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Asymptotic analysis of linearly elastic shells with variable thickness: error estimates in the membrane case

Research Article

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Abstract: We consider a family of linearly elastic membrane shells with variable thickness and derive an error estimate between the solutions of the three dimensional and two dimensional membrane shell equations.

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1. Introduction

Lower dimenionl models of elastic bodies like plates, shells, rods are often preferred over three dimensional models because of their amenability to numerical computations and their simple mathematical strucutre. The asymptotic analysis of the linearly elastic membrane shells with uniform thickness has been done by Ciarlet and Lods [8] and Busse [1] extended it to variable thickness case. The error estimates between the solution of the three dimensional and two dimensional model of a membrane shell with uniform thickness has been derived by Mardare [2].

In this work we analyse the asymptotic behaviour of the scaled three dimensional displacement field of a linearly elastic shell with variable thickness for which the middle surface is elliptic and clamped along its whole lateral face. Our objective here is to complete the asymptotic analysis made by Busse [1] by establishing error estimates between the solutions of the scaled three dimensional shell equations and the solution of the two dimensional membrane shell problem. Latin indices take their values in the set {1,2,3}, Greek indices take their values in the set { 1,2}, and the summation convention is used. For each integer $m \geq 1$, $H^m(\Omega)$ and $\boldsymbol{H}^m(\Omega)$ denote the usual sobolev spaces; bold face letters denote vector valued functions and their associated function spaces.

2. The three dimensional problem

Let ω be an open, bounded and connected subset of $\mathbb{R}^2$, with a Lipschitz continuous boundary γ. Let $y = (y_\alpha)$ denote a generic point in the set $\overline{\omega}$, and let $\partial_\alpha = \frac{\partial}{\partial y_\alpha}$. Let $\boldsymbol{\phi} : \overline{\omega} \to \mathbb{R}^3$ be an injective mapping of class C^3 such that the two vectors $\boldsymbol{a}_\alpha(y) = \partial_\alpha \boldsymbol{\phi}(y)$ are linearly independent at all points $y \in \overline{\omega}$. They form the covariant basis of the tangent plane to the surface $S = \boldsymbol{\phi}(\overline{\omega})$ at the point $\boldsymbol{\phi}(y)$. The vectors $\boldsymbol{a}^\alpha(y)$ defined by the relations

$$\boldsymbol{a}^\alpha(y).\boldsymbol{a}_\beta(y) = \delta_{\beta,\alpha}$$

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constitute the contravarient basis of the same tangent plane. We define

$$\boldsymbol{a}_3(y) = \boldsymbol{a}^3(y) = \frac{\boldsymbol{a}_1(y) \times \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \times \boldsymbol{a}_2(y)|}.$$

The *first fundamental form* also known as the *metric tensor*,$(a_{\alpha\beta})$ or $(a^{\alpha\beta})$ (in covariant or contravariant components), the *second fundamental form*, also known as the *curvature tensor*,$(b_{\alpha\beta})$ or (b_α^β) (in covariant or mixed components) and the *Christoffel symbols* $\Gamma^\sigma_{\alpha\beta}$ of the surface S are defined as

$$a_{\alpha\beta}(y) = \boldsymbol{a}_\alpha(y) \cdot \boldsymbol{a}_\beta(y), \quad a^{\alpha\beta}(y) = \boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}^\beta(y), \tag{1}$$

$$b_{\alpha\beta}(y) = -\boldsymbol{a}_\alpha(y) \cdot \partial_\beta \boldsymbol{a}_3(y), \quad b_\alpha^\beta(y) = a^{\beta\sigma}(y) b_{\sigma\alpha}(y), \tag{2}$$

$$\Gamma^\sigma_{\alpha\beta}(y) = \boldsymbol{a}^\sigma(y) \cdot \partial_\alpha \boldsymbol{a}_\beta(y). \tag{3}$$

Whenever no confusion should arise, we henceforth drop the explicit dependence on the variable $y \in \overline{\omega}$. Note the symmetries

$$a_{\alpha\beta} = a_{\beta\alpha}, a^{\alpha\beta} = a^{\beta\alpha}, b_{\alpha\beta} = b_{\beta\alpha}, \Gamma^\sigma_{\alpha\beta} = \Gamma^\sigma_{\beta\alpha}.$$

The area element along S is $\sqrt{a}\, dy$, where

$$a = det(a_{\alpha\beta}). \tag{4}$$

All the functions defined in eq. (1)-eq. (4) are continuous over the set $\overline{\omega}$. In particular there exists a constant a_0 such that

$$0 < a_0 < a(y) \text{ for all } y \in \overline{\omega}. \tag{5}$$

For each $\epsilon > 0$, we define the sets

$$\Omega^\epsilon = \omega \times (-\epsilon, \epsilon), \quad \Gamma_0^\epsilon = \gamma \times (-\epsilon, \epsilon) \tag{6}$$

where γ_0 is the boundary of ω. Let $x^\epsilon = (y, x_3^\epsilon)$ denote a generic point in the set $\overline{\Omega}^\epsilon$, and let $\partial_i(\epsilon) = \frac{\partial}{\partial x_i^\epsilon}$; hence $\partial_\alpha^\epsilon = \partial_\alpha$ as $x_\alpha^\epsilon = y_\alpha$.

Define the mapping $\Phi : \overline{\Omega}^\epsilon \to \mathbb{R}^3$ by letting

$$\Phi(x^\epsilon) = \phi(y) + x_3^\epsilon e(y) a^3(y) \text{ for all } x^\epsilon = (y, x_3^\epsilon) \in \overline{\Omega}^\epsilon, \tag{7}$$

where $e : \overline{\omega} \to \mathbb{R}$ represents the thickness $2\epsilon e$ of the shell and satisfies for a constant e_0

$$e \in W^{2,\infty}(\omega) \text{ and } 0 < e_0 < e(y) \text{ for all } y \in \overline{\omega}. \tag{8}$$

Then there exists $\epsilon_0 > 0$ such that the three vectors

$$g_i^\epsilon(x^\epsilon) = \partial_i^\epsilon \Phi(x^\epsilon)$$

are linearly independent(cf. Ciarlet and Paumier[4]) at all points $x \in \overline{\Omega}^\epsilon$ and the mapping $\Phi : \overline{\Omega}^\epsilon \to \mathbb{R}^3$ is injective for all $0 < \epsilon < \epsilon_0$. Without loss of generality we can assume that $\epsilon_0 < 1$. The three vectors $g_i^\epsilon(\epsilon)$ form the *covariant basis* of the tangent space to the manifold $\Phi\left(\overline{\Omega}^\epsilon\right)$ at the point $\Phi(x^\epsilon)$. The three vectors defined by

$$g^{j,\epsilon}(x^\epsilon) \cdot g_i^\epsilon(x^\epsilon) = \delta_{i,j}$$

form the *contravariant basis*. We define the *metric tensor* (g_{ij}^ϵ) or $g^{ij,\epsilon}$ (in covariant and contravariant components) and the *Christoffel symbols* of the manifold $\Phi\left(\overline{\Omega}^\epsilon\right)$ as

$$g_{ij}^\epsilon = g_i^\epsilon \cdot g_j^\epsilon \quad g^{ij,\epsilon} = g^{i,\epsilon} \cdot g^{j,\epsilon}, \tag{9}$$

$$\Gamma_{ij}^{p,\epsilon} = g^{p,\epsilon} \cdot \partial_i^\epsilon g_j^\epsilon. \tag{10}$$

Note the symmetries,

$$g_{ij}^\epsilon = g_{ji}^\epsilon, \quad g^{ij,\epsilon} = g^{ji,\epsilon}, \quad \Gamma_{ij}^{p,\epsilon} = \Gamma_{ji}^{p,\epsilon}. \tag{11}$$

The volume element in the set $\Phi(\Omega^\epsilon)$ is $\sqrt{g^\epsilon}dx^\epsilon$ where $g^\epsilon = det(g^\epsilon_{ij})$. For each $0 < \epsilon \le \epsilon_0$, the set $\Phi(\overline{\Omega}^\epsilon)$ is the reference configuration of an elastic shell, with middle surface $S = \phi(\overline{\omega})$ and variable thickness $2\epsilon e$. We assume that the material of the shell is homogeneous and isotropic. The material is charecterized by its two *Lame constants* $\lambda^\epsilon > 0$ and $\mu^\epsilon > 0$. The unknown of the problem is the vector field $\mathbf{u}^\epsilon = (u^\epsilon_i) : \overline{\Omega}^\epsilon \to \mathbb{R}^3$. The three functions $u^\epsilon_i : \overline{\Omega}^\epsilon \to \mathbb{R}$ are the covariant components of the displacement field $u^\epsilon_i g^{i,\epsilon}$ of the points of the shell. This means that $u^\epsilon_i(x^\epsilon)g^{i,\epsilon}(x^\epsilon)$ is the displacement of the point $\Phi(x^\epsilon)$. Then the variational formulation of the corresponding three dimensional problem of linearized elasticity is as follows;

The unknown $\mathbf{u}^\epsilon = (u^\epsilon_i)$ satisfies

$$\mathbf{u}^\epsilon \in \mathbf{V}(\Omega^\epsilon) = \{\mathbf{v}^\epsilon = (v^\epsilon_i) \in \mathbf{H}^1(\Omega^\epsilon); \mathbf{v}^\epsilon = 0 \text{ on } \Gamma^\epsilon_0\}. \tag{12}$$

$$\int_{\Omega^\epsilon} A^{ijkl,\epsilon} e_{k\|l}\left(\epsilon, \mathbf{u}^\epsilon\right) e_{i\|j}\left(\epsilon, \mathbf{v}^\epsilon\right) \sqrt{g^\epsilon}\, dx^\epsilon = \int_{\Omega^\epsilon} f^{i,\epsilon} v^\epsilon_i \sqrt{g^\epsilon}\, dx^\epsilon \tag{13}$$

for all $\mathbf{v}^\epsilon \in \mathbf{V}(\Omega^\epsilon)$, where $A^{ijkl,\epsilon} = \lambda^\epsilon g^{ij,\epsilon} g^{kl,\epsilon} + \mu^\epsilon (g^{ik,\epsilon} g^{jl,\epsilon} + g^{il,\epsilon} g^{jk,\epsilon})$ and $f^{i,\epsilon} \in L^2(\Omega^\epsilon)$. Note the symmetries $A^{ijkl,\epsilon} = A^{jikl,\epsilon} = A^{klij,\epsilon}$.

3. The scaled three dimensional problem

Let $\Omega = \omega \times (-1,1), \Gamma_+ = \omega \times \{1\}, \Gamma_- = \omega \times \{-1\}, \Gamma_0 = \gamma_0 \times (-1,1)$ let $x = (x_i)$ denote a generic point in the set $\overline{\Omega}$ and let $\partial_i = \frac{\partial}{\partial x_i}$. With $x^\epsilon = (x^\epsilon_i) \in \overline{\Omega}^\epsilon$ we associate the point $x = (x_i) \in \overline{\Omega}$ defined by $x_\alpha = x^\epsilon_\alpha$ and $x_3 = \left(\frac{1}{\epsilon}\right) x^\epsilon_3$. Thus we have $\partial^\epsilon_\alpha = \partial_\alpha$ and $\partial^\epsilon_3 = \left(\frac{1}{\epsilon}\right)\partial_3$. With the unknown $\mathbf{u}^\epsilon = (u^\epsilon_i) : \overline{\Omega}^\epsilon \to \mathbb{R}^3$ and the vector fields $\mathbf{v}^\epsilon = (v^\epsilon_i) : \overline{\Omega}^\epsilon \to \mathbb{R}^3$ appearing in the three dimensional problem eq. (13) we associate the scaled unknown $\mathbf{u}(\epsilon) = (u_i(\epsilon)) : \overline{\Omega} \to \mathbb{R}^3$ and scaled vector fields $v = (v_i)$ defined by

$$u_i(\epsilon)(x) = u^\epsilon_i(x^\epsilon) \text{ and } v_i(x) = v^\epsilon_i(x^\epsilon) \text{ for all } x^\epsilon \in \overline{\Omega}^\epsilon. \tag{14}$$

We assume that there exist constants $\lambda > 0$ and $\mu > 0$ independent of ϵ, and there exis $f^i \in L^2(\Omega)$ independent of ϵ such that

$$\lambda^\epsilon = \lambda, \quad \mu^\epsilon = \mu \tag{15}$$

$$, f^{i,\epsilon}(x^\epsilon) = \epsilon^i f^i(x) \text{ for all } x \in \Omega. \tag{16}$$

Also we define the functions $g^{ij}(\epsilon), \Gamma^p_{ij}(\epsilon), g(\epsilon), A^{ijkl}(\epsilon) : \overline{\Omega}^\epsilon \to \mathbb{R}$ by

$$g^{ij}(\epsilon)(x) = g^{ij,\epsilon}(x^\epsilon) \text{ for all } x^\epsilon \in \overline{\Omega}^\epsilon, \tag{17}$$

$$\Gamma^p_{ij}(\epsilon)(x) = \Gamma^{p,\epsilon}_{ij}(x^\epsilon) \text{ for all } x^\epsilon \in \overline{\Omega}^\epsilon, \tag{18}$$

$$g(\epsilon)(x) = g^\epsilon(x^\epsilon) \text{ for all } x^\epsilon \in \overline{\Omega}^\epsilon, \tag{19}$$

$$A^{ijkl}(\epsilon)(x) = A^{ijkl,\epsilon}(x^\epsilon) \text{ for all } x^\epsilon \in \overline{\Omega}^\epsilon \tag{20}$$

Note the symmetries $g^{ij}(\epsilon) = g^{ji}(\epsilon)$, $\Gamma^p_{ij}(\epsilon) = \Gamma^p_{ji}(\epsilon)$, $A^{ijkl}(\epsilon) = A^{jikl}(\epsilon) = A^{klij}(\epsilon)$. We also have the following properties:

$$\Gamma^3_{\alpha 3}(\epsilon) = \Gamma^p_{33}(\epsilon) = 0, \quad A^{\alpha\beta\sigma 3}(\epsilon) = A^{\alpha 333}(\epsilon) = 0. \tag{21}$$

With any vector field $\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega)$, we associate the symmetric tensor $(e_{i\|j}(\epsilon, \mathbf{v})) \in \mathbf{L}^2(\Omega)$ defined by

$$\begin{aligned}
e_{\alpha\|\beta}(\epsilon, \mathbf{v}) &= \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^p_{\alpha\beta}(\epsilon) v_p \\
e_{\alpha\|3}(\epsilon, \mathbf{v}) &= \frac{1}{2}(\partial_\alpha v_3 + \frac{1}{\epsilon}\partial_3 v_\alpha) - \Gamma^\sigma_{\alpha 3}(\epsilon) v_\sigma \\
e_{3\|3}(\epsilon, \mathbf{v}) &= \frac{1}{\epsilon}\partial_3 v_3.
\end{aligned} \tag{22}$$

It is then easy to verify that the scaled unknown $\mathbf{u}(\epsilon)$ solves the scaled three dimensional shell problem, now posed over the fixed domain Ω, defined by

$$\mathbf{u}(\epsilon) \in \mathbf{V}(\Omega) = \{\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega); \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\} \tag{23}$$

$$\int_\Omega A^{ijkl}(\epsilon) e_{k\|l}(\epsilon, \mathbf{u}(\epsilon))\, e_{i\|j}(\epsilon, \mathbf{v}) \sqrt{g}\, dx = \int_\Omega f^i v_i \sqrt{g}\, dx \tag{24}$$

for all $\mathbf{v} \in \mathbf{V}(\Omega)$, where $A^{ijkl}(\epsilon) = \lambda g^{ij}(\epsilon) g^{kl}(\epsilon) + \mu\left(g^{ik}(\epsilon) g^{jl}(\epsilon) + g^{il}(\epsilon) g^{jk}(\epsilon)\right)$.

4. Error estimates

We shall complete the asymptotic analysis made by Stephane Busse in[1] by establishing the error estimates in the case of elliptic shells. A Surface $S=\boldsymbol{\phi}(\overline{\omega})$ with $\boldsymbol{\phi}\in C^2(\overline{\omega};\mathbb{R}^3)$ is elliptic if there exists a constant $b>0$ such that

$$\left|b_{\alpha\beta}(y)\xi^\alpha\xi^\beta\right|\geq b\xi^\alpha\xi^\alpha \tag{25}$$

for all $y\in\overline{\omega}$ and $\boldsymbol{\xi}=(\xi^\alpha)\in\mathbb{R}^3$; equivalently, the two principal radii of curvature are either >0 at all points of S or <0 at all points of S, and their moduli lie in a compact interval of $(0,\infty)$.

If the surface S is elliptic, then

$$\|\mathbf{v}\|_{H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)}\leq C\left(\sum_{i,j}\left\|e_{i\|j}(\epsilon,\mathbf{v})\right\|^2_{L^2(\Omega)}\right)^{1/2} \tag{26}$$

for all $\mathbf{v}\in V(\Omega)$. The two dimensional membrane shell problem found by asymptotic analysis from the three dimensional shell problem is then defined by

$$\boldsymbol{\zeta}\in\mathbf{V}_M(\omega)=H^1_0(\omega)\times H^1_0(\omega)\times L^2(\omega), \tag{27}$$

$$\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})e\sqrt{a}\,dy=\int_\omega p^i\eta_i e\sqrt{a}\,dy \text{ for all } \boldsymbol{\eta}\in\mathbf{V}(\omega), \tag{28}$$

where

$$a^{\alpha\beta\sigma\tau}=\frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau}+2\mu(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}), \tag{29}$$

$$\gamma_{\alpha\beta}(\boldsymbol{\eta})=\frac{1}{2}(\partial_\alpha\eta_\beta+\partial_\beta\eta_\alpha)-\Gamma^\sigma_{\alpha\beta}\eta_\sigma-\frac{1}{e}b_{\alpha\beta}\eta_3 \text{ for all } \boldsymbol{\eta}\in\mathbf{V}(\omega), \tag{30}$$

$$p^i=\int_{-1}^1 f^i(y,x_3)dx_3 \tag{31}$$

This problem is well posed because for an elliptic shell we have the inequality,

$$\left(\sum_{\alpha\beta}\left\|\gamma_{\alpha\beta}(\boldsymbol{\eta})\right\|^2_{L^2(\Omega)}\right)^{1/2}\geq C\left\|\boldsymbol{\eta}\right\|_{H^1(\omega)\times H^1(\omega)\times L^2(\omega)} \tag{32}$$

for all $\boldsymbol{\eta}\in H^1_0\times H^1_0\times L^2(\omega)$ (Ciarlet and Lods [3]). Now let us prove some preliminary lemmas. In the following whenever a symbol C appears in an inequality, it means that there exists a positive constant, denoted by this symbol that depends only on ω and $\boldsymbol{\theta}$.

Lemma 4.1.
There exists $C_1,C_2,C_3>0$ independent of ϵ such that for all $0<\epsilon<\epsilon_0$ we have

$$\begin{aligned}&\left\|\Gamma^\sigma_{\alpha\beta}(\epsilon)-\Gamma^\sigma_{\alpha\beta}\right\|_{0,\infty,\overline{\Omega}}+\left\|\Gamma^3_{\alpha\beta}(\epsilon)-\frac{1}{e}b_{\alpha\beta}\right\|_{0,\infty,\overline{\Omega}}\\&+\left\|\Gamma^\sigma_{\alpha3}(\epsilon)+eb^\sigma_\alpha\right\|_{0,\infty,\overline{\Omega}}+\left\|\Gamma^3_{\alpha3}(\epsilon)-\frac{1}{e}\partial_\alpha e\right\|_{0,\infty,\overline{\Omega}}\leq C_1\epsilon,\end{aligned} \tag{33}$$

$$\Gamma^p_{33}(\epsilon)=0, \tag{34}$$

$$\left\|g(\epsilon)-e^2a\right\|_{0,\infty,\overline{\Omega}}\leq C_2\epsilon, \tag{35}$$

$$\left\|A^{ijkl}(\epsilon)-A^{ijkl}(0)\right\|\leq C_3\epsilon, \tag{36}$$

with

$$A^{\alpha\beta\sigma\tau}(0)=\lambda a^{\alpha\beta}a^{\sigma\tau}+\mu(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}) \tag{37}$$

$$A^{\alpha\beta\sigma3}(0)=0,\ A^{\alpha\beta33}(0)=\frac{1}{e^2}\lambda a^{\alpha\beta} \tag{38}$$

$$A^{\alpha3\sigma3}(0)=\frac{1}{e^2}\mu a^{\alpha\beta},\ A^{\alpha333}(0)=0,\ A^{3333}(0)=\frac{1}{e^4}(\lambda+2\mu) \tag{39}$$

Moreover, there exists a constant $C_4>0$ such that for all $0<\epsilon<\epsilon_0$, for all $x\in\overline{\Omega}$ and any symmetric tensor (t_{ij}) we have the inequality

$$t_{ij}t_{ij}<C_4A^{ijkl}(\epsilon)(x)t_{kl}t_{ij}. \tag{40}$$

Proof. See the proof of Lemma 4.1 of [1] □

Lemma 4.2.
Let ω be a bounded open subset of $\mathbb{R}^2$, $e \in W^{2,\infty}(\omega)$, $\boldsymbol{\theta} \in \boldsymbol{C}^3(\overline{\omega})$ and $\Omega = \omega \times (-1,1)$. For each $\boldsymbol{\xi} = (\xi_i) \in H^2(\omega) \times H^2(\omega) \times H^2(\omega)$ we define $\mathbf{u}^0 = (u_i^0)$ and $\mathbf{u}^1 = (u_i^1)$ in $\boldsymbol{H}^1(\Omega)$ by letting

$$\mathbf{u}^0(x_1,x_2,x_3) = \boldsymbol{\xi}(x_1,x_2) \tag{41}$$

and

$$u_\alpha^1(x_1,x_2,x_3) = -x_3(\partial_\alpha \xi_3 + 2b_\alpha^\sigma \xi_\sigma), \quad u_3^1(x_1,x_2,x_3) = -x_3 \frac{\lambda}{\lambda+2\mu} a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\xi}) \tag{42}$$

Then for all $\mathbf{v} \in \mathbf{V}(\Omega)$,

$$\begin{aligned} &\left| \int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\epsilon, \mathbf{u}^0 + \epsilon \mathbf{u}^1) e_{i||j}(\epsilon, \mathbf{v}) \sqrt{g(\epsilon)}\, dx - \int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\alpha\beta}(\boldsymbol{\xi}) \gamma_{\alpha\beta}(\overline{\mathbf{v}}) e\sqrt{a}\, dw \right| \\ &\qquad \le C\epsilon ||\boldsymbol{\xi}||_{\mathbf{H}^2(\omega)} \left\{ ||\mathbf{v}||_{\mathbf{L}^2(\Omega)} + \left(\sum_{i,j} ||e_{i||j}(\epsilon,\mathbf{v})||^2_{L^2(\Omega)} \right)^{1/2} \right\} \end{aligned} \tag{43}$$

where $\overline{\mathbf{v}} = \frac{1}{2}\int_{-1}^1 \mathbf{v}(x_1,x_2,x_3)dx_3$.

Proof. Using the definition of $e_{i||j}(\epsilon,\mathbf{v})$ we find that

$$e_{i||j}\left(\epsilon, \mathbf{u}^0 + \epsilon \mathbf{u}^1\right) = e^0_{i||j}(\boldsymbol{\xi}) + \epsilon e^1_{i||j}(\epsilon,\boldsymbol{\xi}), \tag{44}$$

where

$$e^0_{\alpha||\beta}(\boldsymbol{\xi}) = \gamma_{\alpha\beta}(\boldsymbol{\xi}), \quad e^0_{\alpha||3}(\boldsymbol{\xi}) = \mathbf{0}, \quad e^0_{3||3}(\boldsymbol{\xi}) = \partial_3 u_3^1 = -\frac{\lambda}{\lambda+2\mu} a^{\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi}). \tag{45}$$

and where the functions $e^1_{i||j}(\epsilon,\boldsymbol{\xi})$ satisfy

$$||e^1_{i||j}(\epsilon,\boldsymbol{\xi})||_{L^2(\Omega)} \le C||\boldsymbol{\xi}||_{\mathbf{H}^2(\omega)}. \tag{46}$$

Then

$$\begin{aligned} \left\| e_{i||j}\left(\epsilon, \mathbf{u}^0 + \epsilon\mathbf{u}^1\right) \right\|_{L^2(\Omega)} &\le \left\| e^0_{i||j}(\boldsymbol{\xi}) \right\|_{L^2(\Omega)} + \epsilon \left\| e^1_{i||j}(\boldsymbol{\xi}) \right\|_{L^2(\Omega)} \\ &\le C \left\{ \sum_{\alpha,\beta} \left\| \gamma_{\alpha\beta}(\boldsymbol{\xi}) \right\|_{L^2(\omega)} + \epsilon \|\boldsymbol{\xi}\|_{\mathbf{H}^2(\omega)} \right\} \\ &\le C\|\boldsymbol{\xi}\|_{\mathbf{H}^2(\omega)} \end{aligned} \tag{47}$$

Using eq. (35) and eq. (36) we find that

$$\begin{aligned} &\left| \int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\epsilon, \mathbf{u}^0 + \epsilon\mathbf{u}^1) e_{i||j}(\epsilon,\mathbf{v}) \sqrt{g(\epsilon)}\, dx - \int_\Omega A^{ijkl}(0) e_{k||l}(\epsilon, \mathbf{u}^0 + \epsilon\mathbf{u}^1) e_{i||j}(\epsilon,\mathbf{v})\, e\sqrt{a}\, dx \right| \\ &\quad = \left| \int_\Omega \left[A^{ijkl}(\epsilon)\sqrt{g(\epsilon)} - A^{ijkl}(0) e\sqrt{a} \right] e_{k||l}(\epsilon, \mathbf{u}^0 + \epsilon\mathbf{u}^1) e_{i||j}(\epsilon,\mathbf{v})\, dx \right| \\ &\quad \le C\epsilon \left(\sum_{k,l} \left\| e_{k||l}(\epsilon, \mathbf{u}^0 + \epsilon\mathbf{u}^1) \right\|^2_{L^2(\Omega)} \right)^{1/2} \left(\sum_{i,j} \left\| e_{i||j}(\epsilon,\mathbf{v}) \right\|^2_{L^2(\Omega)} \right)^{1/2} \\ &\quad \le C\epsilon \|\boldsymbol{\xi}\|_{\mathbf{H}^2(\omega)} \left(\sum_{i,j} \left\| e_{i||j}(\epsilon,\mathbf{v}) \right\|^2_{L^2(\Omega)} \right)^{1/2}. \end{aligned} \tag{48}$$

From eq. (44) we get,

$$\begin{aligned} &\int_\Omega A^{ijkl}(0) e_{k||l}\left(\epsilon, \mathbf{u}^0 + \epsilon\mathbf{u}^1\right) e_{i||j}(\epsilon,\mathbf{v})\, e\sqrt{a}\, dx \\ &\quad = \int_\Omega A^{ijkl}(0) e^0_{k||l}(\boldsymbol{\xi}) e_{i||j}(\epsilon,\mathbf{v})\, e\sqrt{a}\, dx + \epsilon \int_\Omega A^{ijkl}(0) e^1_{k||l}(\epsilon,\boldsymbol{\xi}) e_{i||j}(\epsilon,\mathbf{v})\, e\sqrt{a}\, dx. \end{aligned} \tag{49}$$

we have the identity

$$A^{ijkl}(0)e^0_{k\|l}(\boldsymbol{\xi})t_{ij} = \frac{1}{2}a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})t_{\alpha\beta}, \tag{50}$$

valid for all symmetric tensors t_{ij}. Thus we can write eq. (49) as.

$$\begin{aligned}\int_\Omega A^{ijkl}(0)e_{k\|l}\left(\epsilon,\mathbf{u}^0+\epsilon\mathbf{u}^1\right)e_{i\|j}(\epsilon,\mathbf{v})\,e\sqrt{a}\,dx\\ = \frac{1}{2}\int_\Omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})e_{\alpha\|\beta}(\epsilon,\mathbf{v})\,e\sqrt{a}\,dx + \epsilon\int_\Omega A^{ijkl}(0)e^1_{k\|l}(\epsilon,\boldsymbol{\xi})e_{i\|j}(\epsilon,\mathbf{v})\,e\sqrt{a}\,dx.\end{aligned} \tag{51}$$

From this last relation and eq. (48), we infer that,

$$\begin{aligned}&\left|\int_\Omega A^{ijkl}(\epsilon)e_{k\|l}(\epsilon,\mathbf{u}^0+\epsilon\mathbf{u}^1)e_{i\|j}(\epsilon,\mathbf{v})\sqrt{g(\epsilon)}\,dx - \frac{1}{2}\int_\Omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})e_{\alpha\|\beta}(\epsilon,\mathbf{v})\,e\sqrt{a}\,dx\right|\\
&\leq \left|\int_\Omega\left[A^{ijkl}(\epsilon)\sqrt{g(\epsilon)} - A^{ijkl}(0)e\sqrt{a}\right]e_{k\|l}(\epsilon,\mathbf{u}^0+\epsilon\mathbf{u}^1)e_{i\|j}(\epsilon,\mathbf{v})\,dx\right|\\
&\quad+\left|\int_\Omega A^{ijkl}(0)e_{k\|l}\left(\epsilon,\mathbf{u}^0+\epsilon\mathbf{u}^1\right)e_{i\|j}(\epsilon,\mathbf{v})\,e\sqrt{a}\,dx - \frac{1}{2}\int_\Omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})e_{\alpha\|\beta}(\epsilon,\mathbf{v})\,e\sqrt{a}\,dx\right|\\
&\leq C\epsilon\|\boldsymbol{\xi}\|_{\mathbf{H}^2(\omega)}\left(\sum_{i,j}\left\|e_{i\|j}(\epsilon,\mathbf{v})\right\|^2_{L^2(\Omega)}\right)^{1/2} + \epsilon\left|\int_\Omega A^{ijkl}(0)e^1_{k\|l}(\epsilon,\boldsymbol{\xi})e_{i\|j}(\epsilon,\mathbf{v})\,e\sqrt{a}\,dx\right|\\
&\leq C\epsilon\|\boldsymbol{\xi}\|_{\mathbf{H}^2(\omega)}\left(\sum_{i,j}\left\|e_{i\|j}(\epsilon,\mathbf{v})\right\|^2_{L^2(\Omega)}\right)^{1/2} + C\epsilon\left(\sum_{k,l}\left\|e^1_{k\|l}(\epsilon,\boldsymbol{\xi})\right\|^2_{L^2(\Omega)}\right)^{1/2}\left(\sum_{i,j}\left\|e_{i\|j}(\epsilon,\mathbf{v})\right\|^2_{L^(\Omega)}\right)^{1/2}\\
&\leq C\epsilon\|\boldsymbol{\xi}\|_{\mathbf{H}^2(\omega)}\left(\sum_{i,j}\left\|e_{i\|j}(\epsilon,\mathbf{v})\right\|^2_{L^2(\Omega)}\right)^{1/2},\end{aligned} \tag{52}$$

where the last inequality follows from eq. (46). Define

$$\gamma_{\alpha\beta}(\mathbf{v}) = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\sigma_{\alpha\beta}v_\sigma - \frac{1}{e}b_{\alpha\beta}v_3, \tag{53}$$

so that

$$e_{\alpha\|\beta}(\epsilon,\mathbf{v}) = \gamma_{\alpha\beta}(\mathbf{v}) - \{\Gamma^\sigma_{\alpha\beta}(\epsilon) - \Gamma^\sigma_{\alpha\beta}\}v_\sigma - \left\{\Gamma^3_{\alpha\beta}(\epsilon) - \frac{1}{e}b_{\alpha\beta}\right\}v_3, \tag{54}$$

and using eq. (33) we get

$$\left\|e_{\alpha\|\beta}(\epsilon,\mathbf{v}) - \gamma_{\alpha\beta}(\mathbf{v})\right\|_{L^2(\Omega)} \leq C\epsilon\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \tag{55}$$

We then have

$$\begin{aligned}\left|\frac{1}{2}\int_\Omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})e_{\alpha\|\beta}(\epsilon,\mathbf{v})\,e\sqrt{a}dx - \frac{1}{2}\int_\Omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\xi})\gamma_{\alpha\beta}(\mathbf{v})e\sqrt{a}dx\right|\\ \leq C\epsilon\left\{\sum_{\alpha,\beta}\left\|\gamma_{\alpha\beta}(\boldsymbol{\xi})\right\|^2_{L^2(\omega)}\right\}^{1/2}\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}.\end{aligned} \tag{56}$$

Using eq. (52) and eq. (56) we obtain

$$\begin{aligned}\left|\int_\Omega A^{ijkl}(\epsilon)e_{k\|l}(\epsilon,\mathbf{u}^0+\epsilon\mathbf{u}^1)e_{i\|j}(\epsilon,\mathbf{v})\sqrt{g(\epsilon)}\,dx - \frac{1}{2}\int_\Omega a^{\alpha\beta\sigma\tau}\gamma_{\alpha\beta}(\boldsymbol{\xi})\gamma_{\alpha\beta}(\mathbf{v})e\sqrt{a}\,dx\right|\\ \leq C\epsilon||\boldsymbol{\xi}||_{\mathbf{H}^2(\omega)}\left\{||\mathbf{v}||_{\mathbf{L}^2(\Omega)} + \left(\sum_{i,j}||e_{i\|j}(\epsilon,\mathbf{v})||^2_{L^2(\Omega)}\right)^{1/2}\right\}\end{aligned} \tag{57}$$

for all $\mathbf{v}\in\mathbf{V}(\Omega)$. Using the identity,

$$\overline{\gamma_{\alpha\beta}(\mathbf{v})} = \overline{\frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \Gamma^\sigma_{\alpha\beta}v_\sigma - \frac{1}{e}b_{\alpha\beta}v_3} = \gamma_{\alpha\beta}(\overline{\mathbf{v}}) \tag{58}$$

we get,

$$\begin{aligned}\left|\int_\Omega A^{ijkl}(\epsilon)e_{k\|l}(\epsilon,\mathbf{u}^0+\epsilon\mathbf{u}^1)e_{i\|j}(\epsilon,\mathbf{v})\sqrt{g(\epsilon)}\,dx - \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\alpha\beta}(\boldsymbol{\xi})\gamma_{\alpha\beta}(\overline{\mathbf{v}})e\sqrt{a}\,dw\right|\\ \leq C\epsilon||\boldsymbol{\xi}||_{\mathbf{H}^2(\omega)}\left\{||\mathbf{v}||_{\mathbf{L}^2(\Omega)} + \left(\sum_{i,j}||e_{i\|j}(\epsilon,\mathbf{v})||^2_{L^2(\Omega)}\right)^{1/2}\right\}.\end{aligned} \tag{59}$$

□

Lemma 4.3.
Let $\Omega = \omega \times (-1,1)$, where $\omega \subset \mathbb{R}^2$ is a bounded open set, and let $\mathbf{f} = (f_i) \in \mathbf{L}^2(\Omega)$ with $\partial_\alpha f^\alpha \in L^2(\Omega)$ and let $e \in W^{2,\infty}(\omega)$. Let

$$p^i = \int_{-1}^{1} f^i(y, x_3)\, dx_3. \tag{60}$$

Then

$$\begin{aligned} &\left| \int_\Omega f^i v_i \sqrt{g(\epsilon)}\, dx - \int_\omega p^i \overline{v_i} e\sqrt{a}\, dy \right| \\ &\le C\epsilon \left(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \left\|\partial_\alpha f^\alpha\right\|_{L^2(\Omega)} \right) \left\{ \|\mathbf{v}\|_{L^2(\Omega)} + \left(\sum_{i,j} \left\| e_{i\|j}(\epsilon, \mathbf{v}) \right\|^2_{L^2(\Omega)} \right)^{1/2} \right\} \end{aligned} \tag{61}$$

for all $\mathbf{v} \in \mathbf{V}(\Omega)$, where

$$\overline{v_i} = \frac{1}{2} \int_{-1}^{1} v_i(x_1, x_2, x_3)\, dx_3. \tag{62}$$

Proof. To prove eq. (61) it suffices to consider only smooth functions $\mathbf{v} \in \mathbf{V}(\Omega) \cap \mathbf{C}^1(\overline{\Omega})$, and then use the density argument. We have,

$$\begin{aligned} &\left| \int_\Omega f_i v_i \sqrt{g(\epsilon)}\, dx - \int_\omega p^i \overline{v_i} e\sqrt{a}\, dy \right| \\ &\le \left| \int_\Omega f^i v_i \left(\sqrt{g(\epsilon)} - e\sqrt{a} \right) dx \right| + \left| \int_\Omega f^i v_i e\sqrt{a}\, dy - \int_\Omega f^i \overline{v_i} e\sqrt{a}\, dy \right| \\ &\le C\epsilon \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} + \left| \int_\Omega f_i (v_i - \overline{v_i}) e\sqrt{a}\, dx \right|, \end{aligned} \tag{63}$$

where the last inequality follows from eq. (35). From eq. (22) we can write

$$\partial_3 v_\alpha = \epsilon \left\{ 2e_{\alpha\|3}(\epsilon, \mathbf{v}) + 2\Gamma^\sigma_{\alpha 3}(\epsilon) v_\sigma - \partial_\alpha v_3 \right\}, \partial_3 v_3 = \epsilon e_{3\|3}(\epsilon, \mathbf{v}) \tag{64}$$

By integrating these relations with respect to x_3, we get

$$v_\alpha(y, x_3) = v_\alpha(y, -1) + \epsilon \int_{-1}^{x_3} \left\{ 2e_{\alpha\|3}(\epsilon, \mathbf{v}) + 2\Gamma^\sigma_{\alpha 3}(\epsilon) v_\sigma - \partial_\alpha v_3 \right\} dy_3, \tag{65}$$

$$v_3(y, x_3) = v_3(y, -1) + \epsilon \int_{-1}^{x_3} e_{3\|3}(\epsilon, \mathbf{v})\, dy_3. \tag{66}$$

Consequently we obtain,

$$\begin{aligned} &v_\alpha(y, x_3) - \overline{v_\alpha(y, x_3)} = \\ &\epsilon \left\{ \int_{-1}^{x_3} 2e_{\alpha\|3}(\epsilon, \mathbf{v}) + 2\Gamma^\sigma_{\alpha 3}(\epsilon) v_\sigma - \partial_\alpha v_3\, dy_3 - \overline{\int_{-1}^{x_3} 2e_{\alpha\|3}(\epsilon, \mathbf{v}) + 2\Gamma^\sigma_{\alpha 3}(\epsilon) v_\sigma - \partial_\alpha v_3\, dy_3.} \right\} \end{aligned} \tag{67}$$

and

$$v_3(y, x_3) - \overline{v_3(y, x_3)} = \epsilon \left\{ \int_{-1}^{x_3} e_{3\|3}(\epsilon, \mathbf{v})\, dy_3 - \overline{\int_{-1}^{x_3} e_{3\|3}(\epsilon, \mathbf{v})\, dy_3} \right\}. \tag{68}$$

This last relation implies that

$$\left\| v_3 - \overline{v_3} \right\|_{L^2(\Omega)} \le C\epsilon \left\| e_{3\|3}(\epsilon, \mathbf{v}) \right\|_{L^2(\Omega)} \tag{69}$$

Hence

$$\left| \int_\Omega f^3 (v_3 - \overline{v_3}) e\sqrt{a})\, dx \right| \le C \left\| f^3 \right\|_{L^2(\Omega)} \left\| v_3 - \overline{v_3} \right\|_{L^2(\Omega)} \le C\epsilon \left\| f^3 \right\|_{L^2(\Omega)} \left\| e_{3\|3}(\epsilon, \mathbf{v}) \right\|_{L^2(\Omega)} \tag{70}$$

Using eq. (67), we obtain

$$\begin{aligned}
&\left|\int_\Omega f^\alpha (v_\alpha - \overline{v_\alpha}) e\sqrt{a}\, dx\right| \\
&\le 2\epsilon \left|\int_\Omega f^\alpha \left(\int_{-1}^{x_3} e_{\alpha||3}(\epsilon,\mathbf{v})\, dy_3 - \overline{\int_{-1}^{x_3} e_{\alpha||3}(\epsilon,\mathbf{v})}\, dy_3\right) e\sqrt{a}\, dx\right| \\
&+ 2\epsilon \left|\int_\Omega f^\alpha \left(\int_{-1}^{x_3} \Gamma^\sigma_{\alpha 3}(\epsilon) v_\sigma\, dy_3 - \overline{\int_{-1}^{x_3} \Gamma^\sigma_{\alpha 3}(\epsilon) v_\sigma}\, dy_3\right) e\sqrt{a}\, dx\right| \\
&+ \epsilon \left|\int_\Omega f^\alpha \left(\int_{-1}^{x_3} \partial_\alpha v_3\, dy_3 - \overline{\int_{-1}^{x_3} \partial_\alpha v_3}\, dy_3\right) e\sqrt{a}\, dx\right| \\
&\le 2C\epsilon \left(\sum_\alpha \|f^\alpha\|_{L^2(\Omega)}\right)\left(\sum_\alpha \|e_{\alpha||3}(\epsilon,\mathbf{v})\|_{L^2(\Omega)}\right) \\
&+ 2C\epsilon \left(\sum_\alpha \|f^\alpha\|_{L^2(\Omega)}\right)\left(\sum_\alpha \|\Gamma^\sigma_{\alpha 3} v_\sigma\|_{L^2(\Omega)}\right) \\
&+ \epsilon \left|\int_\Omega f^\alpha \partial_\alpha \left(\int_{-1}^{x_3} v_3\, dy_3 - \overline{\int_{-1}^{x_3} v_3\, dy_3}\right) e\sqrt{a}\, dx\right|.
\end{aligned} \tag{71}$$

We now use the assumption that $\mathbf{v} \in \mathbf{V}(\Omega)$ which shows that $\int_{-1}^{x_3} v_3\, dy_3 - \overline{\int_{-1}^{x_3} v_3\, dy_3}$ vanishes on Γ_0. Therefore by green's formula we get,

$$\int_\Omega f^\alpha \partial_\alpha \left(\int_{-1}^{x_3} v_3\, dy_3 - \overline{\int_{-1}^{x_3} v_3\, dy3}\right) e\sqrt{a}\, dx = -\int_\Omega \partial_\alpha (f^\alpha e\sqrt{a}) \left(\int_{-1}^{x_3} v_3\, dy_3 - \overline{\int_{-1}^{x_3} v_3\, dy3}\right) dx \tag{72}$$

Then the last two inequalities together imply

$$\begin{aligned}
&\left|\int_\Omega f^\alpha (v_\alpha - \overline{v_\alpha}) e\sqrt{a}\, dx\right| \\
&\le 2C\epsilon \left(\sum_\alpha \|f^\alpha\|_{L^2(\Omega)}\right)\left(\sum_\alpha \|e_{\alpha||3}(\epsilon,\mathbf{v})\|_{L^2(\Omega)} + \sum_\alpha \|v_\alpha\|_{L^2(\Omega)}\right) \\
&+ \epsilon \left| -\int_\Omega \partial_\alpha (f^\alpha e\sqrt{a}) \left(\int_{-1}^{x_3} v_3\, dy_3 - \overline{\int_{-1}^{x_3} v_3\, dy_3}\right) dx\right| \\
&\le 2C\epsilon \left(\sum_\alpha \|f^\alpha\|_{L^2(\Omega)}\right)\left(\sum_\alpha \|e_{\alpha||3}(\epsilon,\mathbf{v})\|_{L^2(\Omega)} + \sum_\alpha \|v_\alpha\|_{L^2(\Omega)}\right) \\
&+ C\epsilon \left(\|\partial_\alpha f^\alpha\|_{L^2(\Omega)} + \sum_\alpha \|f^\alpha\|_{L^2(\Omega)}\right) \left\|\int_{-1}^{x_3} v_3\, dy_3 - \overline{\int_{-1}^{x_3} v_3\, dy_3}\right\|_{L^2(\Omega)}
\end{aligned} \tag{73}$$

and consequently,

$$\begin{aligned}
&\left|\int_\Omega f^\alpha (v_\alpha - \overline{v_\alpha}) e\sqrt{a}\, dx\right| \\
&\le C\epsilon \left(\|\partial_\alpha f^\alpha\|_{L^2(\Omega)} + \sum_\alpha \|f^\alpha\|_{L^2(\Omega)}\right)\left(\|\mathbf{v}\|_{L^2(\Omega)} + \left(\sum_{i,j} \|e_{i||j}(\epsilon,\mathbf{v})\|^2_{L^2(\Omega)}\right)^{1/2}\right)
\end{aligned} \tag{74}$$

Using eq. (70) and eq. (74) in eq. (63), we obtain

$$\begin{aligned}
&\left|\int_\Omega f_i v_i \sqrt{g(\epsilon)}\, dx - \int_\omega p^i \overline{v_i} e\sqrt{a}\, dy\right| \\
&\le C\epsilon \left(\|\partial_\alpha f^\alpha\|_{L^2(\Omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\right)\left(\|\mathbf{v}\|_{L^2(\Omega)} + \left(\sum_{i,j} \|e_{i||j}(\epsilon,\mathbf{v})\|^2_{L^2(\Omega)}\right)^{1/2}\right)
\end{aligned} \tag{75}$$

for all $\mathbf{v} \in \mathbf{V}(\Omega)$. □

Lemma 4.4.
Let ω be an open subset of $\mathbb{R}^2$ with a boundary $\gamma = \partial\omega$ of class C^2, let ξ be an element of $H^2(\omega)$, and let $a > 0$ be fixed. Then for all $\epsilon > 0$, There exists a function $\xi(\epsilon) \in H^2(\omega)$ such that

$$\xi(\epsilon)_{|\gamma} = \xi_{|\gamma} \tag{76}$$

and that verifies the following inequalities:

$$\|\xi(\epsilon)\|_{L^2(\Omega)} \le C\epsilon^{a}\,\|\xi\|_{L^2(\Omega)}, \tag{77}$$

$$\|\xi(\epsilon)\|_{H^1(\omega)} \le C\epsilon^{-a}\,\|\xi\|_{H^1(\omega)}, \tag{78}$$

$$\|\xi(\epsilon)\|_{H^2(\omega)} \le C\epsilon^{-3a}\,\|\xi\|_{H^2(\omega)}, \tag{79}$$

where the constant C depends only on ω.

Proof. See the proof of Lemma 4.4 □

Lemma 4.5.
Let ω be an open subset of $\mathbb{R}^2$ with a boundary $\gamma = \partial\omega$ of class C^1. Let $\Omega = \omega \times (-1,1) \subset \mathbb{R}^3$ and let $\Gamma_0 = \gamma \times (-1,1)$ be its lateral face. Let $\mathbf{G} \in \mathbf{H}^1(\Omega)$ and let $b > 0$ be a fixed real number. Then for all $\epsilon > 0$, there exists a function $\mathbf{w}(\epsilon) \in \mathbf{H}^1(\Omega)$ suchthat

$$\mathbf{W}(\epsilon)_{|\Gamma_0} = \mathbf{G}_{\Gamma_0}, \tag{80}$$

and that satisfies the following inequalities

$$\|\mathbf{W}(\epsilon)\|_{\mathbf{L}^2(\Omega)} \le C\epsilon^{b}\,\|\mathbf{G}\|_{\mathbf{L}^2(\Omega)}, \tag{81}$$

$$\|\partial_3\mathbf{W}(\epsilon)\|_{\mathbf{L}^2(\Omega)} \le C\epsilon^{b}\,\|\partial_3\mathbf{G}\|_{\mathbf{L}^2(\Omega)}, \tag{82}$$

$$\|\partial_\alpha\mathbf{W}(\epsilon)\|_{\mathbf{L}^2(\Omega)} \le C\epsilon^{-b}\,\|\mathbf{G}\|_{\mathbf{H}^1(\Omega)}, \tag{83}$$

where the constant C depends only on ω.

Proof. See the proof of Lemma 4.5 of □

We can now state our main result.

Theorem 4.6.
Let $\omega \subset \mathbb{R}^2$ be a bounded open set with a boundary $\gamma = \partial\omega$ of class C^2, let $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ be a mapping of class $C^3(\overline{\omega})$, and let ζ be the solution of the two dimensional membrane shell problem.(number). Assume that the surface $S = \boldsymbol{\theta}(\overline{\omega})$ is elliptic. Let $e \in W^{2,\infty}(\Omega)$ and let $f^i \in L^2(\Omega)$ with $\partial_\alpha f^\alpha \in L^2(\Omega)$ and that $\boldsymbol{\zeta} \in \mathbf{H}^2(\omega)$. Then the following estimate holds for ϵ small enough:

$$\|\mathbf{u}(\epsilon) - \boldsymbol{\zeta}\|_{H^1\Omega\times H^1\Omega\times L^2(\Omega)} \le C\epsilon^{1/6} \tag{84}$$

where the constant C depends on the data $(\omega, \boldsymbol{\theta}, f^i, \boldsymbol{\zeta})$ where $\mathbf{u}(\epsilon)$ is the solution of the three dimensional shell problem,(number) and where $\boldsymbol{\zeta}$ is identified in eq. (84) with an element in $H^1\Omega \times H^1\Omega \times L^2(\Omega)$ by letting $\boldsymbol{\zeta}(y, x_3) = \boldsymbol{\zeta}(y)$ for all $(y, x_3) \in \Omega$.

Proof.

$$\|\mathbf{e}(\epsilon,\mathbf{v})\|_{\mathbf{L}^2(\Omega)} = \left(\sum_{i,j} \|e_{i\|j}(\epsilon,\mathbf{v})\|^2_{L^2(\Omega)}\right)^{(1/2)}, \tag{85}$$

$$B(\epsilon)(\mathbf{u},\mathbf{v}) = \int_\Omega A^{ijkl}(\epsilon)\, e_{k\|l}(\epsilon,\mathbf{u})\, e_{i\|j}(\epsilon,\mathbf{v})\sqrt{g(\epsilon)}\, dx \tag{86}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$. In the case of elliptic shells

$$\|\mathbf{e}(\epsilon,\mathbf{v})\|_{\mathbf{L}^2(\Omega)} \ge C\,\|\mathbf{v}\|_{H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)} \quad \text{for all } \mathbf{v} \in \mathbf{V}(\Omega). \tag{87}$$

Step 1.*Let* $\mathbf{u}^0(x_1,x_2,x_3) = \zeta(x_1,x_2)$, *so that* $\mathbf{u}^0 \in \boldsymbol{H}^2(\Omega)$. *Then there exists* $\mathbf{u}^1 \in \boldsymbol{H}^1(\Omega)$ *such that*

$$\begin{aligned} &\left| B(\epsilon)(\mathbf{u}^0 + \epsilon\mathbf{u}^1, \mathbf{v}) - \int_\Omega f^i v_i \sqrt{g(\epsilon)}\, dx \right| \\ &\quad \le C\epsilon\left(\|\boldsymbol{\zeta}\|_{\boldsymbol{H}^2(\Omega)} + \|\mathbf{f}\|_{\boldsymbol{L}^2(\Omega)} + \left\|\partial_\alpha f^\alpha\right\|_{L^2(\Omega)}\right)\left(\|\boldsymbol{e}(\epsilon,\mathbf{v})\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{v}\|_{\boldsymbol{L}^2(\Omega)}\right) \end{aligned} \tag{88}$$

for all $\mathbf{v} \in \boldsymbol{V}(\Omega)$.
To prove this we define $\mathbf{u}^1 = (u_i^1) \in \mathbf{H}^1(\Omega)$ as

$$u_\alpha^1(x_1,x_2,x_3) = -x_3(\partial_\alpha\zeta_3 + 2b_\alpha^\sigma\zeta_\sigma),$$
$$u_3^1(x_1,x_2,x_3) = -x_3\frac{\lambda}{\lambda+2\mu}a^{\alpha\beta}\gamma_{\alpha\beta}(\zeta) \tag{89}$$

Then from lemma 4.2 we get,

$$\left|B(\epsilon)(\mathbf{u}^0+\epsilon\mathbf{u}^1,\mathbf{v}) - \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\zeta)\gamma_{\alpha\beta}(\overline{\mathbf{v}})e\sqrt{a}\,dy\right| \leq C\epsilon\,\|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)}\left(\|\mathbf{e}(\epsilon,\mathbf{v})\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}\right) \tag{90}$$

for all $\mathbf{v} \in \mathbf{V}(\Omega)$. On the other hand choosing $\eta = \overline{\mathbf{v}}$ in the two-dimensional limit problem we get,

$$\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\overline{\mathbf{v}})e\sqrt{a}\,dy = \int_\omega p^i\overline{v_i}e\sqrt{a}\,dy. \tag{91}$$

Hence

$$\left|\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\overline{\mathbf{v}})e\sqrt{a}\,dy - \int_\Omega f^i v_i\sqrt{g(\epsilon)}\,dx\right| \tag{92}$$
$$= \left|\int_\omega p^i\overline{v_i}e\sqrt{a}\,dy - \int_\Omega f^i v_i\sqrt{g(\epsilon)}\,dx\right| \tag{93}$$
$$\leq C\epsilon\left(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \left\|\partial_\alpha f^\alpha\right\|_{L^2(\Omega)}\right)\left\{\|\mathbf{v}\|_{L^2(\Omega)} + \|\mathbf{e}(\epsilon,\mathbf{v})\|_{\mathbf{L}^2(\Omega)}\right\}. \tag{94}$$

Where the last equality follows from lemma 4.3. The functions $\mathbf{u}^0$ and $\mathbf{u}^1$ do not belong to the space $\mathbf{V}(\Omega)$ in general. To avoid this difficulty we shall introduce in steps 2 and 3 two correctors, $z(\epsilon)$ and $\mathbf{w}(\epsilon)$, such that $\{\mathbf{u}^0+\epsilon\mathbf{u}^1+\mathbf{z}(\epsilon)+\mathbf{w}(\epsilon)\} \in \mathbf{V}(\Omega)$.
Step 2*. We can define a corrector* $\mathbf{z}(\epsilon) = (z_i(\epsilon)) \in \mathbf{H}^1(\Omega)$ *that satisfies the following conditions*

$$z_3(\epsilon)_{|\Gamma_0} = -\zeta_{3|\Gamma_0}, \tag{95}$$
$$\|\mathbf{z}(\epsilon)\|_{H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)} \leq C\left(\epsilon^a + \epsilon^{1-3a}\right)\|\zeta_3\|_{H^2(\omega)} \tag{96}$$
$$|B(\epsilon)(\mathbf{z}(\epsilon),\mathbf{v})| \leq C\left(\epsilon^a + \epsilon^{1-3a}\right)\|\zeta_3\|_{H^2(\omega)}\|\mathbf{e}(\epsilon,\mathbf{v})\|_{\mathbf{L}^2(\Omega)}. \tag{97}$$

To prove this, we first remark that $\zeta_3 \in H^2(\omega)$. lemma 4.5 insures the existence, for all $\epsilon > 0$ and $a > 0$ of $z_3(\epsilon)$ in $H^2(\omega)$ such that

$$z_3(\epsilon)_{|\gamma} = -\zeta_{3|\gamma},$$

and that verifies the following inequalities:

$$\|z_3(\epsilon)\|_{L^2(\omega)} \leq C\epsilon^a\|\zeta_3\|_{L^2(\omega)}, \tag{98}$$
$$\|z_3(\epsilon)\|_{H^1(\omega)} \leq C\epsilon^{-a}\|\zeta_3\|_{H^1(\omega)}, \tag{99}$$
$$\|z_3(\epsilon)\|_{H^2(\omega)} \leq C\epsilon^{-3a}\|\zeta_3\|_{H^2(\omega)}, \tag{100}$$

We then define the corrector $\mathbf{z}(\epsilon) = (z_i(\epsilon)) \in \mathbf{H}^1(\Omega)$, by letting

$$z_\alpha(\epsilon) = -\epsilon x_3\partial_\alpha z_3(\epsilon). \tag{101}$$

From the above inequalities we get,

$$\|\mathbf{z}(\epsilon)\|_{H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)} \leq C\left(\epsilon^a + \epsilon^{1-3a}\right)\|\zeta_3\|_{H^2(\omega)}. \tag{102}$$

Using eq. (98) , eq. (99), eq. (100) ,eq. (101) we can compute the following estimates.

$$\left\|e_{\alpha\|\beta}(\epsilon,\mathbf{z}(\epsilon))\right\|_{L^2(\Omega)} \leq C\left(\sum_\alpha\|z_\alpha(\epsilon)\|_{H^1(\Omega)} + \|z_3(\epsilon)\|_{L^2(\Omega)}\right) \tag{103}$$
$$\leq C\left(\epsilon\|z_3(\epsilon)\|_{H^2(\Omega)} + \|z_3(\epsilon)\|_{L^2(\Omega)}\right) \tag{104}$$
$$\leq C\left(\epsilon^{1-3a}\|\zeta_3\|_{H^2(\omega)} + \epsilon^a\|\zeta_3\|_{L^2(\omega)}\right) \tag{105}$$
$$\leq C\left(\epsilon^{1-3a} + \epsilon^a\right)\|\zeta_3\|_{H^2(\omega)} \tag{106}$$

$$\begin{aligned}\left\|e_{\alpha||3}(\epsilon,\mathbf{z}(\epsilon))\right\|_{L^2(\Omega)} &= \left\|\frac{1}{2}\left(\partial_\alpha z_3(\epsilon)+\frac{1}{\epsilon}\partial_3 z_\alpha(\epsilon)\right)-\Gamma^{\sigma}_{\alpha 3}(\epsilon)z_\sigma(\epsilon)\right\|_{L^2(\omega)} &(107)\\
&= \left\|-\Gamma^{\sigma}_{\alpha 3}(\epsilon)z_\sigma(\epsilon)\right\|_{L^2(\omega)} &(108)\\
&\leq C\sum_{\sigma}\|z_\sigma(\epsilon)\|_{L^2(\Omega)} &(109)\\
&\leq C\epsilon\|z_3(\epsilon)\|_{H^1(\Omega)} &(110)\\
&\leq C\epsilon^{1-a}\|\zeta_3\|_{H^1(\omega)} &(111)\end{aligned}$$

and $e_{3||3}(\epsilon,\mathbf{z}(\epsilon))=0$. Hence we get

$$\|\mathbf{e}(\epsilon,\mathbf{z}(\epsilon))\|_{\mathbf{L}^2(\Omega)} \leq C\left(\epsilon^a+\epsilon^{1-3a}\right)\|\zeta_3\|_{H^2(\omega)} \tag{112}$$

Thus we get,

$$\begin{aligned}|B(\epsilon)(\mathbf{z}(\epsilon),\mathbf{v})| &\leq C\|\mathbf{e}(\epsilon,\mathbf{z}(\epsilon))\|_{\mathbf{L}^2(\Omega)}\|\mathbf{e}(\epsilon,\mathbf{v})\|_{\mathbf{L}^2(\Omega)} &(113)\\
&\leq C\left(\epsilon^a+\epsilon^{1-3a}\right)\|\zeta_3\|_{H^2(\omega)}\|\mathbf{e}(\epsilon,\mathbf{v})\|_{\mathbf{L}^2(\Omega)} &(114)\end{aligned}$$

Step 3*. We can define another corrector* $\mathbf{w}(\epsilon)=(w_i(\epsilon))\in\mathbf{H}^1(\Omega)$ *that satisfies the following conditions.*

$$\begin{aligned}&\{\mathbf{u^0}+\epsilon\mathbf{u^1}+\mathbf{z}(\epsilon)+\mathbf{w}(\epsilon)\}\in\mathbf{V}(\Omega), &(115)\\
&\|\mathbf{w}(\epsilon)\|_{H^1(\Omega)\times H^1(\Omega)\times H^1(\Omega)} \leq C\epsilon^{1-b-3a}\|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)} &(116)\\
&|B(\epsilon)(\mathbf{w}(\epsilon),\mathbf{v})| \leq C\left(\epsilon^{1-b-3a}+\epsilon^{b-a}\right)\|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)}\|\mathbf{e}(\epsilon,\mathbf{v})\|_{\mathbf{L}^2(\omega)} &(117)\end{aligned}$$

To prove this take $\mathbf{G}=-\epsilon\mathbf{u}^1-(z_\alpha(\epsilon),0)$. Then $\{\mathbf{u^0}+\epsilon\mathbf{u^1}+\mathbf{z}(\epsilon)+\mathbf{G}\}\in\mathbf{V}(\Omega)$. Let $b>0$ be a fixed real number. Then there exists a function $\mathbf{w}(\epsilon)=(w_i(\epsilon))\in\mathbf{H}^1(\Omega)$ such that

$$\mathbf{w}(\epsilon)_{|\Gamma_0}=\mathbf{G}_{|\Gamma_0}, \tag{118}$$

and that satisfies the following inequalities

$$\begin{aligned}&\|\mathbf{w}(\epsilon)\|_{\mathbf{L}^2(\Omega)} \leq C\epsilon^b\|\mathbf{G}\|_{\mathbf{L}^2(\Omega)} &(119)\\
&\|\partial_3\mathbf{w}(\epsilon)\|_{\mathbf{L}^2(\Omega)} \leq C\epsilon^b\|\partial_3\mathbf{G}\|_{\mathbf{L}^2(\Omega)} &(120)\\
&\|\partial_\alpha\mathbf{w}(\epsilon)\|_{\mathbf{L}^2(\Omega)} \leq C\epsilon^{-b}\|\mathbf{G}\|_{\mathbf{H}^1(\Omega)} &(121)\end{aligned}$$

From eq. (98) , eq. (99), eq. (100) ,eq. (101) we obtain,

$$\begin{aligned}&\|\mathbf{G}\|_{\mathbf{L}^2(\Omega)} \leq \epsilon\left\|\mathbf{u^1}\right\|_{\mathbf{L}^2(\Omega)}+\sum_\alpha\|z_\alpha(\epsilon)\|_{L^2(\Omega)} \leq C\epsilon^{1-a}\|\boldsymbol{\zeta}\|_{\mathbf{H^1}(\omega)}\\
&\|\partial_3\mathbf{G}\|_{\mathbf{L}^2(\Omega)} \leq \epsilon\left\|\partial_3\mathbf{u^1}\right\|_{\mathbf{L}^2(\Omega)}+\sum_\alpha\|\partial_3 z_\alpha(\epsilon)\|_{L^2(\Omega)} \leq C\epsilon^{1-a}\|\boldsymbol{\zeta}\|_{\mathbf{H^1}(\omega)}\\
&\|\partial_\alpha\mathbf{G}\|_{\mathbf{L}^2(\Omega)} \leq \epsilon\left\|\mathbf{u^1}\right\|_{\mathbf{H^1}(\Omega)}+\sum_\alpha\|z_\alpha(\epsilon)\|_{H^1(\Omega)} \leq C\epsilon^{1-3a}\|\boldsymbol{\zeta}\|_{\mathbf{H^2}(\omega)} &(122)\end{aligned}$$

Hence we obtain,

$$\begin{aligned}&\|\mathbf{w}(\epsilon)\|_{\mathbf{L}^2(\Omega)} \leq C\epsilon^{1+b-a}\|\boldsymbol{\zeta}\|_{\mathbf{H^1}(\omega)}\\
&\|\partial_3\mathbf{w}(\epsilon)\|_{\mathbf{L}^2(\Omega)} \leq C\epsilon^{1+b-a}\|\boldsymbol{\zeta}\|_{\mathbf{H^1}(\omega)}\\
&\|\partial_\alpha\mathbf{w}(\epsilon)\|_{\mathbf{L}^2(\Omega)} \leq C\epsilon^{1-b-3a}\|\boldsymbol{\zeta}\|_{\mathbf{H^2}(\omega)}. &(123)\end{aligned}$$

From eq. (22) and eq. (123) we obtain,

$$\begin{aligned}\|\mathbf{e}(\epsilon,\mathbf{w}(\epsilon))\|_{\mathbf{L}^2(\Omega)} &\leq C\left(\|\partial_\alpha\mathbf{w}(\epsilon)\|_{\mathbf{L}^2(\Omega)}+\frac{1}{\epsilon}\|\partial_3\mathbf{w}(\epsilon))\|_{\mathbf{L}^2(\Omega)}+\|\mathbf{w}(\epsilon)\|\right)\\
&\leq C\left(\epsilon^{1-b-3a}+\epsilon^{b-a}+\epsilon^{1+b-a}\right)\|\boldsymbol{\zeta}\|_{\mathbf{H^2}(\omega)} &(124)\end{aligned}$$

and therefore

$$\begin{aligned}\|B(\epsilon)(\mathbf{w}(\epsilon),\mathbf{v})\| &\leq C\|\mathbf{e}(\epsilon,\mathbf{w}(\epsilon))\|_{\mathbf{L}^2(\Omega)}\|\mathbf{e}(\epsilon,\mathbf{v})\|_{\mathbf{L}^2(\Omega)}\\
&\leq C\left(\epsilon^{1-b-3a}+\epsilon^{b-a}\right)\|\boldsymbol{\zeta}\|_{\mathbf{H^2}(\Omega)}\|\mathbf{e}(\epsilon,\mathbf{v})\|_{\mathbf{L}^2(\Omega)} &(125)\end{aligned}$$

for all $\mathbf{v} \in \mathbf{V}(\Omega)$.
Step 4. Let $\tilde{\mathbf{u}}(\epsilon) = \mathbf{u}^0 + \epsilon \mathbf{u}^1 + \mathbf{z}(\epsilon) + \mathbf{w}(\epsilon)$. Then, $\tilde{\mathbf{u}}(\epsilon) \in \mathbf{V}(\Omega)$ by step 3. Using the inequalities eq. (88),eq. (97) and eq. (117) we get,

$$\begin{aligned}
&|B(\epsilon)(\mathbf{u}(\epsilon) - \tilde{\mathbf{u}}(\epsilon), \mathbf{v})| = \left| \int_\Omega f_i v_i \sqrt{g(\epsilon)}\, dx - B(\epsilon)(\tilde{\mathbf{u}}(\epsilon), \mathbf{v}) \right| \\
&\leq \left| \int_\Omega f_i v_i \sqrt{g(\epsilon)}\, dx - B(\epsilon)\left(\mathbf{u}^0 + \epsilon \mathbf{u}^1, \mathbf{v}\right) \right| + |B(\epsilon)(\mathbf{z}(\epsilon), \mathbf{v})| + |B(\epsilon)(\mathbf{w}(\epsilon), \mathbf{v})| \\
&\leq C\epsilon \left(\|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \left\|\partial_\alpha f^\alpha\right\|_{L^2(\Omega)} \right) \left(\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}(\epsilon, \mathbf{v})\|_{\mathbf{L}^2(\Omega)} \right) \\
&+ C\left(\epsilon^a + \epsilon^{1-3a}\right) \|\zeta_3\|_{H^2(\omega)} \|\mathbf{e}(\epsilon, \mathbf{v})\|_{\mathbf{L}^2(\Omega)} + C\left(\epsilon^{1-b-3a} + \epsilon^{b-a}\right) \|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)} \|\mathbf{e}(\epsilon, \mathbf{v})\|_{\mathbf{L}^2(\Omega)} \\
&\leq C\left(\epsilon^a + \epsilon^{b-a} + \epsilon^{1-b-3a}\right) \left(\|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \left\|\partial_\alpha f^\alpha\right\|_{L^2(\Omega)} \right) \|\mathbf{e}(\epsilon, \mathbf{v})\|_{\mathbf{L}^2(\Omega)} \\
&+ C\epsilon \left(\|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \left\|\partial_\alpha f^\alpha\right\|_{L^2(\Omega)} \right) \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}. \qquad (126)
\end{aligned}$$

We have the coerciveness inequality

$$|B(\epsilon)(\mathbf{v}, \mathbf{v})| \geq C \|\mathbf{e}(\epsilon, \mathbf{v})\|^2_{\mathbf{L}^2(\Omega)} \text{ for all } \mathbf{v} \in \mathbf{V}(\Omega). \qquad (127)$$

Taking $\mathbf{v} = \mathbf{u}(\epsilon) - \tilde{\mathbf{u}}(\epsilon) \in \mathbf{V}(\Omega)$ we get,

$$\begin{aligned}
&\|\mathbf{e}(\epsilon, \mathbf{u}(\epsilon) - \tilde{\mathbf{u}}(\epsilon))\|^2_{\mathbf{L}^2(\Omega)} \qquad (128) \\
&\leq C\epsilon \left(\|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \left\|\partial_\alpha f^\alpha\right\|_{L^2(\Omega)} \right) \|\mathbf{u}(\epsilon) - \tilde{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)} \qquad (129) \\
&+ C\left(\epsilon^a + \epsilon^{b-a} + \epsilon^{1-b-3a}\right) \left(\|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \left\|\partial_\alpha f^\alpha\right\|_{L^2(\Omega)} \right) \|\mathbf{e}(\epsilon, \mathbf{u}(\epsilon) - \tilde{\mathbf{u}}(\epsilon))\|_{\mathbf{L}^2(\Omega)}. \qquad (130)
\end{aligned}$$

Since the shell is elliptic we have

$$\|\mathbf{v}\|_{H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)} \leq C \|\mathbf{e}(\epsilon, \mathbf{v})\|_{\mathbf{L}^2(\Omega)} \text{ for all } \mathbf{v} \in \mathbf{V}(\Omega). \qquad (131)$$

Taking $\mathbf{v} = \mathbf{u}(\epsilon) - \tilde{\mathbf{u}}(\epsilon)$ from eq. (130) we get

$$\begin{aligned}
&\|\mathbf{e}(\epsilon, \mathbf{u}(\epsilon) - \tilde{\mathbf{u}}(\epsilon))\|_{\mathbf{L}^2(\Omega)} \\
&\leq C\left(\epsilon^a + \epsilon^{b-a} + \epsilon^{1-b-3a}\right) \left(\|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \left\|\partial_\alpha f^\alpha\right\|_{L^2(\Omega)} \right). \qquad (132)
\end{aligned}$$

Applying eq. (131) in eq. (132) we get,

$$\begin{aligned}
&\|\mathbf{u}(\epsilon) - \tilde{\mathbf{u}}\|_{H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)} \\
&\leq C\left(\epsilon^a + \epsilon^{b-a} + \epsilon^{1-b-3a}\right) \left(\|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \left\|\partial_\alpha f^\alpha\right\|_{L^2(\Omega)} \right). \qquad (133)
\end{aligned}$$

Also by using eq. (89), eq. (96) eq. (117) we obtain

$$\begin{aligned}
&\left\|\mathbf{u}(\epsilon) - \mathbf{u}^{\mathbf{0}}\right\|_{H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)} \\
&\leq \|\mathbf{u}(\epsilon) - \tilde{\mathbf{u}}(\epsilon)\|_{H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)} + \epsilon \left\|\mathbf{u}^1\right\|_{H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)} \\
&+ \|\mathbf{z}(\epsilon)\|_{H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)} + \|\mathbf{w}(\epsilon)\|_{H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)} \\
&\leq C\left(\epsilon^a + \epsilon^{b-a} + \epsilon^{1-b-3a}\right) \left(\|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \left\|\partial_\alpha f^\alpha\right\|_{L^2(\Omega)} \right) \\
&+ \epsilon \|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)} + C\left(\epsilon^a + \epsilon^{1-3a}\right) \|\zeta_3\|_{H^2(\omega)} + C\epsilon^{1-b-3a} \|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)}. \qquad (134)
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\|\mathbf{u}(\epsilon) - \mathbf{u}^0\right\|_{H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)} \\
&\leq C\left(\epsilon^a + \epsilon^{b-a} + \epsilon^{1-b-3a}\right) \left(\|\boldsymbol{\zeta}\|_{\mathbf{H}^2(\omega)} + \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \left\|\partial_\alpha f^\alpha\right\|_{L^2(\Omega)} \right). \qquad (135)
\end{aligned}$$

Since $\mathbf{u}^0(x_1, x_2, x_3) = \boldsymbol{\zeta}(x_1, x_2)$, by making the optimal choice $a = 1/6$ and $b = 1/3$ we obtain the claimed result. □

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