Int. J. Adv. Appl. Math. and Mech. 2(3) (2015) 119 - 125 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

Finite element solution to transient asymmetric heat conduction in multilayer annulus

Research Article

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Received 04 December 2014; accepted (in revised version) 11 March 2015

Abstract: This study investigate the heat transfer and thermal stresses for the time-dependent asymmetric heat conduction in a multilayer annulus by using finite element method. The realistic problem of isotropic multilayer annulus subjected to heat flux with internal heat generation and initial temperature is solved. The results of temperature and thermal stress have been computed numerically, illustrated graphically and interpreted technically. These results may be used in many engineering problem like combustion chamber, electric generator, compressor, hydraulic pumps etc.

MSC: 35G30 • 35K20 • 74S05 • 80A20 • 80M10

Keywords: Heat conduction • Multilayer annulus • Finite element method • Finite difference method • Thermal stresses

1. Introduction

In present world engineering applications, multilayer components are widely used due to its advantage of combining physical, mechanical, and thermal properties of different materials. Many applications require a detailed knowledge of transient heat transfer (ie. temperature and heat-flux distribution) within the multilayer component. Heat conduction in multilayer solids has a many applications in engineering field such as buildings, industrial furnaces, nuclear reactors, turbines, rockets, space craft, and high-tech devices like instruments etc. Therefore it has been a topic of continued research interest. Ozisik [1] contains a detailed review of one-dimensional composite media, mainly focus on orthogonal expansions by using Greens functions and Laplace transform techniques. Turner et al. [2] developed the basic idea of finite element method to obtain the approximate solution of a complicated problem by replacing it into a simpler problem rather than obtaining the exact solution. In the finite element method it is often be possible to improve or refine the approximate solution by spending more computational effort. Dechaumphai et al. [3] used finite element analysis for predicting temperatures and thermal stresses of heated products. Noon [4] studied the fully discrete formulation of Galerkin partial artificial diffusion finite element method for solving 2-D coupled Burgers problem using Crank-Nicholson method for the time variable. The numerical results are obtained by using MATLAB are compared with the exact solution.

Sun et al. [5] presented of a problem of transient heat conduction in a one-dimensional three-layer composite slab. The Eigen function expansion solution is compared with a finite difference numerical solution. In the early nineteenth century the manufacturing industry had to face a critical problem of designing an advanced product with complex geometries, multi-material and different types of boundary conditions. Yang et al. [6] has

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established a new thermal stability test for heat conduction in one dimensional multilayer composite solid that have internal heat generation at a rate proportional to the interior temperature. Dhawan et al. [7] had carried comparative study of two dimensional transient heat conduction problems with Galerkin finite element method, Euler modified method, Crank-Nicholson method etc. and determined the temperature decay in aluminum plate. Jain et al. [8] presented an analytical double-series solution for the time-dependent asymmetric heat conduction in a multilayer annulus by Eigen values. Singh [9] developed an analytical solution for one-dimensional time dependent multilayer heat conduction problems by using separation of variable, finite integral transform and applied this method in Eigen value problems. Singh et al. [10] presented an analytical double-series solution for the unsteady heat conduction multilayer cylindrical problem by using Separation of variables method and obtained transient temperature distribution in radial direction. Assume spatially non-uniform and time-independent volumetric heat sources in each layer. Singh et al. [11] solved the time-dependent heat conduction problem in a multilayer annulus by using separation of variables method, finite integral transform method and observed the transverse or radial Eigen values for the solution in polar coordinate system. This study analyzes nuclear fuel rod subjected to time-dependent boundaries or heat sources. Patil et al. [12] solve multidimensional unsteady state heat conduction equation with Dirichlet boundary conditions using finite volume numerical technique. The efficiency of this technique is tested with known analytical solutions and the numerical results obtained. Chen et al. [13] obtained the solution of time-fractional partial differential equations in a multi-layer annulus by using the finite integral transform technique and Laplace transform technique.

Thatoi et al. [14] obtained the solution of one dimensional heat flow problem in steady state by using Finite Difference Method, Finite Volume Method and Finite Element Method and done the comparative analysis with desired exact solution. A Matlab program was used to find the numerical solution. Kulkarni et al. [15] obtained the heat transfer and thermal stress analysis of cylinder due to internal heat generation under steady temperature conditions using integral transform methods. The internal heat generation is taken as cylindrical surface heat source in annular region of linear length of cylinder and is situated concentrically inside the cylinder.

This paper deals with the realistic problem of the thermal stresses of isotropic multilayer annulus with initial temperature T_i. The finite element formulation has been developed for the solution of governing heat conduction equation and thermal stress analysis. The Matlab programming is used to evaluate the temperature and thermal stresses in the multilayer circular annulus. The results of temperature and thermal stress have been computed numerically, illustrated graphically and interpreted technically.

2. Formulation of the problem

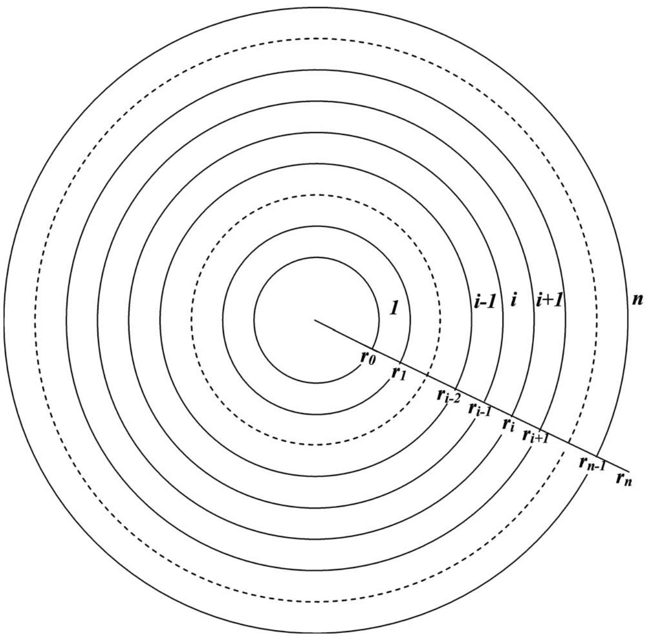

Fig. 1. Schematic diagram of an n-layer annulus.

Consider an n-layer annulus $\{r_0 \le r \le r_n\}$, shown in Fig. 1. All the layers are assumed to be isotropic in thermal properties with perfect thermal contact. Let k_i and α_i be the thermal conductivity and thermal diffusivity of the i^{th} layer and independent on temperature. At initial time, each i^{th} layer is at a specified temperature $f_i(r,\theta)$ and

time-independent heat sources $g_i(r,\theta)$ are switched on for $t > 0$. Both the inner ($i = 1, r = r_0$) as well as the outer ($i = n, r = r_n$) surfaces of the annulus is subjected to any combination of temperature and heat-flux boundary conditions. Since perfect thermal contact between the adjacent layers is seldom observed in real materials, dealing with imperfect contact would require explicit modeling of the thermal resistance at the layer interfaces [[5], [8], [11]]. For such cases, the temperature at the contact interfaces will not be continuous.
The governing heat conduction equation along with the boundary conditions which occupying the space (for $r_{i-1} \le r \le r_i,\ 0 \le \theta \le 2\pi,\ t > 0$, where $i = 1,2.....n$) are given below:

$$\frac{1}{r^2}\frac{\partial^2 T_i(r,\theta,t)}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_i(r,\theta,t)}{\partial r}\right) + \frac{g_i(r,\theta)}{k_i} = \frac{1}{\alpha_i}\frac{\partial T_i(r,\theta,t)}{\partial t} \tag{1}$$

Boundary and initial conditions are given below:
* Inner surface of the first layer (for $0 \le \theta \le 2\pi$ and $t > 0$),

$$A_{in}\frac{\partial T_1(r_0,\theta,t)}{\partial r} + B_{in}T_1(r_0,\theta,t) = C_{in}(\theta) \tag{2}$$

* Outer surface of the n^{th} layer (for $0 \le \theta \le 2\pi$ and $t > 0$),

$$A_{out}\frac{\partial T_n(r_n,\theta,t)}{\partial r} + B_{out}T_n(r_n,\theta,t) = C_{out}(\theta) \tag{3}$$

* Periodic boundary conditions equation (for $r_{i-1} \le r \le r_i$ and $t > 0$, where $i = 1,2.....n$)

$$T_i(r,\theta=0,t) = T_i(r,\theta=2\pi,t) \tag{4}$$

$$\frac{\partial T_i(r,\theta=0,t)}{\partial\theta} = \frac{\partial T_i(r,\theta=2\pi,t)}{\partial\theta} \tag{5}$$

* Interface of the $(i-1)^{th}$ and i^{th} layer equation (for $0 \le \theta \le 2\pi$ and $t > 0$, where $i = 1,2....n$)

$$T_i(r_{i-1},\theta,t) = T_{i-1}(r_{i-1},\theta,t) \tag{6}$$

$$k_i\frac{\partial T_i(r_{i-1},\theta,t)}{\partial r} = k_{i-1}\frac{\partial T_{i-1}(r_{i-1},\theta,t)}{\partial r} \tag{7}$$

* Initial condition (for $r_{i-1} \le r \le r_i, 0 \le \theta \le 2\pi$ and $t = 0$, where $i = 1,2....n$)

$$T_i(r,\theta,t=0) = f_i(r,\theta) \tag{8}$$

Boundary conditions either of the first, second, or third kind may be imposed at $r = r_0$ and $r = r_n$ by choosing the appropriate coefficients in Eqs. (2) and (3). However, the case in which B_{in} and B_{out} are simultaneously zero is not considered. In addition, asymmetric boundary conditions can be applied by choosing dependent C_{in} and C_{out}. Furthermore, multiple layers with zero inner radius ($r_0 = 0$) can be simulated by assigning zero values to constants B_{in} and C_{in} in Eq. (2). Such problem can be solved analytically using finite element method.

3. Solution methodology

3.1. Galerkin finite element approach

Galerkin finite element method is used to convert partial differential equation into algebraic equation [[16], [17], [18]]. Applying Galerkin method to obtain the residual equations corresponding to equation (1) is

$$\int\int\int_V \left\{\frac{1}{r^2}\frac{\partial^2 T_i(r,\theta,t)}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_i(r,\theta,t)}{\partial r}\right) + \frac{g_i(r,\theta)}{k_i} - \frac{1}{\alpha_i}\frac{\partial T_i(r,\theta,t)}{\partial t}\right\}N_j(r)\,r\,dr\,d\theta\,dz$$

$$\forall\ i = 1,2,....n,\ j = 1,2,....M$$

Observing that for the axisymmetric case the integrand is independent of the z co-ordinate equation becomes

$$z\int\int_A \left\{\frac{1}{r^2}\frac{\partial^2 T_i(r,\theta,t)}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_i(r,\theta,t)}{\partial r}\right) + \frac{g_i(r,\theta)}{k_i} - \frac{1}{\alpha_i}\frac{\partial T_i(r,\theta,t)}{\partial t}\right\}N_j(r)\,r\,dr\,d\theta$$

Integrating the first terms with respect to θ and second term with respect to r by parts and rearranging the above equation reduces to

$$z\left\{\int\int A\frac{\partial N_j}{\partial r}\frac{\partial T_i}{\partial r}rdrd\theta+\int\int_A\frac{1}{r}\frac{\partial N_j}{\partial\theta}\frac{\partial T_i}{\partial\theta}drd\theta+\frac{1}{\alpha_i}\int\int_A N_j\frac{\partial T_i}{\partial t}rdrd\theta\right\}$$
$$=z\left\{\frac{1}{k_i}\int\int_A g_iN_jrdrd\theta+\int_{r=r_0}^{r_n}\left\{\frac{1}{r}N_j\frac{\partial T_i}{\partial\theta}\right\}_{\theta=0}^{2\pi}dr+\int_{\theta=0}^{2\pi}\left\{rN_j\frac{\partial T}{\partial r}\right\}_{r=r_0}^{r_n}d\theta\right\}$$

Applying the boundary conditions (2) to (7) the above equation becomes

$$z\left\{\int\int_A\frac{\partial N_j}{\partial r}\frac{\partial T_i}{\partial r}rdrd\theta+\frac{1}{\alpha_i}\int\int_A N_j\frac{\partial T_i}{\partial t}rdrd\theta\right\}$$
$$=z\left\{\frac{1}{k_i}\int\int_A g_iN_jrdrd\theta+\int_{\theta=0}^{2\pi}N_j\frac{r_n}{A_{out}}\{C_{out}(\theta)-B_{out}T_n(r_n,\theta,t)\}d\theta-\int_{\theta=0}^{2\pi}N_j\frac{r_0}{A_{in}}\{C_{in}(\theta)-B_{in}T_1(r_0,\theta,t)\right\}d\theta$$

Substitute the temperature distribution for two node linear element in the above equation $T(r,t)=N_1(r)T_1+N_2(r)T_2=[N]^TT$ and $\frac{\partial T}{\partial t}=\{\dot{T}\}$, equation reduces to

$$z\{\int\int_A\tfrac{\partial[N]}{\partial r}\tfrac{\partial[N]^T}{\partial r}\{T\}rdrd\theta+\tfrac{1}{\alpha_i}\int\int_A[N][N]^T\tfrac{\partial T}{\partial t}rdrd\theta\}=z\{\tfrac{1}{k_i}\int\int_A g_i[N]rdrd\theta+\int_{\theta=0}^{2\pi}[N]\tfrac{r_n}{A_{out}}\{C_{out}(\theta)-B_{out}[N]^T\{T\}\}d\theta-\int_{\theta=0}^{2\pi}[N]\tfrac{r_0}{A_{in}}\{C_{in}(\theta)-B_{in}[N]^T\{T\}\}d\theta$$

Finally above equation can be written in matrix form as

$$[K]\{T\}+[C]\{\dot{T}\}=\{f_Q\}+\{f_g\}$$

3.2. Discretization of multilayer annulus

The multilayer annulus is discretized into 4 numbers of nodes (M) and 3 numbers of linear two nodes elements (NE) [8], which is shown in the Fig. 2.

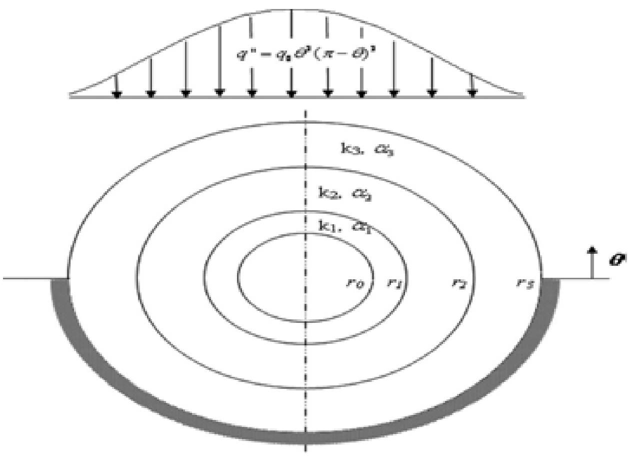

Fig. 2. Discretized model of multilayer circular annulus

All the 4 elements are assembled and the final equation is formed as given below

$$[K]\{T\}+[C]\{\dot{T}\}=\{f_Q\}+\{f_g\}$$

Applying the finite difference method as in [4] and substituting $\{\dot{T}\}=\frac{(T(t+\delta t)-T(t)}{\delta t}$ the above equation reduces to

$$[K]\{T\}+[C]\{\frac{(T(t+\delta t)-T(t)}{\delta t}\}=\{f_Q\}+\{f_g\}$$

this is simplified as under

$$T(t+\delta t)=[C]^{-1}\{f_Q\}\delta t+[C]^{-1}\{f_g\}\delta t-[C]^{-1}[K]\{T\}\delta t+\{T\}$$

4. Illustrative example and results

A three-layer annulus (for $r_0 \le r \le r_3, 0 \le \theta \le 2\pi$) is initially at a uniform zero temperature [8]. For time $t > 0, \theta$-dependent heat flux given by

$$q^n(r = r_3, \theta) = \begin{cases} q_0\ \theta^2(\pi-\theta)^2\ ,\ 0 \le \theta \le \pi \\ 0\ ,\ \pi \le \theta \le 2\pi \end{cases}$$

is applied at the outer surface ($r = r_3$) while the inner surface ($r = r_0$) is maintained isothermal at zero temperature. This leads to the coefficients $A_{in} = 0,\ B_{in} = 1,\ A_{out} = k_3,\ B_{out} = 0,\ C_{in}(\theta) = 0$, and $C_{out}(\theta) = q^n(r_3, \theta)$ in the respective boundary condition equations. There is no volumetric heat generation in any of the layers, i.e. $g_i(r, \theta) = 0$.

To demonstrate the results multilayer annulus is divided into three parts (first layer is cast iron, second layer is aluminum and third layer is of copper). The numerical experimentation has been carried out for the three materials as iron, aluminum and copper layered composite multilayer annulus with no internal heat generation. Parameter values used in this problem are $r_0 = 0,\ q_0 = 1$, length $(L) = 0.03m$, height $(z) = 0.01m$, thermal conductivity $k_1 = 72.7(\frac{w}{m-K^0}),\ k_2 = 204.2(\frac{w}{m-K^0})$ and $k_3 = 386(\frac{w}{m-K^0})$, thermal diffusivity $\alpha_1 = 20.34 * 10^{-6}(\frac{m^2}{s}),\ \alpha_2 = 84.18 * 10^{-6}(\frac{m^2}{s})$ and $\alpha_3 = 112.34 * 10^{-6}(\frac{m^2}{s})$, Young modulus of elasticity $E_1 = 210(GPa),\ E_2 = 69(GPa)$ and $E_3 = 117(GPa)$, coefficient of linear thermal expansion $\beta_1 = 12 * 10^{-6}(\frac{1}{K^0}),\ \beta_2 = 22.2 * 10^{-6}(\frac{1}{K^0})$ and $\beta_3 = 16.5 * 10^{-6}(\frac{1}{K^0})$.

4.1. Temperature distribution

Using Matlab programming and solving the above equation one obtains the temperatures at all the nodes at different time. The numerical value of temperature distribution at different time in the multilayer circular annulus due to point heat source is shown in the Fig. 3.

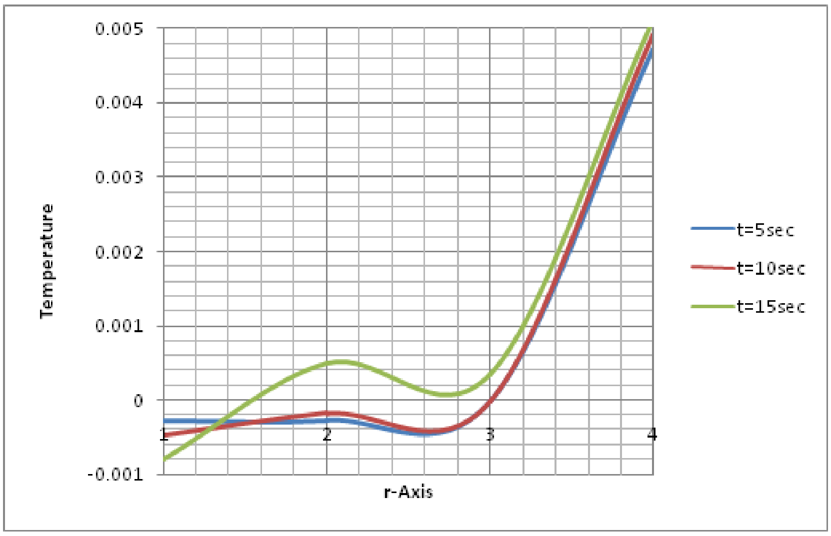

Fig. 3. Transient temperature variation in the radial direction at different time

4.2. Stress analysis

The Strain relation is

$$\epsilon = \beta \delta t$$

Stress-strain relation is

$$\sigma = E\epsilon = E\beta\delta t$$

The elements thermal stresses along the r-axis at $t = 5sec$ are obtained using the temperatures at the nodes which is in the Fir. 4.

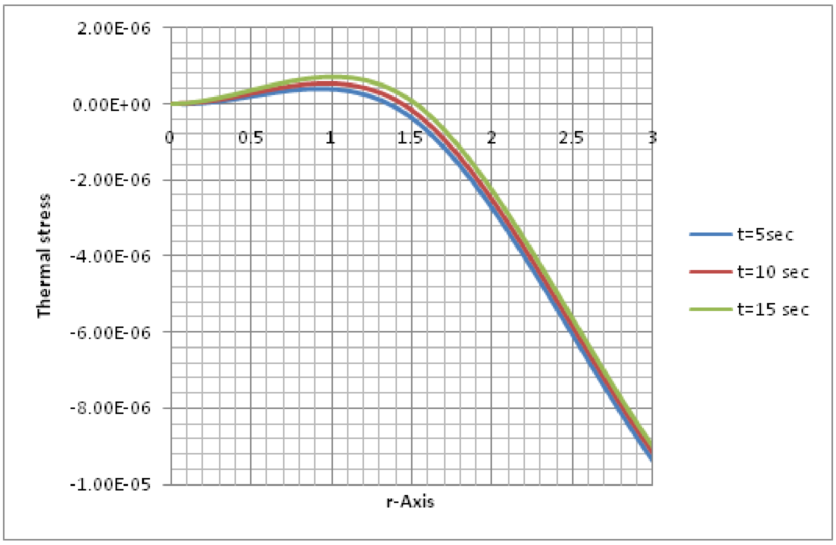

Fig. 4. Thermal Stress variation in the radial direction at different time

5. Concluding remark

In this paper, an analytical solution to the asymmetric transient heat conduction in a layered annulus is presented. Each layer can have spatially varying and time-independent volumetric heat source. Inhomogeneous boundary condition of first, second and third kind can be applied in the radial direction. The proposed solution is applicable to the three layered metallic structure with inner radius $r_0 = 0$ and heat flux $q_0 = 1$. As the thermal diffusivity and conductivity will increase, the maximum temperature variation is seems and least thermal stresses in a material is generated. The results presented in this paper should provide important information and useful guidance for multilayer annulus in many engineering problem like combustion chamber, electric generator, compressor, hydraulic pumps etc. in automobiles field.

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