Int. J. Adv. Appl. Math. and Mech. 2(3) (2015) 1 - 14 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

Stability analysis of an SEIR epidemic model with non-linear saturated incidence and temporary immunity

Research Article

Muhammad Altaf Khan[1], *, Abdul Wahid[2], Saeed Islam[3], Ilyas Khan[4], Sharidan Shafie[5], Taza Gul[1]

[1] *Department of Mathematics, Abdul Wali Khan University Mardan, 23200, Mardan, Pakistan*

[2] *Department of Mathematics, Islamia College University Peshawar, KP, 25000, Peshawar, Pakistan*

[3] *Department of Mathematics, Abdul Wali Khan University Mardan, 23200, Mardan, Pakistan*

[4] *College of Engineering Majmaah University, Majmaah, Kingdom of Saudi Arabia, 11952, Majmaah, Saudi Arabia*

[5] *Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, Skudai, Johor, 81310, Malaysia*

Received 02 February 2015; accepted (in revised version) 28 February 2015

Abstract: In this paper, we consider an SEIR model with non-linear saturated incidence rate and temporary immunity. First, we formulate the model and find its basic reproduction number. For the basic reproduction number $R_0 < 1$, the disease free equilibrium is stable locally as well as globally. In case $R_0 > 1$, the model at the endemic equilibrium point is stable locally as well as globally. Finally, we find the numerical solution of the model which justified the analytical results.

MSC: 92D25 • 49J15 • 93D20

Keywords: SEIR epidemic model • Non-linear incidence• Basic Reproduction Number • Global stability • numerical results

1. Introduction

The modeling of an infectious diseases in host population has a great practical value which predicts and controls the spreading of disease in a community like dengue, West Nile virus, leptospirosis, HIV, hepatitis B and Avian influenza etc [1, 2]. The battle between a human being and a disease has been heavily lopsided through history. The mathematical models not only provides the complicated non-linear process of disease dynamics but also inform the public health about the disease status (spread or control)[3–5].

Much work has been done on the disease transmission such as [6]. They studied a SEIR model with varying population and its global stability. They obtained three threshold parameters to govern disease eradication, which involves the total number of the infected and their proportion in the population, also the reference therein. [7] studied a SEIR epidemic model with seasonal forcing in transmission rate. They consider several varying parameters to analyze the dynamical behavior of the model. [8] studied a SEIR model with saturation contact rate of the individual's contact rate. Using the basic reproduction number the main result is derived. [9] presented a SEIR model with constant inflow of more susceptible, exposed, infected and recovered individuals. They also studied a population size individual dependent on contact rate and the resultant death rate. [10] studied a SEIR model to control the disease by giving pseudo-rational exemption to vaccination, the comparison between the steadily declining risk of infection, and the perceived risk of side effects from the vaccine. The literature of SEIR model is

* Corresponding author.
E-mail address: altafdir@gmail.com

broad. References for the readers [11–20] are given.

In this work, we proposed a mathematical model, which investigate a vaccination strategies for the disease control into a single host population. We assumed that the disease is transmitted horizontally, which can be in the form like direct or indirect contact. The direct contact such as touching, biting, licking, and the indirect contact (vectors or fomites) with no physical contact. We assumed the newborn to be susceptible to the disease such as (Cholera, Polio and Hepatitis A, etc). This study focusses on a SEIR epidemic model of disease transmission with non-linear saturated incidence rate. First, we formulate the model and define its parameters. The model govern a non-linear system of differential equation. Then, we obtain the basic reproduction number called the threshold quantity R_0. Afterwards, we find the local stability of disease free equilibrium (DFE) in case when the threshold quantity $R_0 \leq 1$, an unstable equilibrium exists when the threshold quantity $R_0 \leq 1$. Subsequently, we find the local asymptotical stability of endemic equilibrium for the case when $R_0 > 1$, a stable equilibrium exists. Further, the model is globally asymptotically stable for $R_0 \leq 1$, it is found that the endemic equilibrium is globally stable for $R_0 > 1$. The model is solved numerically by using an iterative numerical technique, which justifies the theoretical results. Finally, we wind up our work with conclusion and references.

The organization of the paper is as follows: In section 2, the basic formulation of the model is presented. The properties of the model, the disease free equilibria, endemic equilibria and the basic reproduction number are presented in section 3. The local stability of disease free and endemic equilibrium are presented in section 4. In section 5, the global stability of disease free and endemic equilibrium are discussed. Finally in section 6, a brief discussion is presented.

2. Mathematical formulation

In the current section, we formulate the epidemic SEIR model with non-linear saturated incidence rate. The total population is divided into four subclasses i.e susceptible-$S(t)$, the exposed-$E(t)$, the infected are denoted by $I(t)$, and the recovered ones $R(t)$. The total population is denoted by $N(t)$. We assume that the total population is constant. New born babies are assumed susceptible. There is no migration of the individuals. A flow diagram is given in Fig. 1. The system that governs the differential equation is given by:

$$\frac{dS}{dt} = \Lambda - \frac{\lambda(1-a)I(t)S(t)}{1+\alpha_1 I(t)} - (1-\delta)a\frac{\lambda S(t)I(t)}{1+\alpha_1 I(t)} - (\delta a+\mu)S(t),$$

$$\frac{dE}{dt} = \frac{\lambda(1-a)I(t)S(t)}{1+\alpha_1 I(t)} + (1-\delta)a\frac{\lambda S(t)I(t)}{1+\alpha_1 I(t)} - (\mu+\epsilon)E(t),$$

$$\frac{dI}{dt} = \epsilon E(t) - (\alpha+\beta+\mu)I(t),$$

$$\frac{dR}{dt} = \delta a S(t) + \beta I(t) - \mu R(t), \tag{1}$$

Subject to initial conditions

$$S(0) = S_o \geq 0,\ E(0) = E_o \geq 0,\ I(0) = I_o \geq 0,\ R(0) = R_o \geq 0. \tag{2}$$

Here, the population growth rate is denoted by Λ. The natural death rate for an individual is shown by μ. The parameter α_1 represents the saturated non-linear incidence rate. The exposed individuals are infected at the rate of ϵ and move to the class of the infected. The recovered people at the rate of β move to the recovered class $R(t)$. The susceptible individuals vaccinated at a constant per capita rate $a(0 \leq a < 1)$. The partial efficacy of vaccine, only $\delta(0 \leq \delta \leq 1)$ fraction of the vaccinated susceptible are recovered. The parameter α shows the resultant death rate at infected class. λ represents the contact rate between susceptible and the infected. The total dynamics of the population is given by

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t) - \alpha I(t). \tag{3}$$

In the next section, we find the basic reproduction number and equilibria of the system (1).

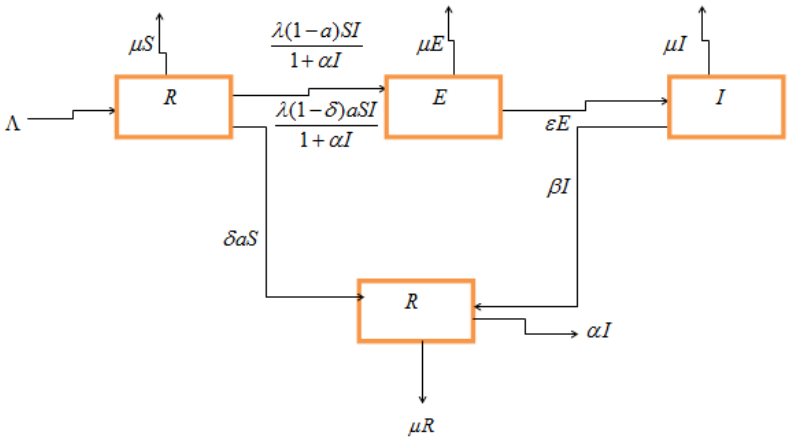

Fig. 1. The graph shows the transfer diagram.

3. Basic reproduction number and equilibrium points

This section investigates the basic reproduction number R_0 for the system (1) in feasible steady states. Observing that the total population size $N(t)$ satisfies the equation $\frac{dN(t)}{dt} = \Lambda - \mu N(t) - \alpha I(t) \leq \Lambda - \mu N(t)$, in the absence of death rate at infected class, so the biological feasible region is given by

$$\Delta = \left\{(S,E,I,R): 0 \leq S,E,I,R, S+E+I+R \leq \frac{\Lambda}{\mu}\right\} \tag{4}$$

is invariant positively for the model (1). So, the only solution with the associated initial conditions will be considered inside the region Δ, where the uniqueness of solutions, usual existence, and continuation results are satisfied.

The system (1) has always the disease free state:

$$E^o = (S^o, 0, 0, 0), \quad S^o = \frac{\Lambda}{(\delta a + \mu)},$$

which represents the level of susceptible populations in the absence of infection.

The basic reproduction number, R_0, is defined as the average number of secondary infections generated by a single infection, which introduced into a purely susceptible population [24]. The basic reproduction number, also called the threshold quantity, is one of the most useful threshold quantity which mathematically characterizes spreading of the disease. This metric is beneficial because it helps to find whether or not an infectious disease will spread through the population. For disease free equilibrium, the threshold quantity $R_0 \leq 1$ holds, the equilibrium is stable and there is no disease spread. When its value exceeds unity, the disease permanently exists in the community and becomes an epidemic. If the threshold quantity exceeds unity, the endemic equilibrium is stable locally and globally. We obtain the basic reproduction number similarly as in [24]. Suppose $x = (E, I)$, then from system (1), it follows:

$$\frac{dx}{dt} = \mathscr{F} - \mathscr{V},$$

where

$$\mathscr{F} = \begin{bmatrix} \frac{\lambda(1-a)SI}{1+\alpha_1 I} + \frac{(1-\delta)a\lambda SI}{1+\alpha_1 I} \\ 0 \end{bmatrix} \text{ and } \mathscr{V} = \begin{bmatrix} (\mu+\epsilon)E(t) \\ -\epsilon E(t) + (\alpha+\beta+\mu)I(t) \end{bmatrix}.$$

We get

$$F = \text{ Jacobian of } \quad F \quad \text{ at } \quad (DFE) = \begin{bmatrix} 0 & \lambda(1-a)S^o + a(1-\delta)\lambda S^o \\ 0 & 0 \end{bmatrix}$$

and

$$V = \text{ Jacobian of } \quad V \quad \text{ at } \quad (DFE) = \begin{bmatrix} \mu+\epsilon & 0 \\ -\epsilon & \alpha+\beta+\mu \end{bmatrix}$$

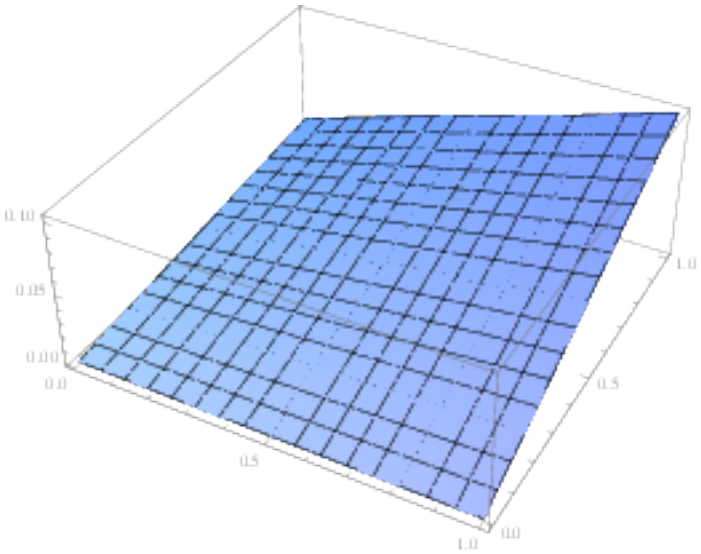

Fig. 2. The graph shows the transfer diagram.

$$V^{-1} = \begin{bmatrix} \frac{1}{\epsilon+\mu} & 0 \\ \frac{\epsilon}{(\alpha+\beta+\mu)(\epsilon+\mu)} & \frac{1}{(\alpha+\beta+\mu)} \end{bmatrix}$$

and

$$T = FV^{-1} = \begin{bmatrix} 0 & \lambda(1-a)S^o + a(1-\delta)\lambda S^o \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\epsilon+\mu} & 0 \\ \frac{\epsilon}{(\alpha+\beta+\mu)(\epsilon+\mu)} & \frac{1}{(\alpha+\beta+\mu)} \end{bmatrix}$$

So, the next generation matrix for the system (1) is given by

$$T = FV^{-1} = \begin{bmatrix} \frac{\lambda(1-a)S^o\epsilon + a\epsilon\lambda S^o(1-\delta)}{(\alpha+\beta+\mu)(\epsilon+\mu)} & \frac{\lambda(1-a)S^o + a\lambda S^o(1-\delta)}{\alpha+\beta+\mu} \\ 0 & 0 \end{bmatrix}$$

Again, the spectral radius R_0 of the matrix $K = FV^{-1}$, is the basic reproduction number of the model,i.e, $R_0 = \rho(FV^{-1})$, so

$$R_0 = \frac{\lambda(1-a)S^o\epsilon + a\epsilon\lambda S^o(1-\delta)}{(\alpha+\beta+\mu)(\epsilon+\mu)} = \frac{\lambda(1-a)\frac{\Lambda}{(\delta a+\mu)}\epsilon + a\epsilon\lambda\frac{\Lambda}{(\delta a+\mu)}(1-\delta)}{(\alpha+\beta+\mu)(\epsilon+\mu)},$$

$$= \frac{\Lambda\lambda\epsilon(1-a\delta)}{(\delta a+\mu)(\alpha+\beta+\mu)(\epsilon+\mu)} \tag{5}$$

The endemic equilibrium of the system (1) is given by

$$\widetilde{E} = (S^*, E^*, I^*, R^*$$

$$S^* = \frac{(\mu+\epsilon)(\alpha+\beta+\gamma)(1+\alpha_1 I^*)}{\epsilon\lambda(1-a\delta)} = \frac{\Lambda(1+\alpha_1 I^*)}{(\delta a+\mu)R_0},$$

$$I^* = \frac{(\delta a+\mu)(R_0-1)}{(1-a\delta)+\alpha(\delta a+\mu)},$$

$$E^* = \frac{(\alpha+\beta+\gamma)I^*}{\epsilon},$$

$$R^* = \frac{\delta a(\mu+\epsilon)(\alpha+\beta+\gamma)(1+\alpha_1 I^*)+\epsilon\lambda(1-a\delta)\beta I^*}{\mu\epsilon\lambda(1-a\delta)}$$

Further, the endemic equilibrium points is written in the following form:

$$S^* = \frac{\Lambda[(1-a\delta)+\alpha(\delta a+\mu)+\Lambda\alpha(\delta+\mu)(R_0-1)]}{(\delta+\mu)[(1-a\delta)+\alpha(\delta a+\mu)]},$$

$$E^* = \frac{(\alpha+\beta+\gamma)(\delta a+\mu)(R_0-1)}{\epsilon[(1-a\delta)+\alpha(\delta a+\mu)]},$$

$$I^* = \frac{(\delta a+\mu)(R_0-1)}{(1-a\delta)+\alpha(\delta a+\mu)},$$

$$R^* = \frac{\Lambda\delta a}{\mu R_0(\delta a+\mu)} + \frac{(\alpha\Lambda\delta a+\beta R_0)(\delta a+\mu)(R_0-1)}{\mu R_0[(1-a\delta)+\alpha(\delta a+\mu)]}, \tag{6}$$

The positive endemic equilibrium exists, if $R_0 > 1$. Figs. 2 and 3 shows the basic reproduction number for the values assumed in Table 1. Fig. 3, is the density plot of the basic reproduction number.

4. Local stability

We find the local stability of the disease free and endemic states in current section. To do this, setting the left hand side of the system (1) equal to zero at the point $E^o = (S^o, 0, 0, 0)$, we present the local stability of disease free equilibrium in the following theorem.

Theorem 4.1.
The disease free equilibrium at E^o is
(i) stable locally asymptotically if, $R_0 < 1$,
(ii)unstable equilibrium exists if, otherwise.

Proof. The variational matrix about the disease free equilibrium point E^o is given by

$$J_0 = \begin{bmatrix} -(\delta a+\mu) & 0 & -\delta(1-a)S^o-(1-\delta)a\lambda S^o & 0 \\ 0 & -(\mu+\epsilon) & \delta(1-a)S^o+(1-\delta)a\lambda S^o & 0 \\ 0 & \epsilon & -(\alpha+\beta+\mu) & 0 \\ \delta a & 0 & \beta & -\mu \end{bmatrix}.$$

Where $Q_1 = -(\mu+\epsilon)(\alpha+\beta+\mu)+\epsilon\lambda(1-a)S^o+\epsilon\lambda aS^o(1-\delta)$, $Q_2 = (\mu+\epsilon)(\delta a+\mu)\beta-\delta a(\mu+\epsilon)(\lambda(1-a)S^o+(1-\delta)a\lambda S^o)$ and $Q_3 = -\mu(\delta a+\mu)(\mu+\epsilon)Q_1$.
The characteristics equation to the Jacobian matrix J_0 is given by

$$(-(\delta a+\mu)-\lambda)(-(\mu+\epsilon)-\lambda)(Q_1-\lambda)(Q_3-\lambda) = 0,$$

The eigenvalues associated to J_0 are, $\lambda_1 = -(\delta a+\mu) < 0$, $\lambda_2 = -(\mu+\epsilon) < 0$, $\lambda_3 = Q_1$. Using the value of Q_1 we obtain,

$$\Lambda\epsilon\lambda(1-a\delta)(R_0-1) < 0.$$

For $\lambda_4 = Q_3$, we use the value of Q_3, is $\lambda_4 = -\mu(\delta a+\mu)(\mu+\epsilon)Q_1$, further use of Q_1 leads to

$$\mu(\delta a+\mu)(\mu+\epsilon)\Lambda\epsilon\lambda(1-a\delta)(R_0-1) < 0.$$

For $R_0 < 1$, it is clear the matrix J_0 has negative real parts. So, the disease free equilibrium is stable locally asymptotically when $R_0 < 1$, and unstable equilibrium exits otherwise. The proof of part (i) is completed. For the proof of part (ii), if $R_0 > 1$, there exists two positive and two negative eigenvalues, so an unstable equilibrium exists other than $R_0 < 1$. The proof of part (ii) completed. □

Further, we find the local asymptotical stability of the system (1), it is better and reasonable that if we reduce the system (1). Omit the fourth equation of system (1), because it is independent of the rest of the system. The reduced system gives the same information as to the system (1), we will better understand the reduced system and their properties. The reduced system is given by,

$$\frac{dS}{dt} = \Lambda - \frac{\lambda(1-a)I(t)S(t)}{1+\alpha_1 I(t)} - (1-\delta)a\frac{\lambda S(t)I(t)}{1+\alpha_1 I(t)} - (\delta a+\mu)S(t),$$

$$\frac{dE}{dt} = \frac{\lambda(1-a)I(t)S(t)}{1+\alpha_1 I(t)} + (1-\delta)a\frac{\lambda S(t)I(t)}{1+\alpha_1 I(t)} - (\mu+\epsilon)E(t),$$

$$\frac{dI}{dt} = \epsilon E(t) - (\alpha+\beta+\mu)I(t), \tag{7}$$

Theorem 4.2.
For $R_0 > 1$, the endemic equilibrium point E^* is locally asymptotically stable, if the following inequalities are satisfied, otherwise it will be unstable.

Proof. The Jacobian matrix J^* about the endemic equilibrium point E^* is given by,

$$J^* = \begin{bmatrix} -(\delta a+\mu) - \frac{\lambda(1-a)I^*}{1+\alpha_1 I^*} - \frac{(1-\delta)\lambda a I^*}{1+\alpha_1 I^*} & 0 & -\frac{\lambda(1-a)S^*}{(1+\alpha_1 I^*)^2} - \frac{(1-\delta)a\lambda S^*}{(1+\alpha_1 I^*)^2} \\ \frac{\lambda(1-a)I^*}{1+\alpha_1 I^*} + \frac{(1-\delta)\lambda a I^*}{1+\alpha_1 I^*} & -(\mu+\epsilon) & \frac{\lambda(1-a)S^*}{(1+\alpha_1 I^*)^2} + \frac{(1-\delta)a\lambda S^*}{(1+\alpha_1 I^*)^2} \\ 0 & \epsilon & -(\alpha+\beta+\mu) \end{bmatrix}.$$

The Jacobian matrix J^* in the echelon form is given by

$$\begin{bmatrix} -Z_1 & 0 & Z_3 \\ 0 & -Z_1(\mu+\epsilon) & Z_3(Z_1-Z_2) \\ 0 & 0 & Z_4 \end{bmatrix},$$

where

$$Z_1 = (\delta a+\mu) + \frac{\lambda(1-a)I^*}{1+\alpha_1 I^*} + \frac{(1-\delta)\lambda a I^*}{1+\alpha_1 I^*},$$

$$Z_2 = \frac{\lambda(1-a)I^*}{1+\alpha_1 I^*} + \frac{(1-\delta)\lambda a I^*}{1+\alpha_1 I^*},$$

$$Z_3 = \frac{\lambda(1-a)S^*}{(1+\alpha_1 I^*)^2} + \frac{(1-\delta)a\lambda S^*}{(1+\alpha_1 I^*)^2},$$

$$Z_4 = Z_1(\alpha+\beta+\mu)(\mu+\epsilon) + \epsilon Z_3(Z_1-Z_2).$$

The characteristics equation related to the Jacobian matrix J^*,

$$P(\hat{\lambda}) = (-Z_1-\hat{\lambda})(-Z_1(\mu+\epsilon)-\hat{\lambda})(Z_4-\hat{\lambda}) = 0,$$

The eigenvalues associated to the J^* are

$$\hat{\lambda_1} = -\{(\delta a+\mu) + \frac{\lambda(1-a)I^*}{1+\alpha_1 I^*} + \frac{(1-\delta)\lambda a I^*}{1+\alpha_1 I^*}\} < 0,$$

$$\hat{\lambda_2} = -\{(\mu+\epsilon)(\delta a+\mu) + \frac{\lambda(1-a)I^*}{1+\alpha_1 I^*} + \frac{(1-\delta)\lambda a I^*}{1+\alpha_1 I^*}\} < 0,$$

$$\hat{\lambda_3} = Z_4.$$

The first two eigenvalues have negative real parts, the third eigenvalues are calculated as

$$\hat{\lambda_3} = -\frac{\lambda I(1-a\delta)}{(1+\alpha_1 I)}\left(\frac{\epsilon\Lambda\alpha}{R_0} + (\alpha+\beta+\mu)(\mu+\epsilon)\right) < 0.$$

Thus, all the eigenvalues associated to the Jacobian matrix J^* have negative real parts and so the system (1) is stable locally asymptotically. □

5. Global stability of DFE and EE

In this section, we obtain the global stability of the disease free and endemic equilibrium. First, we discuss the global stability of the disease free by using the method of Castillo-Chavez et al. [21]. Then, the geometric approach method will be used to obtain the global stability of the endemic equilibrium. For the disease free states, we consider the two conditions which guarantee the global stability of the disease-free state. We rewrite the model (7) as

$$\frac{dX}{dt} = F(X,Z),$$

$$\frac{dZ}{dt} = G(X,Z),\ \ G(X,0)=0,$$

where $X = S$ represent the number of uninfected individuals with $X \in R$, and $Z = (E,I)$ with $Z \in R^2$ shows the number of infected individuals, includes latent and infected. We denote the disease-free state by $T^o = (X^o,0)$. The following two conditions $\mathscr{H}_1$ and $\mathscr{H}_2$ must be met to guarantee a local asymptotic stability:

$$(\mathscr{H}_1)\quad For \frac{dX}{dt} = F(X,0),\ \ X^o\ is\ globally\ asymptotically\ stable,$$

$$(\mathscr{H}_2)\quad G(X,Z) = BZ - G(X,Z,),\ where\ G(X,Z) \geq 0, for\ (X,Z) \in Q,$$

where $B = D_Z G(X^o,0)$ is an M-matrix (the off-diagonal elements of B are non-negative) and Δ is the biological feasible region. We present the following lemma:

Lemma 5.1.
The point $T_0 = (X^0,0)$ called the fixed point is know to be stable globally asymptotic equilibrium of (12), with the addition of $R_0 < 1$ and the conditions $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$ are fulfilled. The following theorem is presented:

Theorem 5.1.
Let $R_0 < 1$. Then the disease free equilibrium at E^o is globally asymptotically stable.

Proof. Let $X = (S)$ and $Z = (E,I)$, and $T^o = (X^o,0)$ where $X^o = \frac{\Lambda}{\delta a+\mu}$.
Then

$$\frac{dX}{dt} = F(X,Z) = \Lambda - \frac{\lambda(1-a)I(t)S(t)}{1+\alpha_1 I(t)} - (1-\delta)a\frac{\lambda S(t)I(t)}{1+\alpha_1 I(t)} - (\delta a+\mu)S(t).$$

At $S = S^o$, $F(X,0) = 0$, and $\frac{dX}{dt} = F(X,0) = \frac{\Lambda}{\delta a+\mu}$.
As $t \to \infty$, $X \to X^o$. So, $X = X^0(S^o)$ is globally asymptotically stable.
Now

$$G(X,Z) = \begin{bmatrix} -(\mu+\epsilon) & \lambda(1-a)S^o + \lambda(1-\delta)aS^o \\ \epsilon & -(\alpha+\beta+\mu) \end{bmatrix} \begin{bmatrix} E \\ I \end{bmatrix}$$

$$- \begin{bmatrix} \lambda(1-a)S^o I + \lambda(1-\delta)aS^o I - \frac{\lambda(1-a)SI}{1+\alpha_1 I} - \frac{\lambda(1-\delta)aSI}{1+\alpha_1 I} \\ 0 \end{bmatrix} = BZ - \widehat{G}(X,Z),$$

where

$$B = \begin{bmatrix} -(\mu+\epsilon) & \lambda(1-a)S^o + \lambda(1-\delta)aS^o \\ \epsilon & -(\alpha+\beta+\mu) \end{bmatrix} and$$

$$\widehat{G}(X,Z) = \begin{bmatrix} \lambda(1-a)I\left(S^o - \frac{S}{1+\alpha_1 I}\right) + \lambda(1-\delta)aI\left(S^o - \frac{S}{1+\alpha_1 I}\right) \\ 0 \end{bmatrix}.$$

In the reduced system, the population is bounded by $S^o = \frac{\Lambda}{\delta a+\mu}$ and $S,E,I \leq S^o$ and $S^o \geq \frac{S}{1+\alpha_1 I}$ at E^o and thus $\widehat{G}(X,Z) \geq 0$. Obviously B is an M-matrix, so the conditions $\mathscr{H}_1$ and $\mathscr{H}_2$ are satisfied, so by lemma 5.1, the disease free equilibrium E^o is globally asymptotically stable, provided that $R_0 < 1$. □

5.1. Global stability of endemic equilibrium

Theorem 5.2.
If $R_0 > 1$, then the system (7), at endemic equilibrium E^* is globally asymptotically stable, and unstable otherwise.

Proof. To show this result, we find the second additive compound matrix $J^{[2]}$ for the reduced system (7) at E^* in the following

$$J^{[2]} = \begin{bmatrix} A_{11} & -\left(\frac{\lambda(1-a)S}{(1+\alpha_1 I)^2} + \frac{(1-\delta)a\lambda S}{(1+\alpha_1 I)^2}\right)\frac{I}{E} & \left(\frac{\lambda(1-a)S}{(1+\alpha_1 I)^2} + \frac{(1-\delta)a\lambda S}{(1+\alpha_1 I)^2}\right)\frac{I}{E} \\ \epsilon\frac{E}{I} & A_{22} & 0 \\ 0 & \frac{\lambda(1-a)I}{1+\alpha_1 I} + \frac{(1-\delta)a\lambda I}{1+\alpha_1 I} & A_{33} \end{bmatrix},$$

where

$$A_{11} = -(\delta a+\mu) - \frac{\lambda(1-a)I}{1+\alpha_1 I} + \frac{(1-\delta)a\lambda I}{1+\alpha_1 I} - (\mu+\epsilon),$$

$$A_{22} = -(\delta a+\mu) - \frac{\lambda(1-a)I}{1+\alpha_1 I} + \frac{(1-\delta)a\lambda I}{1+\alpha_1 I} - (\alpha+\beta+\mu),$$

$$A_{33} = -(\mu+\epsilon) + (\alpha+\beta+\mu),$$

and

$$P = P(S,E,I) = diag(1, \frac{E}{I}, \frac{E}{I}),$$

with

$$P^{-1} = (1, \frac{I}{E}, \frac{I}{E})$$

and

$$P_f = (0, \frac{\dot{E}}{I} - \frac{E\dot{I}}{I^2}, \frac{\dot{E}}{I} - \frac{E\dot{I}}{I^2}),$$

so

$$P_f P^{-1} = (0, \frac{\dot{E}}{E} - \frac{\dot{I}}{I}, \frac{\dot{E}}{E} - \frac{\dot{I}}{I}).$$

and

$$\widehat{B} = P_f P^{-1} + P J^{[2]} P^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\dot{E}}{E} - \frac{\dot{I}}{I} & 0 \\ 0 & 0 & \frac{\dot{E}}{E} - \frac{\dot{I}}{I} \end{bmatrix} + \begin{bmatrix} A_{11} & -\frac{\lambda(1-a)S}{(1+\alpha_1 I)^2} - \frac{(1-\delta)a\lambda S}{(1+\alpha_1 I)^2} & -\frac{\lambda(1-a)S}{(1+\alpha_1 I)^2} - \frac{(1-\delta)a\lambda S}{(1+\alpha_1 I)^2} \\ \epsilon & A_{22} & 0 \\ 0 & \frac{\lambda(1-a)I}{1+\alpha_1 I} + \frac{(1-\delta)a\lambda I}{1+\alpha_1 I} & A_{33} \end{bmatrix}.$$

Let

$$\widehat{B} = \begin{pmatrix} \widehat{B}_{11} & \widehat{B}_{12} \\ \widehat{B}_{21} & \widehat{B}_{22} \end{pmatrix}$$

$$\widehat{B}_{11} = A_{11},$$

$$\widehat{B}_{12} = \left[\frac{\lambda(1-a)S}{(1+\alpha_1 I)^2} + \frac{(1-\delta)a\lambda S}{(1+\alpha_1 I)^2}, \frac{\lambda(1-a)S}{(1+\alpha_1 I)^2} + \frac{(1-\delta)a\lambda S}{(1+\alpha_1 I)^2}\right]\frac{I}{E},$$

$$\widehat{B}_{21} = [\epsilon, 0]^T,$$

$$\widehat{B}_{22} = \begin{bmatrix} A_{22} + \frac{\dot{E}}{E} - \frac{\dot{I}}{I} & 0 \\ \frac{\lambda(1-a)I}{1+\alpha_1 I} \frac{(1-\delta)a\lambda I}{1+\alpha_1 I} & \frac{\dot{E}}{E} - \frac{\dot{I}}{I} + A_{33} \end{bmatrix}$$

Suppose $(\hat{w}_1, \hat{w}_2, \hat{w}_3)$ represent the vector in R^3 and the associated norm is $||.||$, defined as

$$||(\hat{w}_1, \hat{w}_2, \hat{w}_3)|| = max\{|\hat{w}_1|, |\hat{w}_2| + |\hat{w}_3|\}.$$

Suppose $\mu\widehat{B}$ represents Lozinski measure with the above defined norm, so as described in [22], we choose

$$\mu(\widehat{B}) \leq \sup(g_1,\ g_2),$$

where

$$g_1 = \mu(\widehat{B}_{11}) + |\widehat{B}_{12}|, \qquad g_2 = |\widehat{B}_{21}| + \mu(\widehat{B}_{22})$$

$|\widehat{B}_{21}|$ and $|\widehat{B}_{12}|$ are the matrix norm with respect to vector ℓ and μ_1 represent the Lozinski measure with respect to this ℓ norm, then

$$\mu(\widehat{B}_{11}) = -(\delta a + \mu) - \frac{\lambda(1-a)I}{1+\alpha_1 I} - \frac{(1-\delta)a\lambda I}{1+\alpha_1 I} - (\mu+\epsilon),$$

and

$$\mu(\widehat{B}_{22}) = max\left\{A_{22} + \frac{\dot{E}}{E} - \frac{\dot{I}}{I}, A_{33}, [\frac{\lambda(1-a)I}{1+\alpha_1 I}\ \frac{(1-\delta)a\lambda I}{1+\alpha_1 I}]\right\}$$

$$g_1 = \mu(\widehat{B}_{11}) + |\widehat{B}_{12}|,$$

$$= -(\delta a + \mu) - \frac{\lambda(1-a)I}{1+\alpha_1 I} - \frac{(1-\delta)a\lambda I}{1+\alpha_1 I} - (\mu+\epsilon) + \frac{\lambda(1-a)S}{(1+\alpha_1 I^2)}\frac{I}{E} + \frac{\lambda(1-\delta)aS}{(1+\alpha_1 I^2)}\frac{I}{E},$$

$$\leq \frac{\dot{E}}{E} - (\delta a + \mu) - \frac{\lambda(1-a)I}{1+\alpha_1 I} - \frac{(1-\delta)a\lambda I}{1+\alpha_1 I}, \text{ using second equation of system (7)},$$

$$= \frac{\dot{E}}{E} - \delta a + \mu,$$

$$= \frac{\dot{E}}{E} - \mu.$$

and

$$g_2 = \mu(\widehat{B}_{22}) + |\widehat{B}_{21}|,$$

$$= \frac{\epsilon E}{I} + \frac{\dot{E}}{E} - \frac{\dot{I}}{I} - (\mu+\epsilon) + max[A_{22}, A_{33}],$$

$$= \frac{\epsilon E}{I} + \frac{\dot{E}}{E} - \frac{\dot{I}}{I} - (\mu+\epsilon) - (\delta a + \mu) - \frac{\lambda(1-a)I}{1+\alpha_1 I} - \frac{\lambda(1-\delta)a\lambda I}{1+\alpha_1 I} - (\alpha+\beta+\mu),$$

$$\leq \frac{\dot{E}}{E} - \mu - \epsilon - (\delta a + \mu) - \frac{\lambda(1-a)I}{1+\alpha_1 I} - \frac{\lambda(1-\delta)a\lambda I}{1+\alpha_1 I} \text{ Using third equation of system (7)},$$

$$= \frac{\dot{E}}{E} - \mu.$$

so

$$\mu\widehat{B} \leq \sup(g_1, g_2) \leq \frac{\dot{E}}{E} - \mu,$$

then

$$q = \frac{1}{t}\int_0^t \mu\widehat{B}\,ds \leq \frac{1}{t}\int_0^t \left(\frac{\dot{E}}{E} - \mu\right)ds = \frac{1}{t} ln \frac{E(\dot{t})}{E(0)} - \mu$$

implies that $q \leq -\frac{\mu}{2} < 0$. Thus the result [23], implies that the endemic equilibrium E^* of the reduced system (7) is globally asymptotically stable. □

6. Numerical results and conclusion

In this section, we present the numerical simulation of the proposed model (1), using the numerical technique, RK-4 to obtain their numerical results. The parameters used in the numerical simulation, their values are presented in Table 1. Throughout this simulation, the susceptible individuals are shown by bold red, exposed by blue, infected with dashed and recovered with dotted line. The numerical results are presented in the form of graphics, from Figs. 3 to 8.The disease contact rate slowly decreases with decrease in the value of λ(the contact rate). In this work, we have successfully presented an epidemic model of SEIR model with nonlinear saturated incidence rate. Initially, the formulation of the problem obtained, then we obtained the basic reproduction number. The disease free equilibrium for $R_0 < 1$, found to be stable both locally and globally. Further, the stability of endemic equilibrium were presented. Its found that the endemic equilibrium is stable both globally and locally when $R_0 > 1$. Finally, the theoretical results were justified by the numerical simulation.

Table 1. Estimated parameters.

Notation	Parameter Description	Value	Dimension
Λ	Birth rate of the population	1.2	$days^{-1}$
λ	Disease contact rate	0.8	$days^{-1}$
μ	Natural death rate	0.03	$days^{-1}$
α_1	Saturation constants	0.07	$days^{-1}$
a	partial immunity among susceptible individuals	0.008	$days^{-1}$
ϵ	Movement of exposed individuals to infected class	0.03	$days^{-1}$
δ	Individuals with no immunity	0.2	$days^{-1}$
ϵ	Resulting death rate	0.089	$days^{-1}$
β	Rate of recovery	0.007	$days^{-1}$

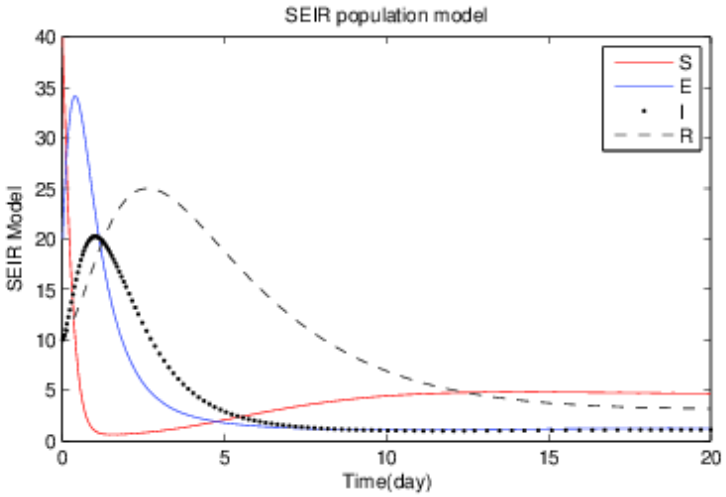

Fig. 3. The graph shows the population of individuals.

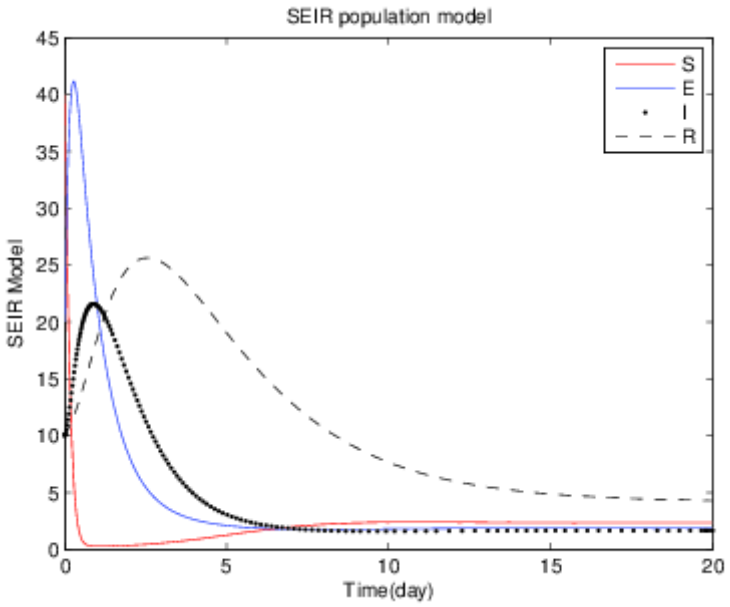

Fig. 4. The graph shows the population of individuals.

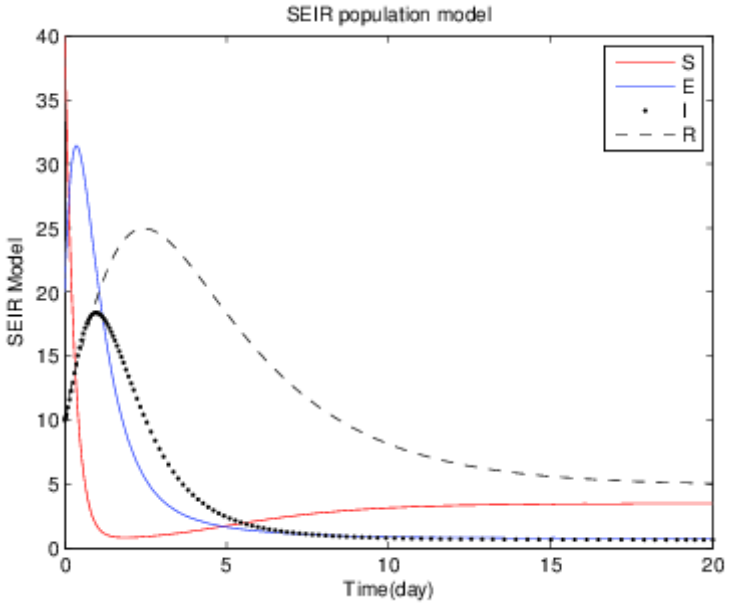

Fig. 5. The graph shows the population of individuals for values $\Lambda = 1.2$, $\lambda = 0.8$, $\delta = 0.08$.

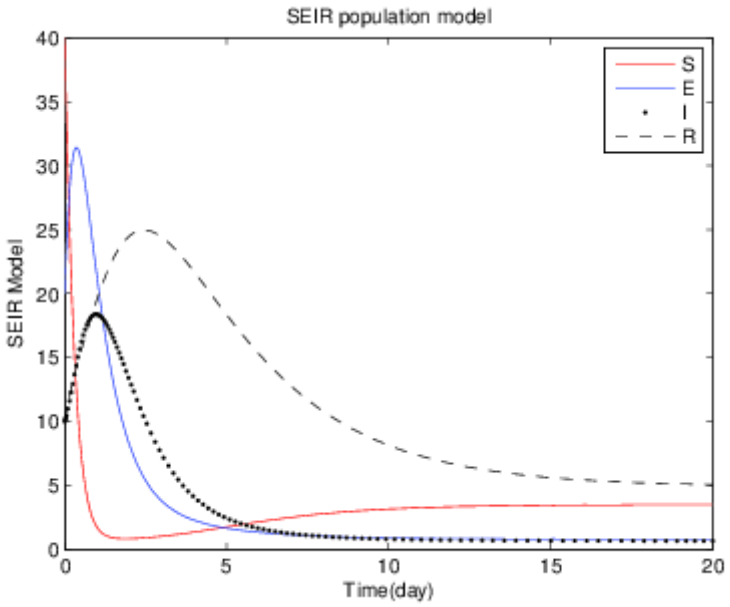

Fig. 6. The graph shows the population of individuals for values $\Lambda = 1.2$, $\lambda = 0.8$, $\delta = 0.08$.

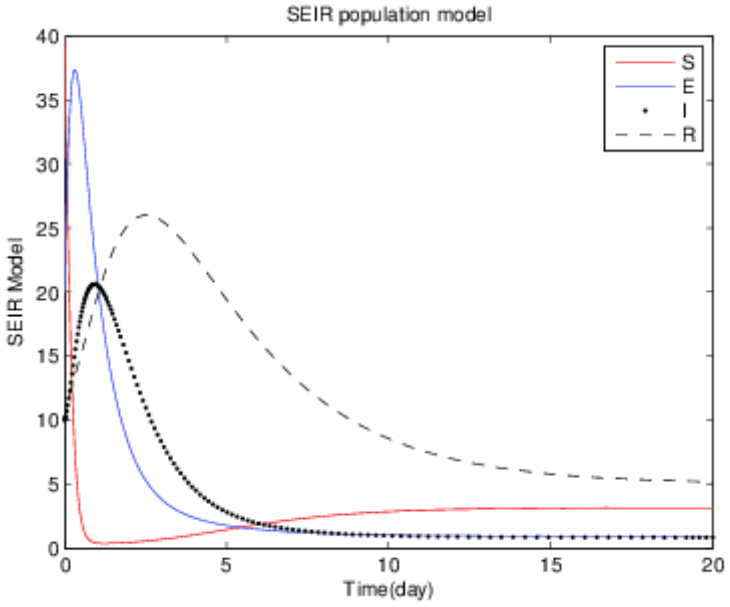

Fig. 7. The graph shows the population of individuals for the values $\Lambda = 1.0$, $\lambda = 0.3$, $\delta = 0.03$ and $\epsilon = 0.4$.

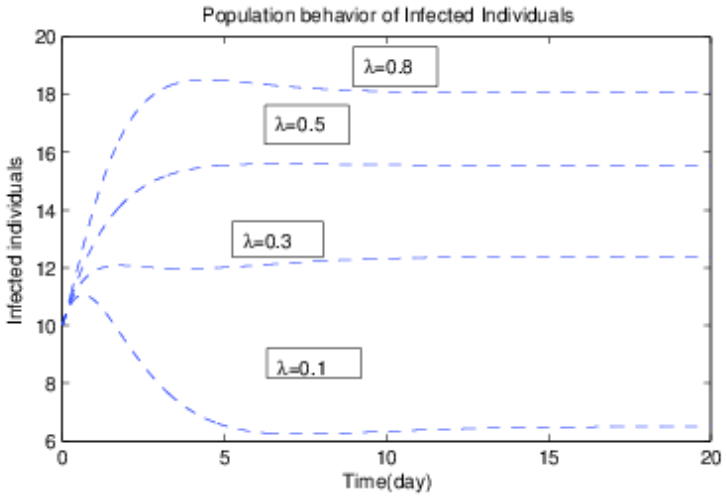

Fig. 8. The graph represents the different values of λ.

Acknowledgments

The author(s) would like to thank University Technologi Malaysia for financial support.

References

[1] A. A. Lashari, S. Aly, K. Hattaf, Gul Zaman,I.l Jung, X.Z. Li, Presentation of Malaria Epidemics Using Multiple Optimal Controls, Jour. Appl. Math. 2012 (2012) 1-17.
[2] G. Zaman, Samir H, Dynamics and control of a system of two non-interacting preys with common predator, Math. Meth. Appl. Sci. 34 (2011) 2259-2273.
[3] F. Fenner, D.A. Henderson, I. Arita, Z. Jezek, I.D. Ladnyi, Smallpox and Its Eradication, WHO, 1998.
[4] R.M. Anderson, R.M. May, Infectious Diseases of Humans, Dynamics and Control, Oxford University Press, Oxford, 1991.
[5] M.A. Garly, P. Aaby, The challenge of improving the efficacy of measles vaccine, Acta Tropica. 85 (2003) 1Ű17.
[6] C. Sun, Y. Hsieh, Global analysis of an SEIR model with varying population size and vaccination, Appl. Math. Model. 34 (2010) 2685-2697.
[7] N. Yi, Q. Zhang, K. Mao, D. Yang, Q. Li, Analysis and control of an SEIR epidemic system with nonlinear transmission rate, Math. Comp. Mod. 50 (2009) 1498-1513.
[8] Juan Zhang, Zhien Ma, Global dynamics of an SEIR epidemic model with saturating contact rate, Math. Bios. 185 (2003) 15-32.
[9] J. Zhang, J. Li, Z. Ma, Global dynamis of an SEIR epidemic model with immigration of treatmnet of different compartments, Act. Math. Sci. 26 (2006) 551-567.
[10] B. Buonomo, A. dŠOnofrio, D. Lacitignola, Modeling of pseudo-rational exemption to vaccination for SEIR diseases, Jour. Math. Anal. Appl. 404 (2013) 385-398.
[11] X. Zhou, Q. Sun, Stability analysis of a fractional-order HBV infection model, Int. J. Adv. Appl. Math. and Mech. 2 (2014) 1-6.
[12] M. A. Khan. S. Islam, S. A. Khan, G. Zaman, Global Stability of Vector-Host Disease with Variable Population Size, Biomed. Res. Int. 2013 (2013) 1-9.
[13] M. A. Khan, S. Islam, M. Arif, Z. Haq, Transmission Model of Hepatitis B Virus with the Migration Effect, Biomed. Res. Int. 2013 (2013) 1-10.
[14] M. A. Khan, S. Islam, S. Shafie, I. Khan, S. A. Khan, Prevention of Leptospirosis Infected Vector and Human Population by Multiple Control Variables, Abs. Appl. Anal. 2014 (2014) 1-9.
[15] N. Özalp, E. Demirci, A fractional order SEIR model with vertical transmission, Math. Comp. Model. 54 (2011) 1-6.
[16] M. De la Sen, S. Alonso-Quesada, Vaccination strategies based on feedback control techniques for a general SEIR-epidemic model, Appl. Math. Comp. 218 (2011) 3888-3904.
[17] X. Wang, Y. Tao, X. Song, Pulse vaccination on SEIR epidemic model with nonlinear incidence rate, Appl. Math. Comp. 210 (2009) 398-404.
[18] X. Zhou, Q. Sun, Stability analysis of a fractional-order HBV infection model, Int. J. Adv. Appl. Math. and Mech. 2 (2014) 1-6.
[19] M. Aslefallah, D. Rostamy, A numerical scheme for solving Space-Fractional equation by finite differences theta-method, 1 (2014) 1-9.
[20] S. P. A. Devi, D. V. kumari, Numerical investigation of slip flow effects on unsteady hydromagnetic flow over a stretching surface with thermal radiation, 1 (2014) 20-32.
[21] C. C. Chavez, Z. Feng, W. Huang, Mathematical Approaches for Emerging and Re-Emerging Infectious Diseases: An Introduction, Springer Verlag, 2002.
[22] R. Martin, Logarithmic norms and projections applied to linear differential systems, J. Math. Anal. Appl. 45 (1974) 432-454.
[23] M.Y. Li, J. S. Muldowney, A geometric approach to global-stability problems, SIAM J. Math. Anal. 27 (1996) 1070-1083.
[24] P. V. Driessche, J. Watmough, Reproduction number and sub-threshold endemic equilibria for compartmental models of disease transmission, Math. Biosci. 180 (2002) 29-48.