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A Study on Generalized Fourth-Order Jacobsthal Sequences

Research Article

Yüksel Soykan*, **Evren Eyican Polatlı**

Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey

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Abstract: In this paper, we investigate the generalized fourth order Jacobsthal sequences and we deal with, in detail, six special cases which we call them as fourth order Jacobsthal, fourth order Jacobsthal-Lucas, modified fourth order Jacobsthal , fourth-order Jacobsthal Perrin, adjusted fourth-order Jacobsthal and modified fourth-order Jacobsthal-Lucas sequences.

MSC: 11B83 • 11B39

Keywords: Jacobsthal numbers • Fourth order Jacobsthal numbers • Fourth order Jacobsthal-Lucas numbers

1. Introduction

It is well-known that the Jacobsthal sequence (sequence A001045 in [28]) $\{J_n\}$ is defined recursively by the equation, for $n \geq 0$

$$J_{n+2} = J_{n+1} + 2J_n$$

in which $J_0 = 0$ and $J_1 = 1$.

Next, we present the first few values of Jacobsthal numbers with positive and negative subscripts:

Table 1. The first few values of the Jacobsthal numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10
J_n	0	1	1	3	5	11	21	43	85	171	341
J_{-n}		$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{3}{8}$	$-\frac{5}{16}$	$\frac{11}{32}$	$-\frac{21}{64}$	$\frac{43}{128}$	$-\frac{85}{256}$	$\frac{171}{512}$	$-\frac{341}{1024}$

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1–3, 10, 14, 15, 17, 18, 21**?** –23]. For higher order Jacobsthal sequences, see [4–9, 12, 26, 32–34].

A generalized Tetranacci sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3; r_1, r_2, r_3, r_4)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = r_1 W_{n-1} + r_2 W_{n-2} + r_3 W_{n-3} + r_4 W_{n-4}, \tag{1}$$

* Corresponding author.
E-mail address(es): yuksel_soykan@hotmail.com (Yüksel Soykan), evreneyican@hotmail.com (Evren Eyican Polatlı).

where the initial values W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers and r_1, r_2, r_3, r_4 are real numbers. Tetranacci sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [16], [24],[25],[27],[30],[35],[36].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r_3}{r_4}W_{-(n-1)} - \frac{r_2}{r_4}W_{-(n-2)} - \frac{r_1}{r_4}W_{-(n-3)} + \frac{1}{r_4}W_{-(n-4)}$$

for $n = 1,2,3,....$ Therefore, recurrence (1) holds for all integer n.

In this paper we consider the case $r_1 = r_2 = r_3$, $r_4 = 2$ and in this case we write $V_n = W_n$. A generalized fourth order Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + 2V_{n-4} \tag{2}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$, $V_3 = c_3$ not all being zero. The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} - \frac{1}{2}V_{-(n-2)} - \frac{1}{2}V_{-(n-3)} + \frac{1}{2}V_{-(n-4)}$$

for $n = 1,2,3,....$ Therefore, recurrence (2) holds for all integer n.

As $\{V_n\}$ is a fourth order recurrence sequence (difference equation), it's characteristic equation is

$$x^4 - x^3 - x^2 - x - 2 = 0. \tag{3}$$

The roots α, β, γ and δ of Equation (3) are given by

$$\begin{aligned}\alpha &= -1,\\ \beta &= 2,\\ \gamma &= i,\\ \delta &= -i.\end{aligned}$$

Note that we have the following identities:

$$\begin{aligned}\alpha+\beta+\gamma+\delta &= 1,\\ \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= -1,\\ \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta &= 1,\\ \alpha\beta\gamma\delta &= -2.\end{aligned}$$

The first few generalized fourth order Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few generalized fourth order Jacobsthal numbers

n	V_n	V_{-n}
0	V_0	...
1	V_1	$\frac{1}{2}V_3 - \frac{1}{2}V_1 - \frac{1}{2}V_2 - \frac{1}{2}V_0$
2	V_2	$\frac{3}{4}V_2 - \frac{1}{4}V_1 - \frac{1}{4}V_0 - \frac{1}{4}V_3$
3	V_3	$\frac{7}{8}V_1 - \frac{1}{8}V_0 - \frac{1}{8}V_2 - \frac{1}{8}V_3$
4	$2V_0 + V_1 + V_2 + V_3$	$\frac{15}{16}V_0 - \frac{1}{16}V_1 - \frac{1}{16}V_2 - \frac{1}{16}V_3$
5	$2V_0 + 3V_1 + 2V_2 + 2V_3$	$\frac{15}{32}V_3 - \frac{17}{32}V_1 - \frac{17}{32}V_2 - \frac{17}{32}V_0$
6	$4V_0 + 4V_1 + 5V_2 + 4V_3$	$\frac{47}{64}V_2 - \frac{17}{64}V_1 - \frac{17}{64}V_0 - \frac{17}{64}V_3$
7	$8V_0 + 8V_1 + 8V_2 + 9V_3$	$\frac{111}{128}V_1 - \frac{17}{128}V_0 - \frac{17}{128}V_2 - \frac{17}{128}V_3$
8	$18V_0 + 17V_1 + 17V_2 + 17V_3$	$\frac{239}{256}V_0 - \frac{17}{256}V_1 - \frac{17}{256}V_2 - \frac{17}{256}V_3$
9	$34V_0 + 35V_1 + 34V_2 + 34V_3$	$\frac{239}{512}V_3 - \frac{273}{512}V_1 - \frac{273}{512}V_2 - \frac{273}{512}V_0$
10	$68V_0 + 68V_1 + 73V_2 + 68V_3$	$\frac{751}{1024}V_2 - \frac{273}{1024}V_1 - \frac{273}{1024}V_0 - \frac{273}{1024}V_3$

Now we define six special case of the sequence $\{V_n\}$. Fourth-order Jacobsthal sequence $\{J_n^{(4)}\}_{n\geq 0}$, fourth-order Jacobsthal-Lucas sequence $\{j_n^{(4)}\}_{n\geq 0}$, modified fourth-order Jacobsthal sequence $\{K_n^{(4)}\}_{n\geq 0}$, fourth-order Jacobsthal

Perrin sequence $\{Q_n^{(4)}\}_{n\geq 0}$, adjusted fourth-order Jacobsthal sequence $\{S_n^{(4)}\}_{n\geq 0}$ and modified fourth-order Jacobsthal-Lucas sequence $\{R_n^{(4)}\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned}
J_{n+4} &= J_{n+3}+J_{n+2}+J_{n+1}+2J_n, \quad J_0=0, J_1=1, J_2=1, J_3=2, &(4)\\
j_{n+4} &= j_{n+3}+j_{n+2}+j_{n+1}+2j_n, \quad j_0=2, j_1=1, j_2=5, j_3=10, &(5)\\
K_{n+4} &= K_{n+3}+K_{n+2}+K_{n+1}+2K_n, \quad K_0=3, K_1=1, K_2=3, K_3=10, &(6)\\
Q_{n+4} &= Q_{n+3}+Q_{n+2}+Q_{n+1}+2Q_n, \quad Q_0=3, Q_1=0, Q_2=2, Q_3=8, &(7)\\
S_{n+4} &= S_{n+3}+S_{n+2}+S_{n+1}+2S_n, \quad S_0=0, S_1=1, S_2=1, S_3=2, &(8)\\
R_{n+4} &= R_{n+3}+R_{n+2}+R_{n+1}+2R_n, \quad R_0=4, R_1=1, R_2=3, R_3=7, &(9)
\end{aligned}$$

The sequences $\{J_n^{(4)}\}_{n\geq 0}$, $\{j_n^{(4)}\}_{n\geq 0}$,$\{K_n^{(4)}\}_{n\geq 0}$, $Q_n^{(4)}\}_{n\geq 0}$, $\{S_n^{(4)}\}_{n\geq 0}$ and $\{R_n^{(4)}\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
J_{-n}^{(4)} &= -\frac{1}{2}J_{-(n-1)}^{(4)}-\frac{1}{2}J_{-(n-2)}^{(4)}-\frac{1}{2}J_{-(n-3)}^{(4)}+\frac{1}{2}J_{-(n-4)}^{(4)}, &(10)\\
j_{-n}^{(4)} &= -\frac{1}{2}j_{-(n-1)}^{(4)}-\frac{1}{2}j_{-(n-2)}^{(4)}-\frac{1}{2}j_{-(n-3)}^{(4)}+\frac{1}{2}j_{-(n-4)}^{(4)}, &(11)\\
K_{-n}^{(4)} &= -\frac{1}{2}K_{-(n-1)}^{(4)}-\frac{1}{2}K_{-(n-2)}^{(4)}-\frac{1}{2}K_{-(n-3)}^{(4)}+\frac{1}{2}K_{-(n-4)}^{(4)}, &(12)\\
Q_{-n}^{(4)} &= -\frac{1}{2}Q_{-(n-1)}^{(4)}-\frac{1}{2}Q_{-(n-2)}^{(4)}-\frac{1}{2}Q_{-(n-3)}^{(4)}+\frac{1}{2}Q_{-(n-4)}^{(4)}, &(13)\\
S_{-n}^{(4)} &= -\frac{1}{2}S_{-(n-1)}^{(4)}-\frac{1}{2}S_{-(n-2)}^{(4)}-\frac{1}{2}S_{-(n-3)}^{(4)}+\frac{1}{2}S_{-(n-4)}^{(4)}, &(14)\\
R_{-n}^{(4)} &= -\frac{1}{2}R_{-(n-1)}^{(4)}-\frac{1}{2}R_{-(n-2)}^{(4)}-\frac{1}{2}R_{-(n-3)}^{(4)}+\frac{1}{2}R_{-(n-4)}^{(4)}. &(15)
\end{aligned}$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences (4)-(9) hold for all integer n.

Next, we present the first few values of the fourth order Jacobsthal, fourth order Jacobsthal-Lucas, modified fourth order Jacobsthal, fourth-order Jacobsthal Perrin, adjusted fourth-order Jacobsthal and modified fourth-order Jacobsthal-Lucas numbers with positive and negative subscripts:

Table 3. The first few values of the special fourth-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10
$J_n^{(4)}$	0	1	1	1	3	7	13	25	51	103	205
$J_{-n}^{(4)}$		$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{8}$	$-\frac{3}{16}$	$-\frac{19}{32}$	$\frac{13}{64}$	$\frac{77}{128}$	$-\frac{51}{256}$	$-\frac{307}{512}$	$\frac{205}{1024}$
$j_n^{(4)}$	2	1	5	10	20	37	77	154	308	613	1229
$j_{-n}^{(4)}$		1	$\frac{1}{2}$	$-\frac{5}{4}$	$\frac{7}{8}$	$\frac{7}{16}$	$\frac{7}{32}$	$-\frac{89}{64}$	$\frac{103}{128}$	$\frac{103}{256}$	$\frac{103}{512}$
$K_n^{(4)}$	3	1	3	10	20	35	71	146	292	579	1159
$K_{-n}^{(4)}$		$\frac{3}{2}$	$-\frac{5}{4}$	$-\frac{9}{8}$	$\frac{31}{16}$	$\frac{31}{32}$	$-\frac{97}{64}$	$-\frac{161}{128}$	$\frac{479}{256}$	$\frac{479}{512}$	$-\frac{1569}{1024}$
$Q_n^{(4)}$	3	0	2	8	16	26	54	112	224	442	886
$Q_{-n}^{(4)}$		$\frac{3}{2}$	$-\frac{5}{4}$	$-\frac{13}{8}$	$\frac{35}{16}$	$\frac{35}{32}$	$-\frac{93}{64}$	$-\frac{221}{128}$	$\frac{547}{256}$	$\frac{547}{512}$	$-\frac{1501}{1024}$
$S_n^{(4)}$	0	1	1	2	4	9	17	34	68	137	273
$S_{-n}^{(4)}$		0	0	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{16}$	$\frac{15}{32}$	$-\frac{17}{64}$	$-\frac{17}{128}$	$-\frac{17}{256}$
$R_n^{(4)}$	4	1	3	7	19	31	63	127	259	511	1023
$R_{-n}^{(4)}$		$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	$\frac{49}{16}$	$-\frac{31}{32}$	$-\frac{63}{64}$	$-\frac{127}{128}$	$\frac{769}{256}$	$-\frac{511}{512}$	$-\frac{1023}{1024}$

In the rest of the paper, for easy writing, we drop the superscripts and write J_n, j_n, K_n, Q_n, S_n and R_n for $J_n^{(4)}, j_n^{(4)}$, $K_n^{(4)}, Q_n^{(4)}, S_n^{(4)}$ and $R_n^{(4)}$, respectively. Note that J_n, j_n and S_n are the sequences A007909, A226309 and A115451 in [28], respectively. K_n, Q_n and R_n sequence aren't in the database of http://oeis.org [28], yet.

2. Generating Functions

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 2.1.
Suppose that $f_{V_n}(x) = \sum\limits_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized fourth-order Jacobsthal sequence

$\{V_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2 + (V_3 - V_2 - V_1 - V_0)x^3}{1 - x - x^2 - x^3 - 2x^4}. \tag{16}$$

Proof. Using the definition of generalized fourth-order Jacobsthal numbers, and substracting $2xf(x)$, $x^2 f(x)$, $x^3 f(x)$ and $x^4 f(x)$ from $f(x)$ we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
& (1 - x - x^2 - x^3 - 2x^4) f_{V_n}(x) \\
= & \sum_{n=0}^{\infty} V_n x^n - x \sum_{n=0}^{\infty} V_n x^n - x^2 \sum_{n=0}^{\infty} V_n x^n - x^3 \sum_{n=0}^{\infty} V_n x^n - 2x^4 \sum_{n=0}^{\infty} V_n x^n \\
= & \sum_{n=0}^{\infty} V_n x^n - \sum_{n=0}^{\infty} V_n x^{n+1} - \sum_{n=0}^{\infty} V_n x^{n+2} - \sum_{n=0}^{\infty} V_n x^{n+3} - 2\sum_{n=0}^{\infty} V_n x^{n+4} \\
= & \sum_{n=0}^{\infty} V_n x^n - \sum_{n=1}^{\infty} V_{n-1} x^n - \sum_{n=2}^{\infty} V_{n-2} x^n - \sum_{n=3}^{\infty} V_{n-3} x^n - 2\sum_{n=4}^{\infty} V_{n-4} x^n \\
= & (V_0 + V_1 x + V_2 x^2 + V_3 x^3) - (V_0 x + V_1 x^2 + V_2 x^3) - (V_0 x^2 + V_1 x^3) - V_0 x^3 \\
& + \sum_{n=4}^{\infty} (V_n - V_{n-1} - V_{n-2} - V_{n-3} - 2V_{n-4}) x^n \\
= & (V_0 + V_1 x + V_2 x^2 + V_3 x^3) - (V_0 x + V_1 x^2 + V_2 x^3) - (V_0 x^2 + V_1 x^3) - V_0 x^3 \\
= & V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2 + (V_3 - V_2 - V_1 - V_0)x^3
\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2 + (V_3 - V_2 - V_1 - V_0)x^3}{1 - x - x^2 - x^3 - 2x^4}.$$

The previous Lemma gives the following results as particular examples.

Corollary 2.1.
Generated functions of fourth-order Jacobsthal, fourth-order Jacobsthal-Lucas , modified fourth-order Jacobsthal, fourth-order Jacobsthal Perrin , adjusted fourth-order Jacobsthal and modified fourth-order Jacobsthal-Lucas sequences are

$$\begin{aligned}
\sum_{n=0}^{\infty} J_n x^n &= \frac{x - x^3}{1 - x - x^2 - x^3 - 2x^4}, \\
\sum_{n=0}^{\infty} j_n x^n &= \frac{2 - x + 2x^2 + 2x^3}{1 - x - x^2 - x^3 - 2x^4}, \\
\sum_{n=0}^{\infty} K_n x^n &= \frac{3 - 2x - x^2 + 3x^3}{1 - x - x^2 - x^3 - 2x^4}, \\
\sum_{n=0}^{\infty} Q_n x^n &= \frac{3 - 3x - x^2 + 3x^3}{1 - x - x^2 - x^3 - 2x^4}, \\
\sum_{n=0}^{\infty} S_n x^n &= \frac{x}{1 - x - x^2 - x^3 - 2x^4}, \\
\sum_{n=0}^{\infty} R_n x^n &= \frac{4 - 3x - 2x^2 - x^3}{1 - x - x^2 - x^3 - 2x^4},
\end{aligned}$$

respectively.

3. Obtaining Binet Formula From Generating Function

We next find Binet formula of generalized fourth order Jacobsthal numbers $\{V_n\}$ by the use of generating function for V_n.

Theorem 3.1.
(Binet formula of generalized fourth order Jacobsthal numbers)

$$V_n = \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \quad (17)$$
$$+\frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{d_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \quad (18)$$

where

$$\begin{aligned}
d_1 &= V_0\alpha^3+(V_1-V_0)\alpha^2+(V_2-V_1-V_0)\alpha+(V_3-V_2-V_1-V_0),\\
d_2 &= V_0\beta^3+(V_1-V_0)\beta^2+(V_2-V_1-V_0)\beta+(V_3-V_2-V_1-V_0),\\
d_3 &= V_0\gamma^3+(V_1-V_0)\gamma^2+(V_2-V_1-V_0)\gamma+(V_3-V_2-V_1-V_0),\\
d_4 &= V_0\delta^3+(V_1-V_0)\delta^2+(V_2-V_1-V_0)\delta+(V_3-V_2-V_1-V_0).
\end{aligned}$$

Proof. Let

$$h(x) = 1-x-x^2-x^3-2x^4.$$

Then for some α,β,γ and δ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)$$

i.e.,

$$1-x-x^2-x^3-2x^4 = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x). \quad (19)$$

Hence $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ ve $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α,β,γ and δ as the roots of

$$h(\frac{1}{x}) = 1-\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}-\frac{2}{x^4} = 0.$$

This implies $x^4-x^3-x^2-x-2=0$. Now, by (16) and (19), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)} \quad (20)$$
$$= \frac{A_1}{(1-\alpha x)}+\frac{A_2}{(1-\beta x)}+\frac{A_3}{(1-\gamma x)}+\frac{A_4}{(1-\delta x)}. \quad (21)$$

So

$$\begin{aligned}
&V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3\\
=\;& A_1(1-\beta x)(1-\gamma x)(1-\delta x)+A_2(1-\alpha x)(1-\gamma x)(1-\delta x)\\
&+A_3(1-\alpha x)(1-\beta x)(1-\delta x)+A_4(1-\alpha x)(1-\beta x)(1-\gamma x).
\end{aligned}$$

If we consider $x=\frac{1}{\alpha}$, we get $V_0+(V_1-V_0)\frac{1}{\alpha}+(V_2-V_1-V_0)\frac{1}{\alpha^2}+(V_3-V_2-V_1-V_0)\frac{1}{\alpha^3} = A_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})(1-\frac{\delta}{\alpha})$. This gives

$$\begin{aligned}
A_1 &= \frac{\alpha^3(V_0+(V_1-V_0)\frac{1}{\alpha}+(V_2-V_1-V_0)\frac{1}{\alpha^2}+(V_3-V_2-V_1-V_0)\frac{1}{\alpha^3})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\\
&= \frac{V_0\alpha^3+(V_1-V_0)\alpha^2+(V_2-V_1-V_0)\alpha+(V_3-V_2-V_1-V_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
A_2 &= \frac{V_0\beta^3+(V_1-V_0)\beta^2+(V_2-V_1-V_0)\beta+(V_3-V_2-V_1-V_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)},\\
A_3 &= \frac{V_0\gamma^3+(V_1-V_0)\gamma^2+(V_2-V_1-V_0)\gamma+(V_3-V_2-V_1-V_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)},\\
A_4 &= \frac{V_0\delta^3+(V_1-V_0)\delta^2+(V_2-V_1-V_0)\delta+(V_3-V_2-V_1-V_0)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.
\end{aligned}$$

Thus (20) can be written as

$$\sum_{n=0}^{\infty} V_n x^n = A_1(1-\alpha x)^{-1} + A_2(1-\beta x)^{-1} + A_3(1-\gamma x)^{-1} + A_4(1-\delta x)^{-1}.$$

This gives

$$\begin{aligned}\sum_{n=0}^{\infty} V_n x^n &= A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n + A_3 \sum_{n=0}^{\infty} \gamma^n x^n + A_4 \sum_{n=0}^{\infty} \delta^n x^n \\ &= \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n) x^n.\end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n.$$

where

$$\begin{aligned}A_1 &= \frac{V_0\alpha^3 + (V_1 - V_0)\alpha^2 + (V_2 - V_1 - V_0)\alpha + (V_3 - V_2 - V_1 - V_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}, \\ A_2 &= \frac{V_0\beta^3 + (V_1 - V_0)\beta^2 + (V_2 - V_1 - V_0)\beta + (V_3 - V_2 - V_1 - V_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ A_3 &= \frac{V_0\gamma^3 + (V_1 - V_0)\gamma^2 + (V_2 - V_1 - V_0)\gamma + (V_3 - V_2 - V_1 - V_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ A_4 &= \frac{V_0\delta^3 + (V_1 - V_0)\delta^2 + (V_2 - V_1 - V_0)\delta + (V_3 - V_2 - V_1 - V_0)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)},\end{aligned}$$

and then we get (17).

Next, using Theorem 3.1, we present the Binet formulas of fourth-order Jacobsthal, Jacobsthal-Lucas, modified Jacobsthal and Jacobsthal Perrin sequences.

Corollary 3.1.

Binet formulas of fourth-order Jacobsthal, fourth-order Jacobsthal-Lucas, modified fourth-order Jacobsthal, fourth-order Jacobsthal Perrin, adjusted fourth-order Jacobsthal and modified fourth-order Jacobsthal-Lucas sequences are

$$\begin{aligned}J_n &= \frac{(\alpha^2-1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n + \frac{(\beta^2-1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^n \\ &\quad + \frac{(\gamma^2-1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n + \frac{(\delta^2-1)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n, \\ j_n &= \frac{(2\alpha^3-\alpha^2+2\alpha+2)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n + \frac{(2\beta^3-\beta^2+2\beta+2)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^n \\ &\quad + \frac{(2\gamma^3-\gamma^2+2\gamma+2)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n + \frac{(2\delta^3-\delta^2+2\delta+2)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n, \\ K_n &= \frac{(3\alpha^3-2\alpha^2-\alpha+3)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n + \frac{(3\beta^3-2\beta^2-\beta+3)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^n \\ &\quad + \frac{(3\gamma^3-2\gamma^2-\gamma+3)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n + \frac{(3\delta^3-2\delta^2-\delta+3)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n, \\ Q_n &= \frac{(3\alpha^3-3\alpha^2-\alpha+3)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n + \frac{(3\beta^3-3\beta^2-\beta+3)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^n \\ &\quad + \frac{(3\gamma^3-3\gamma^2-\gamma+3)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n + \frac{(3\delta^3-3\delta^2-\delta+3)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n, \\ S_n &= \frac{1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^{n+2} + \frac{1}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^{n+2} \\ &\quad + \frac{1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^{n+2} + \frac{1}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^{n+2}, \\ R_n &= \frac{(4\alpha^3-3\alpha^2-2\alpha-1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n + \frac{(4\beta^3-3\beta^2-2\beta-1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^n \\ &\quad + \frac{(4\gamma^3-3\gamma^2-2\gamma-1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n + \frac{(4\delta^3-3\delta^2-2\delta-1)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n.\end{aligned}$$

respectively.

The above formulas can be written as follows

$$
\begin{aligned}
J_n &= \frac{1}{5}\beta^n + (-\frac{1}{10} - \frac{3}{10}i)\gamma^n + (-\frac{1}{10} + \frac{3}{10}i)\delta^n, \\
j_n &= \frac{1}{2}\alpha^n + \frac{6}{5}\beta^n + (\frac{3}{20} + \frac{9}{20}i)\gamma^n + (\frac{3}{20} - \frac{9}{20}i)\delta^n, \\
K_n &= \frac{1}{6}\alpha^n + \frac{17}{15}\beta^n + (\frac{17}{20} + \frac{11}{20}i)\gamma^n + (\frac{17}{20} - \frac{11}{20}i)\delta^n, \\
Q_n &= \frac{1}{3}\alpha^n + \frac{13}{15}\beta^n + (\frac{9}{10} + \frac{7}{10}i)\gamma^n + (\frac{9}{10} - \frac{7}{10}i)\delta^n, \\
S_n &= -\frac{1}{6}\alpha^n + \frac{4}{15}\beta^n + (-\frac{1}{20} - \frac{3}{20}i)\gamma^n + (-\frac{1}{20} + \frac{3}{20}i)\delta^n, \\
R_n &= \alpha^n + \beta^n + \gamma^n + \delta^n.
\end{aligned}
$$

We can also find Binet formulas by using matrix method which is given in [20]. Take $k = i = 4$ in Corollary 3.1 in [20]. Let

$$
\Lambda = \begin{pmatrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \\ \delta^3 & \delta^2 & \delta & 1 \end{pmatrix}, \Lambda_1 = \begin{pmatrix} \alpha^{n-1} & \alpha^2 & \alpha & 1 \\ \beta^{n-1} & \beta^2 & \beta & 1 \\ \gamma^{n-1} & \gamma^2 & \gamma & 1 \\ \delta^{n-1} & \delta^2 & \delta & 1 \end{pmatrix}, \Lambda_2 = \begin{pmatrix} \alpha^3 & \alpha^{n-1} & \alpha & 1 \\ \beta^3 & \beta^{n-1} & \beta & 1 \\ \gamma^3 & \gamma^{n-1} & \gamma & 1 \\ \delta^3 & \delta^{n-1} & \delta & 1 \end{pmatrix},
$$

$$
\Lambda_3 = \begin{pmatrix} \alpha^3 & \alpha^2 & \alpha^{n-1} & 1 \\ \beta^3 & \beta^2 & \beta^{n-1} & 1 \\ \gamma^3 & \gamma^2 & \gamma^{n-1} & 1 \\ \delta^3 & \delta^2 & \delta^{n-1} & 1 \end{pmatrix}, \Lambda_4 = \begin{pmatrix} \alpha^3 & \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^3 & \beta^2 & \beta & \beta^{n-1} \\ \gamma^3 & \gamma^2 & \gamma & \gamma^{n-1} \\ \delta^3 & \delta^2 & \delta & \delta^{n-1} \end{pmatrix}.
$$

Then the Binet formula for fourth-order Jacobsthal numbers is

$$
\begin{aligned}
J_n &= \frac{1}{\det(\Lambda)} \sum_{j=1}^{4} P_{5-j} \det(\Lambda_j) \\
&= \frac{1}{\Lambda}(J_4 \det(\Lambda_1) + J_3 \det(\Lambda_2) + J_2 \det(\Lambda_3) + J_1 \det(\Lambda_4)) \\
&= \frac{1}{\det(\Lambda)}(3\det(\Lambda_1) + \det(\Lambda_2) + \det(\Lambda_3) + \det(\Lambda_4)) \\
&= \left(3 \begin{vmatrix} \alpha^{n-1} & \alpha^2 & \alpha & 1 \\ \beta^{n-1} & \beta^2 & \beta & 1 \\ \gamma^{n-1} & \gamma^2 & \gamma & 1 \\ \delta^{n-1} & \delta^2 & \delta & 1 \end{vmatrix} + \begin{vmatrix} \alpha^3 & \alpha^{n-1} & \alpha & 1 \\ \beta^3 & \beta^{n-1} & \beta & 1 \\ \gamma^3 & \gamma^{n-1} & \gamma & 1 \\ \delta^3 & \delta^{n-1} & \delta & 1 \end{vmatrix} + \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha^{n-1} & 1 \\ \beta^3 & \beta^2 & \beta^{n-1} & 1 \\ \gamma^3 & \gamma^2 & \gamma^{n-1} & 1 \\ \delta^3 & \delta^2 & \delta^{n-1} & 1 \end{vmatrix} \right. \\
&\quad \left. + \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^3 & \beta^2 & \beta & \beta^{n-1} \\ \gamma^3 & \gamma^2 & \gamma & \gamma^{n-1} \\ \delta^3 & \delta^2 & \delta & \delta^{n-1} \end{vmatrix} \right) / \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \\ \delta^3 & \delta^2 & \delta & 1 \end{vmatrix}.
\end{aligned}
$$

Similarly, we obtain the Binet formula for fourth-order Jacobsthal, fourth-order Jacobsthal-Lucas, modified fourth-order Jacobsthal, fourth-order Jacobsthal Perrin, adjusted fourth-order Jacobsthal and modified fourth-order Jacobsthal-Lucas sequences as

$$
\begin{aligned}
j_n &= \frac{1}{\det\Lambda}(20\det(\Lambda_1) + 10\det(\Lambda_2) + 5\det(\Lambda_3) + \det(\Lambda_4)), \\
K_n &= \frac{1}{\det\Lambda}(20\det(\Lambda_1) + 10\det(\Lambda_2) + 3\det(\Lambda_3) + \det(\Lambda_4)), \\
Q_n &= \frac{1}{\det\Lambda}(16\det(\Lambda_1) + 8\det(\Lambda_2) + 2\det(\Lambda_3)), \\
S_n &= \frac{1}{\det\Lambda}(4\det(\Lambda_1) + 2\det(\Lambda_2) + \det(\Lambda_3) + \det(\Lambda_4)), \\
R_n &= \frac{1}{\det\Lambda}(19\det(\Lambda_1) + 7\det(\Lambda_2) + 3\det(\Lambda_3) + \det(\Lambda_4)),
\end{aligned}
$$

respectively.

4. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following Theorem gives generalization of this result to the generalized Tetranacci sequence $\{W_n\}$.

Theorem 4.1 (Simson Formula of Generalized Tetranacci Numbers).
For all integers n we have

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n r_4^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \tag{22}$$

Proof. (22) is given in Soykan [29].

A special case of the above theorem is the following Theorem which gives Simson formula of the generalized fourth-order Jacobsthal sequence $\{V_n\}$.

Theorem 4.2 (Simson Formula of Generalized Fourth-Order Pell Numbers).
For all integers n we have

$$\begin{vmatrix} V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} \end{vmatrix} = (-1)^n 2^n \begin{vmatrix} V_3 & V_2 & V_1 & V_0 \\ V_2 & V_1 & V_0 & V_{-1} \\ V_1 & V_0 & V_{-1} & V_{-2} \\ V_0 & V_{-1} & V_{-2} & V_{-3} \end{vmatrix}.$$

The previous Theorem gives the following results as particular examples.

Corollary 4.1.
Simson formula of fourth-order Jacobsthal sequence, fourth-order Jacobsthal-Lucas sequence, modified fourth-order Jacobsthal sequence, fourth-order Jacobsthal Perrin sequence, adjusted fourth-order Jacobsthal sequence and modified fourth-order Jacobsthal-Lucas sequence are given as

$$\begin{vmatrix} J_{n+3} & J_{n+2} & J_{n+1} & J_n \\ J_{n+2} & J_{n+1} & J_n & J_{n-1} \\ J_{n+1} & J_n & J_{n-1} & J_{n-2} \\ J_n & J_{n-1} & J_{n-2} & J_{n-3} \end{vmatrix} = 0,$$

$$\begin{vmatrix} j_{n+3} & j_{n+2} & j_{n+1} & j_n \\ j_{n+2} & j_{n+1} & j_n & j_{n-1} \\ j_{n+1} & j_n & j_{n-1} & j_{n-2} \\ j_n & j_{n-1} & j_{n-2} & j_{n-3} \end{vmatrix} = \frac{243}{4}(-1)^n 2^n,$$

$$\begin{vmatrix} K_{n+3} & K_{n+2} & K_{n+1} & K_n \\ K_{n+2} & K_{n+1} & K_n & K_{n-1} \\ K_{n+1} & K_n & K_{n-1} & K_{n-2} \\ K_n & K_{n-1} & K_{n-2} & K_{n-3} \end{vmatrix} = \frac{697}{8}(-1)^n 2^n,$$

$$\begin{vmatrix} Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} \\ Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} \\ Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} \end{vmatrix} = 169(-1)^n 2^n,$$

$$\begin{vmatrix} S_{n+3} & S_{n+2} & S_{n+1} & S_n \\ S_{n+2} & S_{n+1} & S_n & S_{n-1} \\ S_{n+1} & S_n & S_{n-1} & S_{n-2} \\ S_n & S_{n-1} & S_{n-2} & S_{n-3} \end{vmatrix} = (-1)^{n+1}2^{n-1},$$

$$\begin{vmatrix} R_{n+3} & R_{n+2} & R_{n+1} & R_n \\ R_{n+2} & R_{n+1} & R_n & R_{n-1} \\ R_{n+1} & R_n & R_{n-1} & R_{n-2} \\ R_n & R_{n-1} & R_{n-2} & R_{n-3} \end{vmatrix} = 450(-1)^n 2^n,$$

respectively.

5. Some Identities

In this section, we obtain some identities of fourth order Jacobsthal, fourth order Jacobsthal-Lucas and modified fourth order Jacobsthal numbers. Next, we can give a few basic relations between $\{J_n\}$ and $\{j_n\}$.

Lemma 5.1.
The following equalities are true:

$$\begin{aligned} 36J_n &= -7j_{n+4}+5j_{n+3}+17j_{n+2}+5j_{n+1}, \quad (23) \\ 18J_n &= -j_{n+3}+5j_{n+2}-j_{n+1}-7j_n, \\ 9J_n &= 2j_{n+2}-j_{n+1}-4j_n-j_{n-1}, \\ 9J_n &= j_{n+1}-2j_n+j_{n-1}+4j_{n-2}, \\ 9J_n &= -j_n+2j_{n-1}+5j_{n-2}+2j_{n-3} \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (23). To show (23), writing

$$J_n = a\times j_{n+4}+b\times j_{n+3}+c\times j_{n+2}+d\times j_{n+1}$$

and solving the system of equations

$$\begin{aligned} J_0 &= a\times j_4+b\times j_3+c\times j_2+d\times j_1 \\ J_1 &= a\times j_5+b\times j_4+c\times j_3+d\times j_2 \\ J_2 &= a\times j_6+b\times j_5+c\times j_4+d\times j_3 \\ J_3 &= a\times j_7+b\times j_6+c\times j_5+d\times j_4 \end{aligned}$$

we find that $a=-\frac{7}{36}, b=\frac{5}{36}, c=\frac{17}{36}, d=\frac{5}{36}$. The other equalities can be proved similarly.

Note that all the identities in the above Lemma can be proved by induction as well.

We present a few basic relations between $\{J_n\}$ and $\{K_n\}$.

Lemma 5.2.
The following equalities are true:

$$\begin{aligned} 1394J_n &= -121K_{n+4}+185K_{n+3}+219K_{n+2}-87K_{n+1} \\ 697J_n &= 32K_{n+3}+49K_{n+2}-104K_{n+1}-121K_n, \\ 697J_n &= 81K_{n+2}-72K_{n+1}-89K_n+64K_{n-1}, \\ 697J_n &= 9K_{n+1}-8K_n+145K_{n-1}+162K_{n-2}, \\ 697J_n &= K_n+154K_{n-1}+171K_{n-2}+18K_{n-3}. \end{aligned}$$

We give a few basic relations between $\{J_n\}$ and $\{Q_n\}$

Lemma 5.3.
The following equalities are true:

$$\begin{aligned}
26J_n &= -2Q_{n+4}+3Q_{n+3}+4Q_{n+2}-Q_{n+1}, \\
26J_n &= Q_{n+3}+2Q_{n+2}-3Q_{n+1}-4Q_n, \\
26J_n &= 3Q_{n+2}-2Q_{n+1}-3Q_n+2Q_{n-1}, \\
26J_n &= Q_{n+1}+5Q_{n-1}+6Q_{n-2}, \\
26J_n &= Q_n+6Q_{n-1}+7Q_{n-2}+2Q_{n-3}.
\end{aligned}$$

We present a few basic relations between $\{J_n\}$ and $\{S_n\}$.

Lemma 5.4.
The following equalities are true:

$$\begin{aligned}
8J_n &= 5S_{n+4}-3S_{n+3}-11S_{n+2}-3S_{n+1}, \\
4J_n &= S_{n+3}-3S_{n+2}+S_{n+1}+5S_n, \\
2J_n &= -S_{n+2}+S_{n+1}+3S_n+S_{n-1}, \\
J_n &= S_n-S_{n-2}.
\end{aligned}$$

We give a few basic relations between $\{J_n\}$ and $\{R_n\}$

Lemma 5.5.
The following equalities are true:

$$\begin{aligned}
150J_n &= -8R_{n+4}+22R_{n+3}+7R_{n+2}-23R_{n+1}, \\
150J_n &= 14R_{n+3}-R_{n+2}-31R_{n+1}-16R_n, \\
150J_n &= 13R_{n+2}-17R_{n+1}-2R_n+28R_{n-1}, \\
150J_n &= -4R_{n+1}+11R_n+41R_{n-1}+26R_{n-2}, \\
150J_n &= 7R_n+37R_{n-1}+22R_{n-2}-8R_{n-3}.
\end{aligned}$$

We present a few basic relations between $\{K_n\}$ and $\{j_n\}$.

Lemma 5.6.
The following equalities are true:

$$\begin{aligned}
697j_n &= 470K_{n+4}-805K_{n+3}+215K_{n+2}-601K_{n+1}, \\
697j_n &= -335K_{n+3}+685K_{n+2}-131K_{n+1}+940K_n-10K_{n-1}, \\
697j_n &= 350K_{n+2}-466K_{n+1}+605K_n-670K_{n-1}, \\
697j_n &= -116K_{n+1}+955K_n-320K_{n-1}+700K_{n-2}, \\
697j_n &= 839K_n-436K_{n-1}+584K_{n-2}-232K_{n-3}
\end{aligned}$$

and

$$\begin{aligned}
24K_n &= -3j_{n+4}-11j_{n+3}+571j_{n+2}-431j_{n+1}, \\
54K_n &= 31j_{n+3}-41j_{n+2}-41j_{n+1}+49j_n, \\
27K_n &= -5j_{n+2}-5j_{n+1}+40j_n+31j_{n-1}, \\
27K_n &= -10j_{n+1}+35j_n+26j_{n-1}-10j_{n-2}, \\
27K_n &= 25j_n+16j_{n-1}-20j_{n-2}-20j_{n-3}.
\end{aligned}$$

We give a few basic relations between $\{j_n\}$ and $\{Q_n\}$.

Lemma 5.7.
The following equalities are true:

$$\begin{aligned}
26j_n &= 11Q_{n+4} - 16Q_{n+3} + 2 \times Q_{n+2} - 10Q_{n+1},\\
26j_n &= 11Q_{n+4} - 16Q_{n+3} + 2Q_{n+2} - 10Q_{n+1},\\
26j_n &= -5Q_{n+3} + 13Q_{n+2} + Q_{n+1} + 22Q_n,\\
26j_n &= 8Q_{n+2} - 4Q_{n+1} + 17Q_n - 10Q_{n-1},\\
26j_n &= 4Q_{n+1} + 25Q_n - 2Q_{n-1} + 16Q_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
108Q_n &= 65j_{n+4} - 7j_{n+3} - 151j_{n+2} - 151j_{n+1},\\
54Q_n &= 29j_{n+3} - 43j_{n+2} - 43j_{n+1} + 65j_n,\\
54Q_n &= 29j_{n+3} - 43j_{n+2} - 43j_{n+1} + 65j_n,\\
27Q_n &= -7j_{n+2} - 7j_{n+1} + 47j_n + 29j_{n-1},\\
27Q_n &= -14j_{n+1} + 40j_n + 22j_{n-1} - 14j_{n-2}.
\end{aligned}$$

We present a few basic relations between $\{j_n\}$ and $\{S_n\}$.

Lemma 5.8.
The following equalities are true:

$$\begin{aligned}
4j_n &= -5S_{n+4} + 7S_{n+3} + 7S_{n+2} + 7S_{n+1}\\
2j_n &= S_{n+3} + S_{n+2} + S_{n+1} - 5S_n\\
j_n &= S_{n+2} + S_{n+1} - 2S_n + S_{n-1}\\
j_n &= 2S_{n+1} - S_n + 2S_{n-1} + 2S_{n-2}\\
j_n &= S_n + 4S_{n-1} + 4S_{n-2} + 4S_{n-3}
\end{aligned}$$

and

$$\begin{aligned}
27S_n &= -4j_{n+4} + 5j_{n+3} + 5j_{n+2} + 5j_{n+1}\\
27S_n &= j_{n+3} + j_{n+2} + j_{n+1} - 8j_n\\
27S_n &= 2j_{n+2} + 2j_{n+1} - 7j_n + 2j_{n-1}\\
27S_n &= 4j_{n+1} - 5j_n + 4j_{n-1} + 4j_{n-2}\\
27S_n &= -j_n + 8j_{n-1} + 8j_{n-2} + 8j_{n-3}
\end{aligned}$$

We give a few basic relations between $\{j_n\}$ and $\{R_n\}$.

Lemma 5.9.
The following equalities are true:

$$\begin{aligned}
300j_n &= 64R_{n+4} - 101R_{n+3} + 19R_{n+2} + 34R_{n+1},\\
300j_n &= -37R_{n+3} + 83R_{n+2} + 98R_{n+1} + 128R_n,\\
300j_n &= 46R_{n+2} + 61R_{n+1} + 91R_n - 74R_{n-1},\\
300j_n &= 107R_{n+1} + 137R_n - 28R_{n-1} + 92R_{n-2},\\
300j_n &= 244R_n + 79R_{n-1} + 199R_{n-2} + 214R_{n-3},
\end{aligned}$$

and

$$\begin{aligned}
36R_n &= 13j_{n+4} + j_{n+3} - 11j_{n+2} - 71j_{n+1},\\
18R_n &= 7j_{n+3} + j_{n+2} - 29j_{n+1} + 13j_n,\\
9R_n &= 4j_{n+2} - 11j_{n+1} + 10j_n + 7j_{n-1},\\
9R_n &= -7j_{n+1} + 14j_n + 11j_{n-1} + 8j_{n-2},\\
9R_n &= 7 \times j_n + 4j_{n-1} + j_{n-2} - 14j_{n-3}.
\end{aligned}$$

We give a few basic relations between $\{K_n\}$ and $\{Q_n\}$.

Lemma 5.10.
The following equalities are true:

$$
\begin{aligned}
26K_n &= 10Q_{n+4} - 7Q_{n+3} - 13Q_{n+2} - 9Q_{n+1}, \\
26K_n &= 3Q_{n+3} - 3Q_{n+2} + Q_{n+1} + 2Q_n, \\
26K_n &= 4Q_{n+1} + 23Q_n + 6Q_{n-1}, \\
26K_n &= 27Q_n + 10Q_{n-1} + 4Q_{n-2} + 8Q_{n-3},
\end{aligned}
$$

and

$$
\begin{aligned}
1394Q_n &= 983K_{n+4} - 1261K_{n+3} - 581K_{n+2} - 1125K_{n+1}, \\
697Q_n &= -139K_{n+3} + 201K_{n+2} - 71K_{n+1} + 983K_n, \\
697Q_n &= 62K_{n+2} - 210K_{n+1} + 844K_n - 278K_{n-1}, \\
697Q_n &= -148K_{n+1} + 906K_n - 216K_{n-1} + 124K_{n-2}, \\
697Q_n &= 758K_n - 364K_{n-1} - 24K_{n-2} - 296K_{n-3}.
\end{aligned}
$$

We present a few basic relations between $\{S_n\}$ and $\{R_n\}$.

Lemma 5.11.
The following equalities are true:

$$
\begin{aligned}
900S_n &= -44R_{n+4} + 121R_{n+3} + R_{n+2} - 14R_{n+1}, \\
900S_n &= 77R_{n+3} - 43R_{n+2} - 58R_{n+1} - 88R_n, \\
900S_n &= 34R_{n+2} + 19R_{n+1} - 11R_n + 154R_{n-1}, \\
900S_n &= 53R_{n+1} + 23R_n + 188R_{n-1} + 68R_{n-2}, \\
900S_n &= 76R_n + 241R_{n-1} + 121R_{n-2} + 106R_{n-3},
\end{aligned}
$$

and

$$
\begin{aligned}
8R_n &= -7S_{n+4} + S_{n+3} + 9S_{n+2} + 49S_{n+1}, \\
4R_n &= -3S_{n+3} + S_{n+2} + 21S_{n+1} - 7S_n, \\
2R_n &= -S_{n+2} + 9S_{n+1} - 5S_n - 3S_{n-1}, \\
R_n &= 4S_{n+1} - 3S_n - 2S_{n-1} - S_{n-2}, \\
R_n &= S_n + 2S_{n-1} + 3S_{n-2} + 8S_{n-3}.
\end{aligned}
$$

We now present a few special identities for the fourth order Jacobsthal sequence J_n.

Theorem 5.1.
(Catalan's identity) For all natural numbers n and m, the following identity holds

$$J_{n+m}J_{n-m} - J_n^2 = (S_{n+m} - S_{n+m-2})(S_{n-m} - S_{n-m-2}) - (S_n - S_{n-2})^2.$$

Proof. We use the identity

$$J_n = S_n - S_{n-2}.$$

Note that for $m = 1$ in Catalan's identity, we get the Cassini identity for the modified fourth order Jacobsthal sequence.

Corollary 5.1.
(Cassini's identity) For all natural numbers n and m, the following identity holds

$$J_{n+1}J_{n-1} - J_n^2 = (S_{n+1} - S_{n-1})(S_{n-1} - S_{n-3}) - (S_n - S_{n-2})^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using $J_n = S_n - S_{n-2}$.The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified fourth order Jacobsthal sequence $\{J_n\}$.

Theorem 5.2.
Let n and m be any integers. Then the following identities are true:

(a) *(d'Ocagne's identity)*

$$J_{m+1}J_n - J_mJ_{n+1} = (S_{m+1} - S_{m-1})(S_n - S_{n-2}) - (S_m - S_{m-2})(S_{n+1} - S_{n-1}).$$

(b) *(Gelin-Cesàro's identity)*

$$J_{n+2}J_{n+1}J_{n-1}J_{n-2} - J_n^4 = (S_{n+2} - S_n)(S_{n+1} - S_{n-1})(S_{n-1} - S_{n-3})(S_{n-2} - S_{n-4}) - (S_n - S_{n-2})^4.$$

(c) *(Melham's identity)*

$$J_{n+1}J_{n+2}J_{n+6} - J_{n+3}^3 = (S_{n+1} - S_{n-1})(S_{n+2} - S_n)(S_{n+6} - S_{n+4}) - (S_{n+3} - S_{n+1})^3.$$

Proof. Use the identity $J_n = S_n - S_{n-2}$.

6. Linear Sums

The following Theorem presents summing formulas of generalized fourth order Jacobsthal numbers.

Proposition 6.1.
If $r = 1, s = 1, t = 1, u = 2$ then for $n \geq 0$ we have the following formula:

$$\sum_{k=0}^{n} W_k = \frac{1}{4}(W_{n+4} - W_{n+2} - 2W_{n+1} - W_3 + W_1 + 2W_0).$$

Proof. This is given in [31].

Corollary 6.1.
We have the following properties:

(a) $\sum_{k=0}^{n} J_k = \frac{1}{4}(J_{n+4} - J_{n+2} - 2J_{n+1})$.

(b) $\sum_{k=0}^{n} j_k = \frac{1}{4}(j_{n+4} - j_{n+2} - 2j_{n+1} - 5)$.

(c) $\sum_{k=0}^{n} K_k = \frac{1}{4}(K_{n+4} - K_{n+2} - 2K_{n+1} - 3)$.

(d) $\sum_{k=0}^{n} Q_k = \frac{1}{4}(Q_{n+4} - Q_{n+2} - 2Q_{n+1} - 2)$.

(e) $\sum_{k=0}^{n} S_k = \frac{1}{4}(S_{n+4} - S_{n+2} - 2S_{n+1} - 1)$.

(f) $\sum_{k=0}^{n} R_k = \frac{1}{4}(R_{n+4} - R_{n+2} - 2R_{n+1} + 2)$.

Proof.

(a) Take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1$ in the last proposition.

(b) Take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10$ in the last proposition.

(c) Take $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10$ in the last proposition.

(d) Take $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8$ in the last proposition.

(e) Take $S_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2$ in the last proposition.

(f) Take $S_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ in the last proposition.

The following proposition presents summing formulas of generalized fourth order Jacobsthal numbers with negative subscripts.

Proposition 6.2.
If $r=s=t=1, u=2$ then for $n\geq 1$ we have the following formula:

$$\sum_{k=1}^{n} W_{-k}=\frac{1}{4}(-W_{-n+3}+W_{-n+1}+2W_{-n}+W_3-W_1-2W_0).$$

Proof. This is given in [31].

Corollary 6.2.
We have the following properties:

(a) $\sum_{k=1}^{n} J_{-k}=\frac{1}{4}(-J_{-n+3}+J_{-n+1}+2J_{-n})$.

(b) $\sum_{k=1}^{n} j_{-k}=\frac{1}{4}(-j_{-n+3}+j_{-n+1}+2j_{-n}+5)$.

(c) $\sum_{k=1}^{n} K_{-k}=\frac{1}{4}(-K_{-n+3}+K_{-n+1}+2K_{-n}+3)$.

(d) $\sum_{k=1}^{n} Q_{-k}=\frac{1}{4}(-Q_{-n+3}+Q_{-n+1}+2Q_{-n}+2)$.

(e) $\sum_{k=1}^{n} S_{-k}=\frac{1}{4}(-S_{-n+3}+S_{-n+1}+2S_{-n}+1)$.

(f) $\sum_{k=1}^{n} R_{-k}=\frac{1}{4}(-R_{-n+3}+R_{-n+1}+2R_{-n}-2)$.

7. Matrices Related with Generalized Fourth-Order Jacobsthal numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix}=\begin{pmatrix} r_1 & r_2 & r_3 & r_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{24}$$

For matrix formulation (24), see [19]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ r_1 & r_2 & r_3 & r_4 \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix}.$$

We define the square matrix A of order 4 as:

$$A=\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A=-2$. From (2) we have

$$\begin{pmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix}. \tag{25}$$

and from (24) (or using (25) and induction) we have

$$\begin{pmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n\begin{pmatrix} V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V=S$ in (25) we have

$$\begin{pmatrix} S_{n+3} \\ S_{n+2} \\ S_{n+1} \\ S_n \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} S_{n+2} \\ S_{n+1} \\ S_n \\ S_{n-1} \end{pmatrix}. \tag{26}$$

We also define

$$B_n = \begin{pmatrix} S_{n+1} & S_n + S_{n-1} + 2S_{n-2} & S_n + 2S_{n-1} & 2S_n \\ S_n & S_{n-1} + S_{n-2} + 2S_{n-3} & S_{n-1} + 2S_{n-2} & 2S_{n-1} \\ S_{n-1} & S_{n-2} + S_{n-3} + 2S_{n-4} & S_{n-2} + 2S_{n-3} & 2S_{n-2} \\ S_{n-2} & S_{n-3} + S_{n-4} + 2S_{n-5} & S_{n-3} + 2S_{n-4} & 2S_{n-3} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & V_n + V_{n-1} + 2V_{n-2} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + V_{n-2} + 2V_{n-3} & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} + 2V_{n-4} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \\ V_{n-2} & V_{n-3} + V_{n-4} + 2V_{n-5} & V_{n-3} + 2V_{n-4} & 2V_{n-3} \end{pmatrix}.$$

Theorem 7.1.
For all integer $m, n \geq 0$, we have

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a) By expanding the vectors on the both sides of (26) to 4-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1} B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_n & V_{n-1} + V_{n-2} + 2V_{n-3} & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} + 2V_{n-4} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \\ V_{n-2} & V_{n-3} + V_{n-4} + 2V_{n-5} & V_{n-3} + 2V_{n-4} & 2V_{n-3} \\ V_{n-3} & V_{n-4} + V_{n-5} + 2V_{n-6} & V_{n-4} + 2V_{n-5} & 2V_{n-4} \end{pmatrix} \\ &= \begin{pmatrix} V_{n+1} & V_n + V_{n-1} + 2V_{n-2} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + V_{n-2} + 2V_{n-3} & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} + 2V_{n-4} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \\ V_{n-2} & V_{n-3} + V_{n-4} + 2V_{n-5} & V_{n-3} + 2V_{n-4} & 2V_{n-3} \end{pmatrix} = C_n. \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1} C_1$. Now we obtain

$$C_{n+m} = A^{n+m-1} C_1 = A^{n-1} A^m C_1 = A^{n-1} C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

Some properties of A^n matris can be given as

$$A^n = A^{n-1} + A^{n-2} + A^{n-3} + 2A^{n-4}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

for all integer m and n.

Theorem 7.2.
For $m, n \geq 0$ we have

$$V_{n+m} = V_n S_{m+1} + (V_{n-1} + V_{n-2} + 2V_{n-3}) S_m \quad (27)$$
$$+ (V_{n-1} + 2V_{n-2}) S_{m-1} + 2V_{n-1} S_{m-2}. \quad (28)$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m.We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof.

Remark 7.1.
By induction, it can be proved that for all integers $m, n \leq 0$, (27) holds. So for all integers m, n (27) is true.

Corollary 7.1.
For all integers m, n, we have

$$J_{n+m} = J_n S_{m+1} + (J_{n-1} + J_{n-2} + 2J_{n-3}) S_m + (J_{n-1} + 2J_{n-2}) S_{m-1} + 2J_{n-1} S_{m-2},$$
$$j_{n+m} = j_n S_{m+1} + \left(j_{n-1} + j_{n-2} + 2j_{n-3}\right) S_m + \left(j_{n-1} + 2j_{n-2}\right) S_{m-1} + 2j_{n-1} S_{m-2},$$
$$K_{n+m} = K_n S_{m+1} + (K_{n-1} + K_{n-2} + 2K_{n-3}) S_m + (K_{n-1} + 2K_{n-2}) S_{m-1} + 2K_{n-1} S_{m-2},$$
$$Q_{n+m} = Q_n S_{m+1} + (Q_{n-1} + Q_{n-2} + 2Q_{n-3}) S_m + (Q_{n-1} + 2Q_{n-2}) S_{m-1} + 2Q_{n-1} S_{m-2},$$
$$S_{n+m} = S_n S_{m+1} + (S_{n-1} + S_{n-2} + 2S_{n-3}) S_m + (S_{n-1} + 2S_{n-2}) S_{m-1} + 2S_{n-1} S_{m-2},$$
$$R_{n+m} = R_n S_{m+1} + (R_{n-1} + R_{n-2} + 2R_{n-3}) S_m + (R_{n-1} + 2R_{n-2}) S_{m-1} + 2R_{n-1} S_{m-2}.$$

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