrix}$

and

$$[A(s)]^{-1} = \frac{1}{L_1 M_2 - L_2 M_1} \begin{bmatrix} M_2 & -L_2 \\ -M_1 & L_1 \end{bmatrix}$$

$$\therefore T(x) A(s)^{-1} = \begin{bmatrix} \dfrac{M_2 T_1 - L_2 T_2}{L_1 M_2 - L_2 M_1} \\ \\ \dfrac{L_1 T_2 - M_1 T_1}{L_1 M_2 - L_2 M_1} \end{bmatrix}$$

We shall use the spectral decomposition of the matrix A(s) and the well-known Cayley- Hamiltonian theorem to find the matrix form of the expression $\exp[-\sqrt{A(s)}x]$. The characteristic equation of the matrix $A(s)$ can be written as follows:

$$k^2 - k(L_1 + M_2) + (L_1 M_2 - L_2 M_1) = 0 \tag{21}$$

The roots of the equation namely k_1 and k_2 satisfies the following relations:

$$k_1 + k_2 = L_1 + M_2 \tag{21a}$$

$$k_1 k_2 = L_1 M_2 - L_2 M_1 \tag{21b}$$

The Taylor series expansion of the matrix exponentials in Eq. (20) has the form

$$\exp[-\sqrt{A(s)}x] = \sum_{n=0}^{\infty} \frac{[-\sqrt{A(s)}x]^n}{n!} \tag{22}$$

and

$$\exp[\sqrt{A(s)}x] = \sum_{n=0}^{\infty} \frac{[\sqrt{A(s)}x]^n}{n!} \tag{23}$$

Using the Cayley-Hamilton theorem, we can express A^2 and higher powers of the matrix A in terms of I and A, where I is the unit matrix of second order.

Thus, the infinite series in Eqs. (22) and (23) can be reduced to

$$\exp[-\sqrt{A(s)}x] = b_0(x,s)I + b_1(x,s)\sqrt{A(s)} \tag{24}$$

and

$$\exp[\sqrt{A(s)}x] = d_0(x,s)I + d_1(x,s)\sqrt{A(s)} \tag{25}$$

where b_0, b_1, d_0 and d_1 are the coefficients depending on x and s.

By the Cayley-Hamilton theorem, the characteristic roots $\sqrt{k_1}$ and $\sqrt{k_2}$ of the matrix $\sqrt{A}$ must satisfy Eqs. (24) and (25), thus

$$\exp(-\sqrt{k_1}x) = b_0 + b_1\sqrt{k_1} \tag{26}$$

and

$$\exp(-\sqrt{k_2}x) = b_0 + b_1\sqrt{k_2} \tag{27}$$

also,

$$\exp(\sqrt{k_1}x) = d_0 + d_1\sqrt{k_1} \tag{28}$$

and

$$\exp(\sqrt{k_2}x) = d_0 + d_1\sqrt{k_2} \tag{29}$$

The solution of the system of Eqs. (26) and (27) is given by

$$b_0 = \frac{\sqrt{k_1}e^{-\sqrt{k_2}x} - \sqrt{k_2}e^{-\sqrt{k_1}x}}{\sqrt{k_1} - \sqrt{k_2}} \text{ and } b_1 = \frac{e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x}}{\sqrt{k_1} - \sqrt{k_2}}$$

and the solution of the system of Eqs. (28) and (29) is given by

$$d_0 = \frac{\sqrt{k_1}e^{\sqrt{k_2}x} - \sqrt{k_2}e^{\sqrt{k_1}x}}{\sqrt{k_1}-\sqrt{k_2}} \text{ and } d_1 = \frac{e^{\sqrt{k_1}x} - e^{\sqrt{k_2}x}}{\sqrt{k_1}-\sqrt{k_2}}$$

Hence, we have

$$\exp[-\sqrt{A(s)}x] = L_{ij}, \quad i,j = 1,2$$

where

$$L_{11} = \frac{e^{-\sqrt{k_2}x}(k_1-L_1) - e^{-\sqrt{k_1}x}(k_2-L_1)}{k_1-k_2}, \; L_{12} = \frac{L_2(e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x})}{k_1-k_2}$$

$$L_{21} = \frac{M_1(e^{-\sqrt{k_1}x} - e^{-\sqrt{k_2}x})}{k_1-k_2}, \; L_{22} = \frac{e^{-\sqrt{k_2}x}(k_1-M_2) - e^{-\sqrt{k_1}x}(k_2-M_2)}{k_1-k_2}$$

Again we have

$$\exp[\sqrt{A(s)}x] = N_{ij}, \quad i,j = 1,2$$

where

$$N_{11} = \frac{e^{\sqrt{k_2}x}(k_1-L_1) - e^{\sqrt{k_1}x}(k_2-L_1)}{k_1-k_2}, \; N_{12} = \frac{L_2(e^{\sqrt{k_1}x} - e^{\sqrt{k_2}x})}{k_1-k_2}$$

$$N_{21} = \frac{M_1(e^{\sqrt{k_1}x} - e^{\sqrt{k_2}x})}{k_1-k_2}, \; N_{22} = \frac{e^{\sqrt{k_2}x}(k_1-M_2) - e^{\sqrt{k_1}x}(k_2-M_2)}{k_1-k_2}$$

Therefore, the solution of Eq. (20) can be written in the form

$$\bar{v}(x,s) = \left[\begin{bmatrix}\bar{\phi}_0\\ \bar{\sigma}_0\end{bmatrix} + A^{-1}T_0 - (2\sqrt{A}L)^{-1}\left\{\begin{bmatrix}\bar{\phi}_0\\ \bar{\sigma}_0\end{bmatrix}(I+\sqrt{A}L) - \begin{bmatrix}\bar{\phi}_L\\ \bar{\sigma}_L\end{bmatrix} + A^{-1}T_0\sqrt{A}L - A^{-1}T_L\right\}\right]\cdot$$
$$(N_{ij}) + (2\sqrt{A}L)^{-1}\left[\begin{bmatrix}\bar{\phi}_0\\ \bar{\sigma}_0\end{bmatrix}(I+\sqrt{A}L) - \begin{bmatrix}\bar{\phi}_L\\ \bar{\sigma}_L\end{bmatrix} + A^{-1}T_0\sqrt{A}L - A^{-1}T_L\right](L_{ij}) - A^{-1}T$$

where

$$e^{\sqrt{A}L} - I = \sqrt{A}L \;, \; e^{\sqrt{A}L} - e^{-\sqrt{A}L} = 2\sqrt{A}L$$

Hence we obtain

$$\begin{bmatrix}\bar{\theta}(x,s)\\ \bar{\sigma}(x,s)\end{bmatrix} = \begin{bmatrix}\bar{\theta}_0\\ \bar{\sigma}_0\end{bmatrix}\begin{bmatrix}N_{11} & N_{12}\\ N_{21} & N_{22}\end{bmatrix} + \begin{bmatrix}\frac{M_2T_{10}-L_2T_{20}}{L_1M_2-L_2M_1}\\ \frac{L_1T_{20}-M_1T_{10}}{L_1M_2-L_2M_1}\end{bmatrix}\begin{bmatrix}N_{11} & N_{12}\\ N_{21} & N_{22}\end{bmatrix}$$

$$+\frac{1}{2L}\frac{(\sqrt{k_1}+\sqrt{k_2})^2}{(L_1M_2-L_2M_1)+(L_1+M_2)\sqrt{k_1k_2}+k_1k_2}\begin{bmatrix}\frac{M_2+\sqrt{k_1k_2}}{\sqrt{k_1}+\sqrt{k_2}} & -\frac{L_2}{\sqrt{k_1}+\sqrt{k_2}}\\ -\frac{M_1}{\sqrt{k_1}+\sqrt{k_2}} & \frac{L_1+\sqrt{k_1k_2}}{\sqrt{k_1}+\sqrt{k_2}}\end{bmatrix}\times$$

$$\left\{\begin{bmatrix}\bar{\theta}_0\\ \bar{\sigma}_0\end{bmatrix}\begin{bmatrix}1+\frac{LL_1+L\sqrt{k_1k_2}}{\sqrt{k_1}+\sqrt{k_2}} & \frac{LL_2}{\sqrt{k_1}+\sqrt{k_2}}\\ \frac{LM_1}{\sqrt{k_1}+\sqrt{k_2}} & 1+\frac{LM_2+L\sqrt{k_1k_2}}{\sqrt{k_1}+\sqrt{k_2}}\end{bmatrix} - \begin{bmatrix}\bar{\theta}_L\\ \bar{\sigma}_L\end{bmatrix}\right.$$

$$\left.+\begin{bmatrix}\frac{M_2T_{10}-L_2T_{20}}{L_1M_2-L_2M_1}\\ \frac{L_1T_{20}-M_1T_{10}}{L_1M_2-L_2M_1}\end{bmatrix}\begin{bmatrix}\frac{LL_1+L\sqrt{k_1k_2}}{\sqrt{k_1}+\sqrt{k_2}} & \frac{LL_2}{\sqrt{k_1}+\sqrt{k_2}}\\ \frac{LM_1}{\sqrt{k_1}+\sqrt{k_2}} & \frac{LM_2+L\sqrt{k_1k_2}}{\sqrt{k_1}+\sqrt{k_2}}\end{bmatrix} - \begin{bmatrix}\frac{M_2T_{1L}-L_2T_{2L}}{L_1M_2-L_2M_1}\\ \frac{L_1T_{2L}-M_1T_{1L}}{L_1M_2-L_2M_1}\end{bmatrix}\right\}$$

$$\times\begin{bmatrix}L_{11}-N_{11} & L_{12}-N_{12}\\ L_{21}-N_{21} & L_{22}-N_{22}\end{bmatrix} - \begin{bmatrix}\frac{M_2T_1-L_2T_2}{L_1M_2-L_2M_1}\\ \frac{L_1T_2-M_1T_1}{L_1M_2-L_2M_1}\end{bmatrix}$$

$$\therefore \bar{\theta}(x,s) = (N_{11}\bar{\theta_0} + N_{12}\bar{\sigma_0}) + \left(\frac{M_2T_{10} - L_2T_{20}}{L_1M_2 - L_2M_1}N_{11} + \frac{L_1T_{20} - M_1T_{10}}{L_1M_2 - L_2M_1}N_{12}\right)$$
$$-\frac{M_2T_1 - L_2T_2}{L_1M_2 - L_2M_1} + \frac{1}{2L}\frac{(\sqrt{k_1} + \sqrt{k_2})^2}{(L_1M_2 - L_2M_1) + (L_1 + M_2)\sqrt{k_1k_2} + k_1k_2}\times$$
$$[P\{\left(1 + \frac{LL_1 + L\sqrt{k_1k_2}}{\sqrt{k_1} + \sqrt{k_2}}\right)\bar{\theta_0} + \left(\frac{LL_2}{\sqrt{k_1} + \sqrt{k_2}}\right)\bar{\sigma_0} - \bar{\theta_L} - \frac{M_2T_{1L} - L_2T_{2L}}{L_1M_2 - L_2M_1}$$
$$+\left(\frac{LL_1 + L\sqrt{k_1k_2}}{\sqrt{k_1} + \sqrt{k_2}}\right)\left(\frac{M_2T_{10} - L_2T_{20}}{L_1M_2 - L_2M_1}\right) + \left(\frac{LL_2}{\sqrt{k_1} + \sqrt{k_2}}\right)\left(\frac{L_1T_{20} - M_1T_{10}}{L_1M_2 - L_2M_1}\right)\} \tag{30}$$
$$+Q\{\left(\frac{LM_1}{\sqrt{k_1} + \sqrt{k_2}}\right)\bar{\theta_0} + \left(1 + \frac{LM_2 + L\sqrt{k_1k_2}}{\sqrt{k_1} + \sqrt{k_2}}\right)\bar{\sigma_0} - \bar{\sigma_L} - \frac{L_1T_{2L} - M_1T_{1L}}{L_1M_2 - L_2M_1}$$
$$+\left(\frac{LM_1}{\sqrt{k_1} + \sqrt{k_2}}\right)\left(\frac{M_2T_{10} - L_2T_{20}}{L_1M_2 - L_2M_1}\right) + \left(\frac{LM_2 + L\sqrt{k_1k_2}}{\sqrt{k_1} + \sqrt{k_2}}\right)\left(\frac{L_1T_{20} - M_1T_{10}}{L_1M_2 - L_2M_1}\right)\}]$$

and

$$\sigma(x,s) = (N_{21}\bar{\theta_0} + N_{22}\bar{\sigma_0}) + \frac{M_2T_{10} - L_2T_{20}}{L_1M_2 - L_2M_1}N_{21} + \frac{L_1T_{20} - M_1T_{10}}{L_1M_2 - L_2M_1}N_{22}$$
$$-\frac{L_1T_2 - M_1T_1}{L_1M_2 - L_2M_1} + \frac{1}{2L}\frac{(\sqrt{k_1} + \sqrt{k_2})^2}{(L_1M_2 - L_2M_1) + (L_1 + M_2)\sqrt{k_1k_2} + k_1k_2}\times$$
$$[R\{(1 + \frac{LL_1 + L\sqrt{k_1k_2}}{\sqrt{k_1} + \sqrt{k_2}})\bar{\theta_0} + (\frac{LL_2}{\sqrt{k_1} + \sqrt{k_2}})\bar{\sigma_0} - \bar{\theta_L} - (\frac{M_2T_{1L} - L_2T_{2L}}{L_1M_2 - L_2M_1})$$
$$+(\frac{LL_1 + L\sqrt{k_1k_2}}{\sqrt{k_1} + \sqrt{k_2}})(\frac{M_2T_{10} - L_2T_{20}}{L_1M_2 - L_2M_1}) + (\frac{LL_1}{\sqrt{k_1} + \sqrt{k_2}})(\frac{L_1T_{20} - M_1T_{10}}{L_1M_2 - L_2M_1})\} \tag{31}$$
$$+S\{\frac{LM_1}{\sqrt{k_1} + \sqrt{k_2}}\bar{\theta_0} + (1 + \frac{LM_2 + L\sqrt{k_1k_2}}{\sqrt{k_1} + \sqrt{k_2}})\bar{\sigma_0} - \bar{\sigma_L} - (\frac{L_1T_{2L} - M_1T_{1L}}{L_1M_2 - L_2M_1})$$
$$+(\frac{LM_1}{\sqrt{k_1} + \sqrt{k_2}})(\frac{M_2T_{10} - L_2T_{20}}{L_1M_2 - L_2M_1}) + (\frac{LM_2 + L\sqrt{k_1k_2}}{\sqrt{k_1} + \sqrt{k_2}})(\frac{L_1T_{20} - M_1T_{10}}{L_1M_2 - L_2M_1})\}]$$

where

$$\frac{M_2 + \sqrt{k_1k_2}}{\sqrt{k_1} + \sqrt{k_2}}(L_{11} - N_{11}) - \frac{M_1}{\sqrt{k_1} + \sqrt{k_2}}(L_{12} - N_{12}) = P$$

$$-\frac{L_2}{\sqrt{k_1} + \sqrt{k_2}}(L_{11} - N_{11}) + \frac{L_1 + \sqrt{k_1k_2}}{\sqrt{k_1} + \sqrt{k_2}}(L_{12} - N_{12}) = Q$$

$$\frac{M_2 + \sqrt{k_1k_2}}{\sqrt{k_1} + \sqrt{k_2}}(L_{21} - N_{21}) - \frac{M_1}{\sqrt{k_1} + \sqrt{k_2}}(L_{22} - N_{22}) = R$$

$$-\frac{L_2}{\sqrt{k_1} + \sqrt{k_2}}(L_{21} - N_{21}) + \frac{L_1 + \sqrt{k_1k_2}}{\sqrt{k_1} + \sqrt{k_2}}(L_{22} - N_{22}) = S$$

Using Laplace transformation to the Eqs. (10), (11), (12) and (13) we obtained

$$\bar{\theta}_0 = \frac{\theta_0}{s} \tag{32}$$

$$\bar{\theta}_L = \frac{\theta_L}{s} \tag{33}$$

$$\bar{\sigma}_0 = 0 \tag{34}$$

$$\bar{\sigma}_L = 0 \tag{35}$$

Hence, we can use the conditions (32)-(35) into Eqs. (30) and (31) to get the exact solution in the Laplace transform domain as

$$\begin{aligned}\bar{\theta}(x,s) = {} & N_{11}\cdot\frac{\theta_0}{s} + \frac{M_2T_{10}-L_2T_{20}}{L_1M_2-L_2M_1}N_{11} + \frac{L_1T_{20}-M_1T_{10}}{L_1M_2-L_2M_1}N_{12} \\ & -\frac{M_2T_1-L_2T_2}{L_1M_2-L_2M_1} + \frac{1}{2L}\frac{(\sqrt{k_1}+\sqrt{k_2})^2}{(L_1M_2-L_2M_1)+(L_1+M_2)\sqrt{k_1k_2}+k_1k_2}\times \\ & [P\{\left(1+\frac{LL_1+L\sqrt{k_1k_2}}{\sqrt{k_1}+\sqrt{k_2}}\right)\cdot\frac{\theta_0}{s}-\frac{\theta_L}{s}-\frac{M_2T_{1L}-L_2T_{2L}}{L_1M_2-L_2M_1} \\ & +\left(\frac{LL_1+L\sqrt{k_1k_2}}{\sqrt{k_1}+\sqrt{k_2}}\right)\left(\frac{M_2T_{10}-L_2T_{20}}{L_1M_2-L_2M_1}\right)+\left(\frac{LL_2}{\sqrt{k_1}+\sqrt{k_2}}\right)\left(\frac{L_1T_{20}-M_1T_{10}}{L_1M_2-L_2M_1}\right)\} \\ & +Q\{\left(\frac{LM_1}{(\sqrt{k_1}+\sqrt{k_2})}\cdot\frac{\theta_0}{s}\right)-\left(\frac{L_1T_{2L}-M_1T_{1L}}{L_1M_2-L_2M_1}\right)+\left(\frac{LM_1}{\sqrt{k_1}+\sqrt{k_2}}\right)\left(\frac{M_2T_{10}-L_2T_{20}}{L_1M_2-L_2M_1}\right) \\ & +\left(\frac{LM_2+L\sqrt{k_1k_2}}{\sqrt{k_1}+\sqrt{k_2}}\right)\left(\frac{L_1T_{20}-M_1T_{10}}{L_1M_2-L_2M_1}\right)\}]\end{aligned} \tag{36}$$

$$\begin{aligned}\bar{\sigma}(x,s) = {} & N_{21}\cdot\frac{\theta_0}{s} + \frac{M_2T_{10}-L_2T_{20}}{L_1M_2-L_2M_1}N_{21} + \frac{L_1T_{20}-M_1T_{10}}{L_1M_2-L_2M_1}N_{22} \\ & -\frac{L_1T_2-M_1T_1}{L_1M_2-L_2M_1} + \frac{1}{2L}\frac{(\sqrt{k_1}+\sqrt{k_2})^2}{(L_1M_2-L_2M_1)+(L_1+M_2)\sqrt{k_1k_2}+k_1k_2}\times \\ & [R\{\left(1+\frac{LL_1+L\sqrt{k_1k_2}}{\sqrt{k_1}+\sqrt{k_2}}\right)\cdot\frac{\theta_0}{s}-\frac{\theta_L}{s}-\left(\frac{M_2T_{1L}-L_2T_{2L}}{L_1M_2-L_2M_1}\right) \\ & +\left(\frac{LL_1+L\sqrt{k_1k_2}}{\sqrt{k_1}+\sqrt{k_2}}\right)\left(\frac{M_2T_{10}-L_2T_{20}}{L_1M_2-L_2M_1}\right)+\left(\frac{LL_1}{\sqrt{k_1}+\sqrt{k_2}}\right)\left(\frac{L_1T_{20}-M_1T_{10}}{L_1M_2-L_2M_1}\right)\} \\ & +S\{\left(\frac{LM_1}{(\sqrt{k_1}+\sqrt{k_2})}\cdot\frac{\theta_0}{s}\right)-\left(\frac{L_1T_{2L}-M_1T_{1L}}{L_1M_2-L_2M_1}\right) \\ & +\left(\frac{LM_1}{\sqrt{k_1}+\sqrt{k_2}}\right)\left(\frac{M_2T_{10}-L_2T_{20}}{L_1M_2-L_2M_1}\right)+\left(\frac{LM_2+L\sqrt{k_1k_2}}{\sqrt{k_1}+\sqrt{k_2}}\right)\left(\frac{L_1T_{20}-M_1T_{10}}{L_1M_2-L_2M_1}\right)\}]\end{aligned} \tag{37}$$

4. Inversion of the Laplace transform

In order to invert the Laplace transforms in the above equations we shall use a numerical technique based on Fourier expansions of functions.

Let $\bar{g}(s)$ be the Laplace transform of a given function $g(t)$. The inversion formula of Laplace transforms states that

$$g(t) = \frac{1}{2\pi i}\int_{d+i\infty}^{d-i\infty} e^{st}\bar{g}(s)\,ds,$$

where d is an arbitrary positive constant greater than all the real parts of the singularities of $\bar{g}(s)$. Taking $s = d + iy$, we get

$$g(t) = \frac{e^{dt}}{2\pi}\int_{\infty}^{-\infty} e^{ity}\bar{g}(d+iy)\,dy.$$

This integral can be approximated by

$$g(t) = \frac{e^{dt}}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikt\Delta y} \bar{g}(d + ik\Delta y)\Delta y.$$

Taking $\Delta y = \dfrac{\pi}{t_1}$, we obtain

$$g(t) = \frac{e^{dt}}{t_1} \left[\frac{1}{2}\bar{g}(d) + Re\left(\sum_{k=1}^{\infty} e^{ik\pi t/t_1} \bar{g}(d + ik\pi/t_1) \right) \right].$$

For numerical purpose this is approximated by the function

$$g_N(t) = \frac{e^{dt}}{t_1} \left[\frac{1}{2}\bar{g}(d) + Re\left(\sum_{k=1}^{N} e^{ik\pi t/t_1} \bar{g}(d + ik\pi/t_1) \right) \right], \tag{38}$$

where N is a sufficiently large integer chosen in such a way that

$$\frac{e^{dt}}{t_1} Re[e^{iN\pi t/t_1} \bar{g}(d + iN\pi/t_1)] < \eta$$

where η is a preselected small positive number that corresponds to the degree of accuracy to be achieved. Formula (A1) is the numerical inversion formula valid for $0 \le t \le 2t_1$ (Honig and Hirdes 1984). In particular, we choose $t = t_1$, getting

$$g_N(t) = \frac{e^{dt}}{t} \left[\frac{1}{2}\bar{g}(d) + Re\left(\sum_{k=1}^{N} (-1)^k \bar{g}(d + ik\pi/t) \right) \right]. \tag{39}$$

5. Numerical Result and Discussion

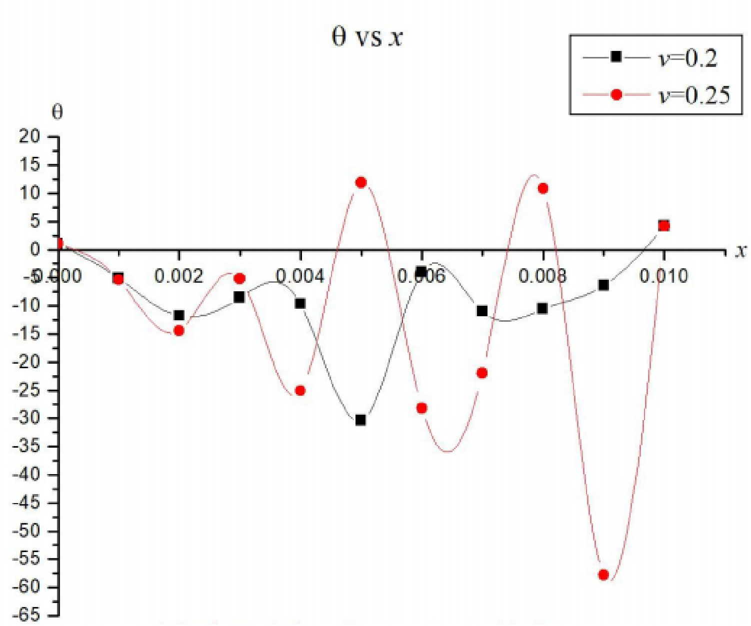

Fig. 1. Variation of temperature with distance

In order to study the variation of temperature within blood vessel in the context of bio heat model under the action of external moving heat source on skin tissue, we now present the obtained results in the form of their graphical representations. To get the solution for temperature and stress in the space-time domain, we have to apply the Laplace inversion formula to Eqs. (36)-(37) respectively. This has been done numerically using a method based on the Fourier series expansion technique. The numerical code has been prepared using programming language. We consider a tissue having normal temperature 37 $^{\circ}C$. The computation is performed at $t = 50s$. For the numerical computations, exemplary values of different factors for human skin have been considered.

$$\epsilon = 0.0168,\ \omega_b = 0.00186\ s^{-1},\ \rho_b = 1060\ \ Kg/m^3,\ p_1 = 8.829$$

$$Q_0 = 2000\ W.m^{-2},\ Q_{met} = 1190\ W.m^{-2},\ \tau_0 = 0.01,\ \theta_0 = 1.0,\ \theta_L = 3.9$$

Fig. 1 is plotted to show the space variation of the temperature θ for two different velocities of applied heat source ($v = 0.2m/s$ and $v = 0.25m/s$). For $v = 0.2m/s$, it is observed that temperature shows negative behavior everywhere except boundary because of the cooling effect within blood vessel. Human body is divided into two components the 'core' and the 'shell'. Core temperature T_c represents internal or deep body temperature. The temperature of the shell is represented by mean skin temperature T_{sk}. The average temperature of the body T_b at any time is a weighted balance between these temperatures, i.e. $T_b = kT_c + (1-k)T_{sk}$ where the weighting factor k varies from about 0.67 to 0.90. So if intensity of applied heat source is very high on skin then cooling effect within blood vessel may be observed. The oscillatory nature of temperature is attributed due to the reflection between the two boundaries for $v = 0.25m/s$.

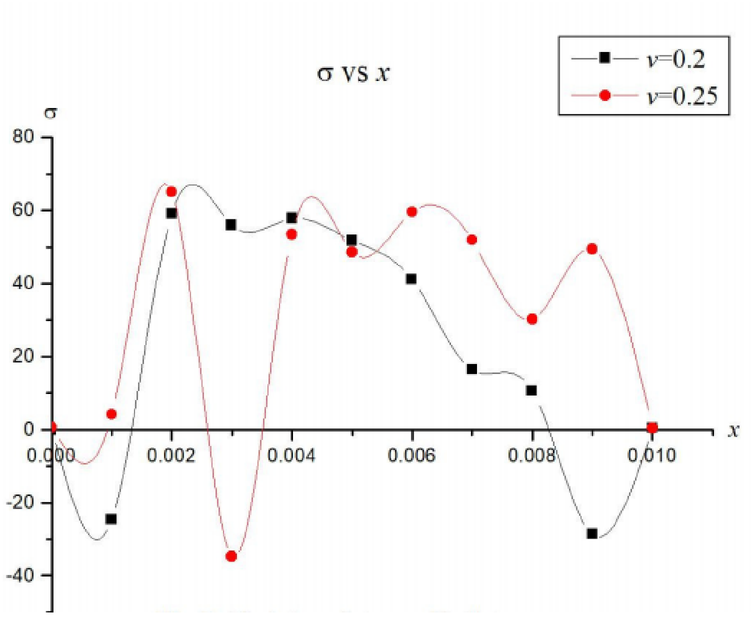

Fig. 2. Variation of stress with distance

Fig. 2 represents the variation of the stress σ against distance x for $v = 0.2m/s$ and $v = 0.25m/s$. From this figure, it can be observed that the stress wave propagate from outer boundary to inner boundary since the heat source is applied on the outer boundary only. It is observed that the stress σ is negative in the regions $0 \le x \le 0.00135$ and $0.0083 \le x \le 0.01$, positive in the region $0.00135 \le x \le 0.0083$ and vanishes on both the boundaries, which agrees with the imposed boundary condition for $v = 0.2m/s$ whereas stress remains positive everywhere except in the region $0.0026 \le x \le 0.0034$ for $v = 0.25m/s$.

Acknowledgements

We express our sincere thanks to the editor for his valuable suggestions in the improvement of the paper.

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