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Analysis of thermoelastic response in a functionally graded infinite space subjected to a Mode-I crack

Research Article

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Abstract: This paper is concerned with the investigation of thermoelastic stresses, displacement and the temperature in a functionally graded (i.e. material with spatially varying material properties) infinite space weakened by a finite linear opening Mode-I crack. The crack is subjected to prescribed temperature and stress distribution in the context of Green-Naghdi theory of generalized thermoelasticity. The analytical expressions of the thermophysical quantities are obtained in the physical domain using the normal mode analysis. The solution to the analogous problem for homogeneous isotropic material is obtained by taking nonhomogeneity parameter suitably. Finally the results obtained are presented graphically to show the effect of nonhomogeneity on displacements, temperature and stresses for both types II and III of Green-Naghdi theory. It is found that a Mode-I crack influences strongly on the distribution of the field quantities with energy dissipation and without energy dissipation also.

MSC: 74F05

Keywords: Wave propagation • Mode-I crack • Green-Naghdi theory • Normal mode analysis • Functionally graded material

1. Introduction

Physical observations and the results of the conventional coupled dynamic thermoelasticity theories involving infinite speed of thermal signals, which are based on the mixed parabolic-hyperbolic governing equations of Biot [1] and Chadwick [2] are mismatched. To remove this inherence, the conventional theories have been generalized, where the generalization is in the sense that these theories involve a hyperbolic-type heat transport equation supported by experiments which exhibit the actual occurrence of wave-type heat transport in solids, called second sound effect. The first generalization proposed by Lord and Shulman [3] involves one relaxation time parameter in the heat flux-temperature gradient relation. In this theory, a flux-rate term has been introduced into Fourier's heat conduction equation to formulate it in a generalized form that involves a hyperbolic-type heat transport equation admitting finite speed of thermal signals. Another model is the temperature-rate-dependent theory of thermoelasticity proposed by Green and Lindsay [4], which involves two relaxation time parameters. The theory obeys the Fourier law of heat conduction and asserts that heat propagates with finite speed. Three models (Models I, II and III) for generalized thermoelasticity of homogeneous and isotropic materials have been developed later by Green and Naghdi [5–7]. The linearized version of Model I reduces to the classical heat conduction theory (based on Fourier's law) and those of Models II and III permit thermal waves to propagate with finite speed. Experimental studies by Tzou [8] and Mitra et al. [9] of the so-called generalized theory show that the relaxation times play a significant role in the cases involving shock wave propagation, laser technique, nuclear reactor, a rapidly propagating crack tip etc. So, when there are problems involving very large heat fluxes at short intervals of time, the conventional theory of thermoelasticity fails to be a suitable model and the generalized thermoelasticity theory is the right mathematical tool to apply [10].

In recent years, considerable effort has been devoted to the study of cracks in solids, due to their applications in industry, in general, and in the fabrication of electronic components, in particular, as well as in geophysics and

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earthquake engineering. They occur for many reasons including natural defects in materials, as a result of fabrication process, uncertainties in the loading or environment, inadequacies in design and deficiencies in construction or maintenance. Consequently all structures contain cracks as manufacturing defects or because of service loading which can be either mechanical or thermal. If the load is frequently applied, the crack may grow in fatigue to a final fracture. As the size of the crack increases, the residual strength of structure ceases. In the final stages of the crack growth, the rate increases suddenly leading to a catastrophic structure failure.

Presence of a hole or crack in a solid causes disturbance in heat flow and the local temperature gradient around the discontinuity increases. Thermal disturbances of this type can produce material failure through crack propagation. Florence and Goodier [11] studied flow induced thermal stresses in infinite isotropic solids. In addition to these several authors including Sih [12], Kassir and Bergman [13], Prasad et al. [14], Hosseini-Teherani et al. [15], Abdel-Halim and Elfalaky [16] and Chaudhuri and Ray [17] solved different thermoelastic crack problems. Recently, several researchers have solved the problems on Mode I crack. Some remarkable works have been studied by Guo et al [18, 19]. Elfalaky and Abdel-Halim [20], Ueda [21], Sherief and El-Maghraby [22] have solved several problems which have high impacts in the Fracture mechanics. Some remarkable works can also be found in the following literature [23–25].

Functionally graded material (FGM) as a new kind of composites were initially designed as thermal barrier materials for aerospace structures, in which the volume fractions of different constituents of composites vary continuously from one side to another [26]. These novel nonhomogeneous materials have excellent thermo-mechanical properties to withstand high temperature and have extensive applications to important structures, such as aerospace, nuclear reactors, pressure vessels and pipes, chemicals plants, etc. The use of FGMs can eliminate or control thermal stresses in structural components [27–29].

Shaw and Mukhopadhyay [30] have studied the thermoelastic response in a functionally graded micro-elongated medium. Further, some recent enlightened discussions on FGM can be found in the existing literatures [31–36]. Mallik and Kanoria [37] have studied the thermoelastic interaction in an unbounded functionally graded medium due to the presence of a heat source. Recently, some remarkable problems on the functionally graded materials have been solved by Banik and Kanoria [38] and also by Sur and Kanoria [39–41].
In this work we have considered a functionally graded isotropic thermoelastic medium having a Mode-I crack employing Green Naghdi models II and III respectively. The medium is subjected to a prescribed temperature and thermal stresses. Using the normal mode analysis, the problem has been solved for an infinite space weakened by a finite linear opening Mode-I crack. Then, the analytical expression of the displacements, temperature and stresses have been found numerically for copper-like material. The significant differences in the two models and the effect of nonhomogeneity have been discussed.

2. Basic equations

The stress-strain-temperature relation is

$$\tau_{ij} = 2\mu e_{ij} + \left[\lambda\Delta - \gamma(\theta - \theta_0)\right]\delta_{ij}, \quad i, j = 1, 2, 3 \tag{1}$$

where τ_{ij} is the stress tensor, λ, μ are Lame's constants, $\gamma = (3\lambda + 2\mu)\alpha_t$, α_t is the coefficient of linear thermal expansion, θ is the temperature field over the reference temperature θ_0, the cubical dilatation $\Delta = e_{ii}$ and e_{ij} is the strain tensor given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2}$$

Stress equations of motion in absence of body forces are

$$\rho \ddot{u}_i = \tau_{ij,j}\,;\; i, j = 1, 2, 3; \tag{3}$$

where u_i $(i = 1, 2, 3)$ are the displacement components and ρ is the density.
Heat equation corresponding to generalized thermoelasticity based on the Green Naghdi type II and III in absence of heat sources are given by

$$K^\star \nabla^2\theta + \chi K \nabla^2\dot{\theta} = \rho C_v \ddot{\theta} + \gamma\theta_0 \nabla \ddot{u}. \tag{4}$$

where $K^\star$ is an additional material constant, K is the thermal conductivity, c_v is the specific heat at constant strain.
For Green Naghdi model II, we have $\chi = 0$ and for Green Naghdi model III $\chi = 1$.

With the effects of functionally graded solid, the parameters λ, μ, K, $K^\star$ and ρ are no longer constant but become space-dependent. Thus, we replace λ, μ, K, $K^\star$, γ, ρ by $\lambda_0 f(\vec{x})$, $\mu_0 f(\vec{x})$, $K_0 f(\vec{x})$, $K_0^\star f(\vec{x})$ and $\rho_0 f(\vec{x})$ respectively, where, λ_0, μ_0, K_0, $K_0^\star$ and ρ_0 are assumed to be constants and $f(\vec{x})$ is a given nondimensional function of the space variable $\vec{x} = (x, y, z)$. Then the corresponding equations take the following form

$$\tau_{ij} = f(\vec{x})[2\mu_0 e_{ij} + \left\{\lambda_0\Delta - \gamma_0(\theta - \theta_0)\right\}\delta_{ij}], \tag{5}$$

$$f(\vec{x})\rho_0\ddot{u}_i = f(\vec{x})[2\mu_0 e_{ij} + \{\lambda_0\Delta - \gamma_0(\theta-\theta_0)\}\delta_{ij}]_{,j} + f(\vec{x})_{,j}[2\mu_0 e_{ij} + \{\lambda_0\Delta - \gamma_0(\theta-\theta_0)\}\delta_{ij}], \tag{6}$$

and

$$[K_0^{\star} f(\vec{x})\theta_{,i}]_{,i} + [\chi K_0 f(\vec{x})\dot{\theta}_{,i}]_{,i} = \rho_0 f(\vec{x}) c_v \ddot{\theta} + \gamma_0 f(\vec{x})\theta_0 \nabla.\ddot{u}, \tag{7}$$

where $\gamma_0 = (3\lambda_0 + 2\mu_0)\alpha_t$.

3. Formulation of the problem

We consider an infinite functionally graded isotropic thermoelastic space occupying the region G given by $G = \{(x,y,z)| -\infty < x < \infty, -\infty < y < \infty, -\infty < z < \infty\}$ with a mode-I crack subjected to prescribed stress and temperature distributions. For the plane strain problem parallel to $xy-$plane, the crack is defined by $|x| \leqslant a$, $z = 0$ and all the functions are assumed to be functions of x, y and t only. So, the displacement components are given by

$$u_1 = u(x,y,t), \quad u_2 = v(x,y,t), \quad u_3 = 0, \tag{8}$$

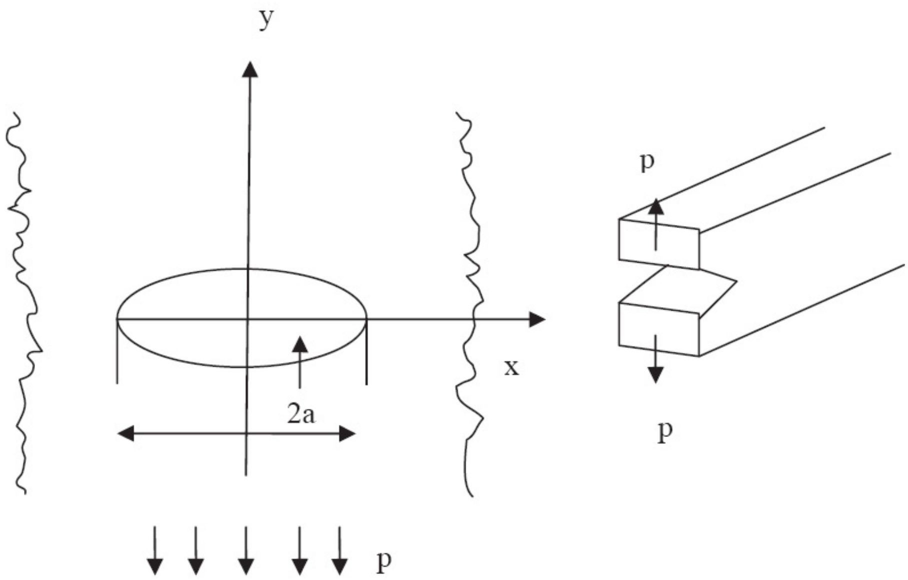

Fig. 1. Displacement of an external Mode-I crack

It is assumed that the material properties depend only on the $x-$coordinate. So, we can take $f(\vec{x})$ as $f(x)$. In the context of linear theory of generalized thermoelasticity based on Green-Naghdi models II and III, the equation of motion, heat equation and the constitute equation can be expressed as

$$\tau_{xx} = f(x)\left[(\lambda_0 + 2\mu_0)\frac{\partial u}{\partial x} + \lambda_0 \frac{\partial v}{\partial y} - \gamma_0(\theta - \theta_0)\right], \tag{9}$$

$$\tau_{yy} = f(x)\left[(\lambda_0 + 2\mu_0)\frac{\partial v}{\partial y} + \lambda_0 \frac{\partial u}{\partial x} - \gamma_0(\theta - \theta_0)\right], \tag{10}$$

$$\tau_{xy} = \mu_0 f(x)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{11}$$

$$f(x)\left[(\lambda_0 + 2\mu_0)\frac{\partial^2 u}{\partial x^2} + \lambda_0\frac{\partial^2 v}{\partial x \partial y} - \gamma_0\frac{\partial \theta}{\partial x}\right] + \left[(\lambda_0 + 2\mu_0)\frac{\partial u}{\partial x} + \lambda_0\frac{\partial v}{\partial y} - \gamma_0(\theta - \theta_0)\right]\frac{\partial f(x)}{\partial x}$$
$$+ \mu_0 f(x)\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) = \rho_0 f(x)\frac{\partial^2 u}{\partial t^2}, \tag{12}$$

$$\mu_0 f(x)\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2}\right) + \mu_0\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial f(x)}{\partial x} + f(x)\left[(\lambda_0 + 2\mu_0)\frac{\partial^2 v}{\partial y^2} + \lambda_0\frac{\partial^2 u}{\partial x \partial y} - \gamma_0\frac{\partial \theta}{\partial y}\right] = \rho_0 f(x)\frac{\partial^2 v}{\partial t^2}, \tag{13}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)[K_0 f(x)\nabla\dot{\theta}] + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)[\chi K_0^{\star} f(x)\nabla\theta] = \rho_0 f(x) c_v \ddot{\theta} + \gamma_0 f(x)\theta_0 \ddot{e}, \tag{14}$$

3.1. Exponential variation of non-homogenity

For a functionally graded solid, we assume $f(x) = e^{-nx}$, where n is a constant and introducing the following nondimensional variables

$$x' = \frac{\omega^\star}{C_2}x, \quad a' = \frac{\omega^\star}{C_2}a, \quad y' = \frac{\omega^\star}{C_2}y, \quad t' = \omega^\star t, \quad u' = \frac{\rho C_2 \omega^\star}{\gamma\theta_0}u, \quad v' = \frac{\rho C_2 \omega^\star}{\gamma\theta_0}v, \quad \theta' = \frac{\theta-\theta_0}{\theta_0},$$

$$n' = \frac{C_2}{\omega^\star}n, \quad \sigma'_{ij} = \frac{\sigma_{ij}}{\gamma\theta_0}, \quad \omega^\star = \frac{\rho c_v C_2^2}{K_0}, \quad C_2^2 = \frac{\mu}{\rho}, \quad f'(x') = f(x),$$

then, after omitting primes, the above equations can be expressed in non-dimensional form as follows

$$\tau_{xx} = e^{-nx}\left[\frac{\partial u}{\partial x} + a_3\frac{\partial v}{\partial y} - \theta\right], \tag{15}$$

$$\tau_{yy} = e^{-nx}\left[\frac{\partial v}{\partial y} + a_3\frac{\partial u}{\partial x} - \theta\right], \tag{16}$$

$$\tau_{xy} = a_4 e^{-nx}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right], \tag{17}$$

where

$$a_2 = \frac{\lambda_0 + 2\mu_0}{\rho_0 C_2^2}, \quad a_3 = \frac{\lambda_0}{\rho_0 C_2^2}, \quad a_4 = \frac{\mu_0}{\rho_0 C_2^2}.$$

In terms of nondimensional quantities, the equation of motions and the heat equation are given by

$$h_{33}\nabla^2 u + h_{22}\frac{\partial e}{\partial x} - \frac{\partial\theta}{\partial x} - n\left\{a_5\frac{\partial u}{\partial x} + a_6\frac{\partial v}{\partial y} - a_7\theta\right\} = \frac{\partial^2 u}{\partial t^2}, \tag{18}$$

$$h_{33}\nabla^2 v + h_{22}\frac{\partial e}{\partial y} - \frac{\partial\theta}{\partial y} - na_4\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\} = \frac{\partial^2 v}{\partial t^2}, \tag{19}$$

$$\ddot{\theta} + \varepsilon_1\ddot{e} = \varepsilon_3\left(\frac{\partial^2\dot{\theta}}{\partial x^2} + \frac{\partial^2\dot{\theta}}{\partial y^2}\right) + \varepsilon_2\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) - n\varepsilon_4\left(\frac{\partial\dot{\theta}}{\partial x} + \frac{\partial\dot{\theta}}{\partial y}\right) - n\varepsilon_5\left(\frac{\partial\theta}{\partial x} + \frac{\partial\theta}{\partial y}\right), \tag{20}$$

where

$$h_{22} = a_3 + a_4, \quad h_{33} = a_4, \quad a_5 = \frac{\lambda_0 + 2\mu_0}{\rho_0 C_2^2}, \quad a_6 = \frac{\lambda_0}{\rho_0 C_2^2}, \quad a_7 = 1,$$

$$\varepsilon_1 = \frac{\gamma_0^2\theta_0}{\rho_0^2 c_v C_2^2}, \quad \varepsilon_2 = \frac{\chi K_0^\star}{\rho_0 c_v C_2^2}, \quad \varepsilon_3 = \frac{K_0\omega^\star}{\rho_0 c_v C_2^2}, \quad \varepsilon_4 = \frac{K_0\omega^\star}{\rho_0 c_v C_2^2}, \quad \varepsilon_5 = \frac{\chi K_0^\star}{\rho_0 c_v C_2 \omega^\star}.$$

The term ε_1 is usually the thermoelastic coupling factor, ε_2 is the characteristic parameter of the GN theory of type II and ε_3 is the characteristic parameter of the GN theory of type III. The solution of the considered physical variable can be decomposed in terms of normal modes in the following form as follows

$$\left[u, v, \theta, \tau_{ij}\right](x, y, t) = \left[u^\star, v^\star, \theta^\star, \tau_{ij}^\star\right](x)\exp(\omega t + iby), \tag{21}$$

where ω is a complex number, $i = \sqrt{-1}$, b is the wave number in the $y-$direction and $u^\star(x)$, $v^\star(x)$, $\theta^\star(x)$, $\tau_{ij}^\star(x)$ are the amplitudes of the field quantities. Using Eqn. (21), Eqs. (18)-(20) take the form

$$\left[h_{11}D^2 + (h_{22}ib - na_5)D - \left(b^2h_{33} + \omega^2\right)\right]u^\star - na_6ibv^\star = (D - na_7)\theta^\star, \tag{22}$$

$$\left[h_{33}D^2 - na_4D - \left(h_{11}b^2 + \omega^2\right)\right]v^\star + (h_{22}ibD - na_4ib)u^\star = ib\theta^\star, \tag{23}$$

$$A_{11}D^2\theta^\star - A_{12}D\theta^\star - A_{33}\theta^\star = \varepsilon_1\omega^2 Du^\star + \varepsilon_1\omega^2 ibv^\star, \tag{24}$$

where

$$h_{11} = \frac{\lambda_0 + 2\mu_0}{\rho_0 C_2^2}, \quad A_{11} = \varepsilon_3\omega + \varepsilon_2, \quad A_{12} = n(\varepsilon_4\omega + \varepsilon_5), \quad A_{33} = b^2 A_{11} + ibA_{12} + \omega^2.$$

Eliminating $u^\star(x)$ and $\theta^\star(x)$ from (22)-(24), the differential equation satisfied by $u^\star(x)$,

$$\left[D^6 + \wp_1 D^5 + \wp_2 D^4 + \wp_3 D^3 + \wp_4 D^2 + \wp_5 D + \wp_6\right] v^\star(x) = 0, \tag{25}$$

where

$$\wp_1 = \frac{1}{ibA_{11}h_{33}(1+h_{22})}\{h_{41}A_{11}h_{33} - h_{61}h_{33} - ib(h_{22}A_{11}h_{34} + h_{33}h_{51})\},$$

$$\wp_2 = \frac{1}{ibA_{11}h_{33}(1+h_{22})}\{ib(h_{33}h_{52} + A_{11}h_{22}h_{35}) - h_{41}h_{51} + h_{61}h_{34} - h_{33}(h_{62} + A_{11}h_{42})\},$$

$$\wp_3 = \frac{1}{ibA_{11}h_{33}(1+h_{22})}\{ib(h_{33}h_{53} + A_{11}h_{22}h_{36}) + h_{41}h_{52} + h_{42}h_{51} - h_{61}h_{35} + h_{62}h_{34} + h_{63}h_{33}\},$$

$$\wp_4 = \frac{1}{ibA_{11}h_{33}(1+h_{22})}\{ibh_{33}h_{54} + h_{41}h_{53} - h_{42}h_{52} - h_{61}h_{36} - h_{62}h_{35} - h_{63}h_{34}\},$$

$$\wp_5 = \frac{1}{ibA_{11}h_{33}(1+h_{22})}\{h_{41}h_{54} - h_{42}h_{53} + h_{63}h_{35} - h_{62}h_{36}\},$$

$$\wp_6 = \frac{1}{ibA_{11}h_{33}(1+h_{22})}\{h_{42}h_{54} + h_{63}h_{36}\}.$$

In a similar manner, we can show that $u^\star(x)$ and $\theta^\star(x)$ satisfy the equation

$$\left[D^6 + \wp_1 D^5 + \wp_2 D^4 + \wp_3 D^3 + \wp_4 D^2 + \wp_5 D + \wp_6\right]\{u^\star(x), \theta^\star(x)\} = 0. \tag{26}$$

where

$$h_{34} = n(a_4 + a_7 h_{33}), \quad h_{35} = n^2 a_4 a_7 - (h_{11}b^2 + \omega^2) + b^2 h_{22}, \quad h_{36} = na_7(h_{11}b^2 + \omega^2) - b^2 na_6, \quad h_{41} = na_7 h_{22} ib,$$

$$h_{42} = ib(b^2 h_{33} + \omega^2) + n^2 a_4 a_7 ib, \quad h_{51} = na_4 A_{11} + A_{12}h_{33}, \quad h_{52} = -A_{11}(h_{11}b^2 + \omega^2) + na_4 A_{12} - A_{33}h_{33},$$

$$h_{53} = A_{12}(h_{11}b^2 + \omega^2) + na_4 A_{33}, \quad h_{54} = A_{33}(h_{11}b^2 + \omega^2) + b^2\varepsilon_1\omega^2,$$

$$h_{61} = ib(na_4 A_{11} + A_{12}h_{22}), \quad h_{62} = ib(A_{33}h_{22} - A_{12}na_4 + \varepsilon_1\omega^2), \quad h_{63} = na_4 A_{33} ib.$$

The solution of Eqn. (25) is given by

$$v^\star(x) = \sum_{\substack{j=1 \\ \mathrm{Re}(\alpha_j)>0}}^{3} N_j(b,\omega) e^{-\alpha_j x} \quad \text{for} \quad x \geqslant 0, \tag{27}$$

Similarly,

$$u^\star(x) = \sum_{\substack{j=1 \\ \mathrm{Re}(\alpha_j)>0}}^{3} M_j(b,\omega) e^{-\alpha_j x} \quad \text{for} \quad x \geqslant 0, \tag{28}$$

$$\theta^\star(x) = \sum_{\substack{j=1 \\ \mathrm{Re}(\alpha_j)>0}}^{3} P_j(b,\omega) e^{-\alpha_j x} \quad \text{for} \quad x \geqslant 0, \tag{29}$$

where $N_j(b,\omega)$, $M_j(b,\omega)$ and $P_j(b,\omega)$ are the parameters depending upon b and ω. Substituting from Eqs. (28) and (29) in Eqs. (22)-(24), we get

$$M_j(b,\omega) = H_{1j} N_j(b,\omega), \quad j = 1,2,3. \tag{30}$$

$$P_j(b,\omega) = H_{2j} N_j(b,\omega), \quad j = 1,2,3. \tag{31}$$

Thus, we have

$$u^\star(x) = \sum_{\substack{j=1\\ \mathrm{Re}(\alpha_j)>0}}^{3} H_{1j} N_j(b,\omega) e^{-\alpha_j x} \quad \text{for} \quad x \geqslant 0, \tag{32}$$

$$\theta^\star(x) = \sum_{\substack{j=1\\ \mathrm{Re}(\alpha_j)>0}}^{3} H_{2j} N_j(b,\omega) e^{-\alpha_j x} \quad \text{for} \quad x \geqslant 0, \tag{33}$$

$$\tau^\star_{xx}(x) = \sum_{\substack{j=1\\ \mathrm{Re}(\alpha_j)>0}}^{3} H_{3j} N_j(b,\omega) e^{-\alpha_j x - nx} \quad \text{for} \quad x \geqslant 0, \tag{34}$$

$$\tau^\star_{yy}(x) = \sum_{\substack{j=1\\ \mathrm{Re}(\alpha_j)>0}}^{3} H_{4j} N_j(b,\omega) e^{-\alpha_j x - nx} \quad \text{for} \quad x \geqslant 0, \tag{35}$$

$$\tau^\star_{xy}(x) = \sum_{\substack{j=1\\ \mathrm{Re}(\alpha_j)>0}}^{3} H_{5j} N_j(b,\omega) e^{-\alpha_j x - nx} \quad \text{for} \quad x \geqslant 0, \tag{36}$$

where

$$H_{1j} = \frac{h_{33}\alpha_j^3 + h_{34}\alpha_j^2 + h_{35}\alpha_j - h_{36}}{h_{42} + h_{41}\alpha_j - ibh_{33}\alpha_j^2},$$

$$H_{2j} = \frac{\varepsilon_T\omega^2\alpha_j^3 h_{33} + na_4\varepsilon_T\omega^2\alpha_j^2 - \varepsilon_T\omega^2\alpha_j(b^2+\omega^2+b^2h_{22}) - \varepsilon_T\omega^2 b^2 na_4}{A_{11}h_{22}ib\alpha_j^3 + ib(A_{11}na_4 + A_{12}h_{22})\alpha_j^2 + ib(na_4A_{12} - A_{33}h_{22} + \varepsilon_T\omega^2)\alpha_j - A_{33}na_4ib},$$

$$H_{3j} = a_3 ib - a_2\alpha_j H_{1j} - H_{2j},$$

$$H_{4j} = a_2 ib - a_3\alpha_j H_{1j} - H_{2j},$$

$$H_{5j} = a_4(-\alpha_j + ibH_{1j}), \quad j = 1,2,3.$$

4. Application, instantaneous mechanical source acting on the surface

The plane boundary subjects to an instantaneous normal point force and the boundary surface is isothermal. The boundary conditions are given by [42]

(i) Thermal boundary condition

The thermal boundary condition is that the surface of the space subjects to a thermal shock

$$\theta = f(y,t) \quad \text{on} \quad |x| < a, \tag{37}$$

(ii) Mechanical boundary condition

The mechanical boundary condition is that the surface of the space obeys

$$\tau_{yy} = -p(y,t) \quad \text{on} \quad |x| < a, \tag{38}$$

$$\tau_{xy} = 0 \quad \text{on} \quad -\infty < x < \infty, \tag{39}$$

where $f(y,t)$ is an arbitrary function of y and t. Substituting the expression of the variables considered into the above boundary conditions, we can obtain the following equations satisfied by the parameters

$$\sum_{j=1}^{3} H_{4j} N_j = -p^\star, \tag{40}$$

$$\sum_{j=1}^{3} H_{5j} N_j = 0, \tag{41}$$

$$\sum_{j=1}^{3} H_{2j} N_j = f^\star. \tag{42}$$

Solving Eqs. (40)-(42), we obtain the parameters $N_j (j = 1,2,3)$ defined as follows

$$N_1 = \frac{\triangle_1}{\triangle}, \quad N_2 = \frac{\triangle_2}{\triangle}, \quad N_3 = \frac{\triangle_3}{\triangle},$$

where

$$\triangle = H_{41}(H_{52}H_{23} - H_{22}H_{53}) - H_{42}(H_{51}H_{23} - H_{21}H_{52}) + H_{42}(H_{51}H_{22} - H_{21}H_{52}),$$

$$\triangle_1 = p^\star(H_{22}H_{53} - H_{52}H_{23}) + f^\star(H_{42}H_{53} - H_{52}H_{43}),$$

$$\triangle_2 = f^\star(H_{43}H_{51} - H_{53}H_{41}) + p^\star(H_{51}H_{23} - H_{21}H_{53}),$$

$$\triangle_3 = f^\star(H_{41}H_{52} - H_{42}H_{51}) + p^\star(H_{21}H_{52} - H_{51}H_{22}).$$

5. Numerical results and discussions

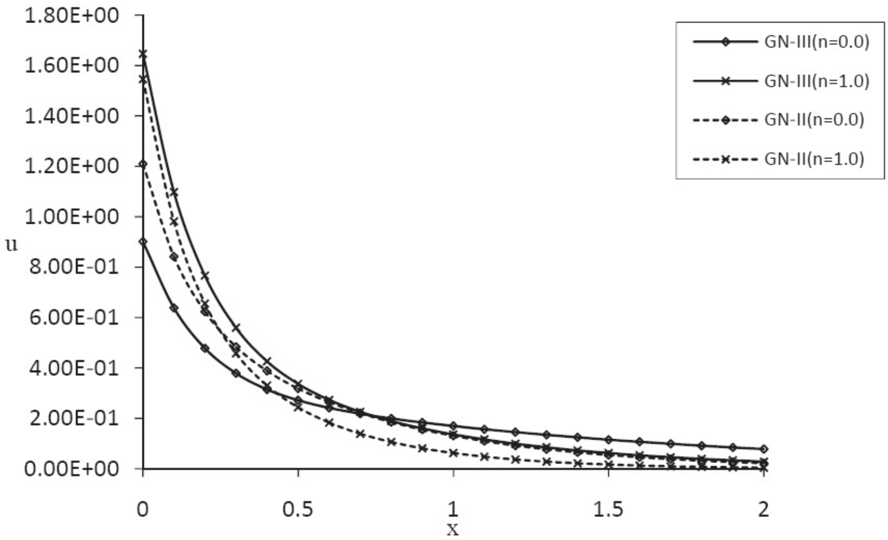

Fig. 2. variation of u vs. x for $y = 0.1$, $t = 0.1$ and $n = 0,1$

In order to illustrate our theoretical results obtained in the preceding section and to compare these under Green-Naghdi theories, we now present the results numerically. For the purpose of illustration, we have chosen a copper crystal as the material subjected to mechanical and thermal disturbances. Since, ω is the complex constant. Then we have $\omega = \omega_0 + i\eta$ with $\omega_0 = 2$ and $\eta = 1$. The material constants are given by

$$\rho = 8954 \text{ kg m}^{-1}, \quad \lambda = 7.76\times10^{10} \text{ N/m}^2, \quad \mu = 3.86\times10^{10} \text{ N/m}^2, \quad \alpha_t = 1.78\times10^{-5} \text{ K}^{-1},$$

$$K = 0.6\times10^{-2}\text{cal/ cm}\quad \text{s}^\circ\text{C}, \quad K^\star = 0.9\times10^{-2}\text{cal/cm}\quad \text{s}^\circ\text{C}, \quad \theta_0 = 293 \text{ K}, \quad f^\star = 0.5,$$

$$c_v = 383.1 \text{ J}\quad (\text{kg})^{-1}\text{K}^{-1}, \quad p^\star = 4, \quad b = 2.$$

Figs. 2-7 are plotted to depict the variation of u, v, θ, τ_{xy}, τ_{xx}, τ_{yy} for $y = 0.1$ and $t = 0.1$ along the crack's edge for Green-Naghdi (GN) models of type II and III respectively. It should noted that in this problem, the crack's size x, is taken to be the length, i.e., $0 \leqslant x \leqslant 2$, $y = 0$ represents the plane of the crack that is symmetric with x to the $y-$plane. In the following figures, the continuous lines represent the figures. corresponding to GN-III model whereas the dotted lines correspond to the GN-II model.

Fig. 2 depicts the variation of the horizontal displacement u versus x for $t = 0.1$ and $y = 0.1$. As seen from the figure, the horizontal displacement u has a maximum value at the center of the crack ($x = 0$) for different nonhomogeneity parameter n ($= 0.0, 1.0$), then it begins to fall just near the crack edge ($x = 2$) for both models. As seen from the figure, on the plane $x = 0$, the magnitude of u is larger for $n = 1$ than that of $n = 0$ for both the models. Also, for $n = 1$, the magnitude of u decays sharply than $n = 0$ and fall near the crack edge. Fig. 3 depicts the variation

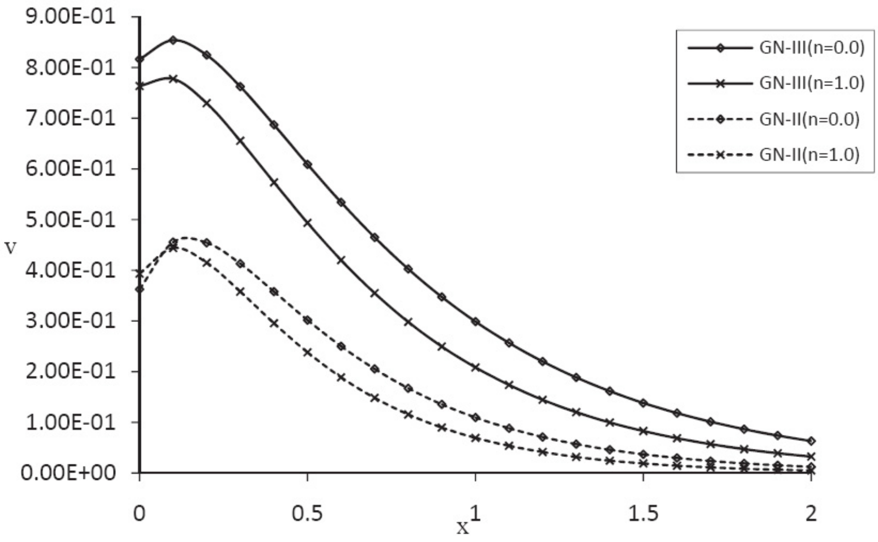

Fig. 3. variation of v vs. x for $y = 0.1, t = 0.1$ and $n = 0, 1$

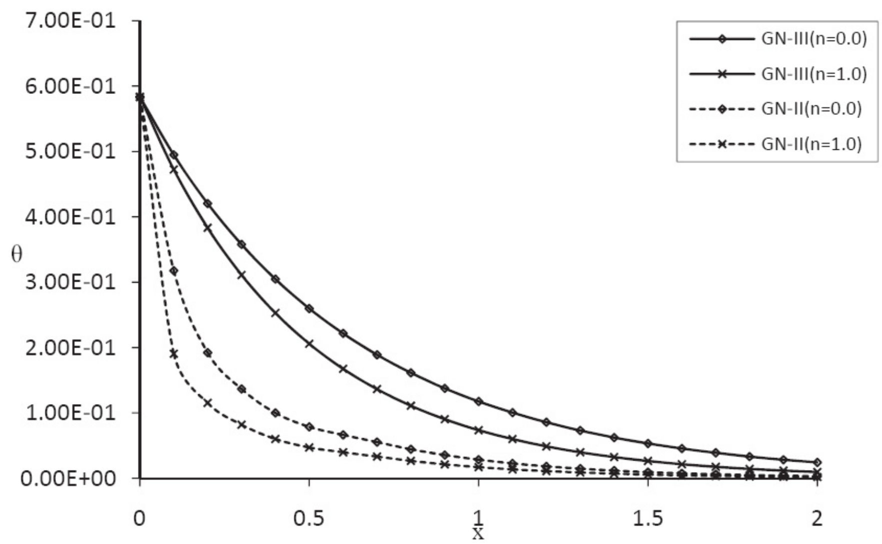

Fig. 4. variation of θ vs. x for $y = 0.1, t = 0.1$ and $n = 0, 1$

of the vertical displacement v versus x for $t = 0.1$ and for GN models of type II and III respectively. Is is observed from the figure that the vertical displacement v has a maximum value near $x = 0.1$ for both models. Though, the magnitude for GN III model is greater compared to GN II model for nonhomogeneity parameter $n = 0,\ 1$ respectively. The magnitude of v begin to fall just near the crack edge ($x = 2$), where it experiences sharp increases in the range $0 < x < 0.1$ and has a decreasing effect in the range $0.1 < x < 2$. Further, it is seen that increase in the nonhomogeneity parameter will decrease the magnitude if the vertical displacement.

Fig. 4 depicts the distribution of the temperature θ versus distance x in both type II and III for $y = 0.1$. It is observed that, on the plane $x = 0$, $\theta \simeq 0.583$, which satisfies our thermal boundary condition (validates the expression

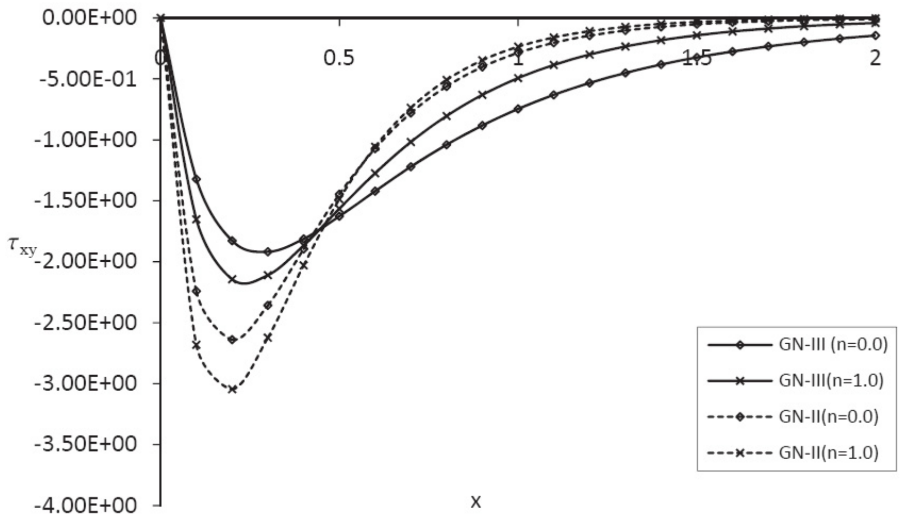

Fig. 5. variation of τ_{xy} vs. x for $y = 0.1, t = 0.1$ and $n = 0,1$

seen in eqn. (37)). Further, it is seen that temperature is maximum at the center of the crack ($x = 0$) and it shows a decreasing effect near the crack edge. For GN II model, the magnitude of θ decay very sharply compared to GN III model and finally it asymptotically tend to zero near the crack edge. For nonhomogeneity parameter $n = 1$, the decay of the temperature θ od faster compared to that of $n = 0$, which is observed for both the models.

Fig. 5 is plotted to show the variation of thermal stress τ_{xy} versus x for $y = 0.1$. In the figure, all lines of the stress component τ_{xy} reach coincidence with zero values and satisfy our mechanical boundary condition for both type II and III models. It is seen that the stress wave is compressive near the center of the crack. The maximum magnitude of τ_{xy} is obtained near $x = 0.25$ and beyond this, the magnitude decays gradually and then tends to zero. It is seen that for different values of the nonhomogeneity parameter n, near $x = 0$, τ_{xy} decays sharply for GN II compared to that of GN III model and in $0 < x < 0.48$, the magnitude of τ_{xy} is larger for GN II compared to GN III whereas in $0.49 < x < 2$, the decay of τ_{xy} for GN III is slower compared to that of GN II and near the crack edge, the stress component almost disappears for both models. For nonhomogeneity parameter $n = 1$, for GN II model, the stress component is maximum near $x = 0.25$. the presence of the nonhomogeneity parameter influences the decay of the stress component τ_{xy}.

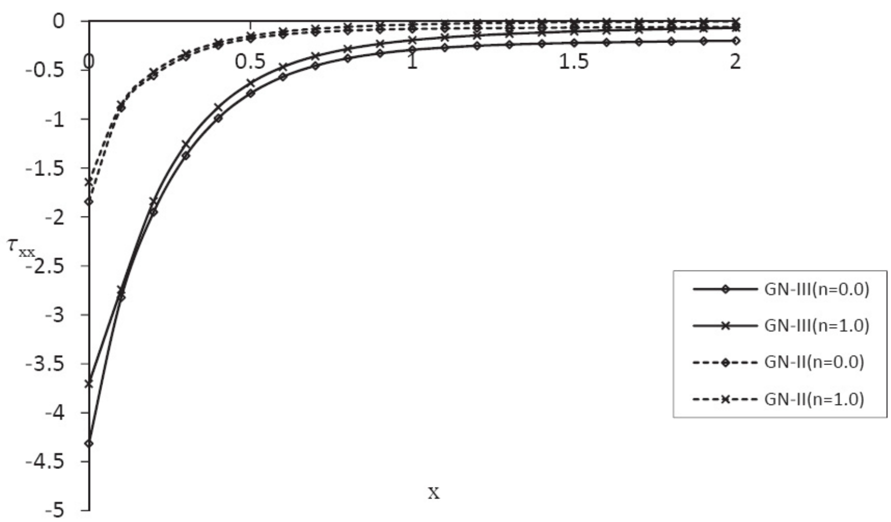

Fig. 6. variation of τ_{xx} vs. x for $y = 0.1, t = 0.1$ and $n = 0,1$

Fig. 6 depicts the variation of the horizontal stress τ_{xx} versus x for both models when $t = 0.1$ and for $y = 0.1$. It is seen that near the center of the crack, the maximum magnitude of the thermal stress is seen for nonhomogeneity parameter $n = 1$ and then the magnitude will decay and decays gradually near the crack edge. The rate of decay of the magnitude is faster for GN II compared to that of GN III model.

Fig. 7 is plotted to show the variation of of the vertical stress τ_{yy} against the distance x for GN models of types

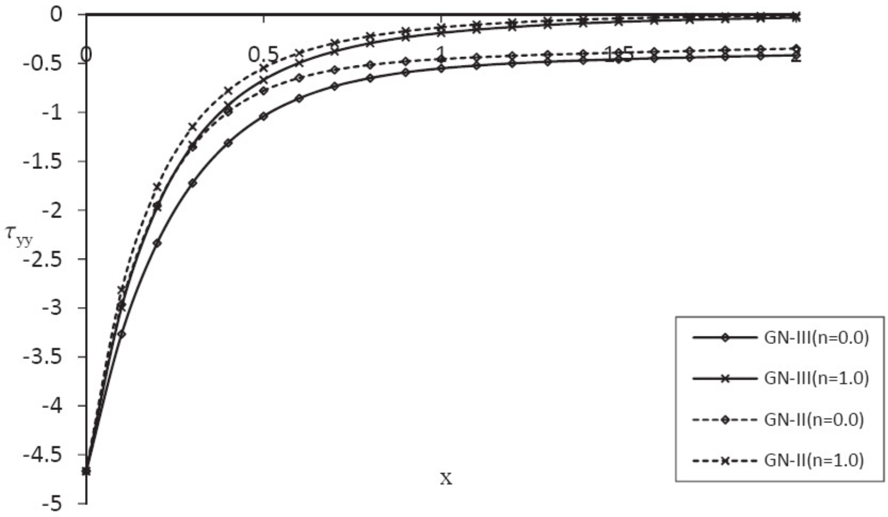

Fig. 7. variation of τ_{yy} vs. x for $y = 0.1, t = 0.1$ and $n = 0, 1$

II and III respectively for the same set of parameters. From the figure, it is seen that on the plane $x = 0$, for both the models, the thermal stress satisfies our mechanical boundary condition. Also, it is observed that with the increase of the nonhomogeneity parameter n, as the distance increases, the magnitude decays sharply and almost coincide with the line $\tau_{yy} = 0$ whereas for a lesser value of the nonhomogeneity parameter n, the decay is more slower.

6. Conclusions

In this paper we have presented a model of the generalization of a Mode-I crack in a thermoelastic functionally graded solid under Green Naghdi theories (GN models II and III) of nonclassical thermoelasticity. The analytical solutions based upon normal mode analysis for the thermoelastic problem in solids have been developed and utilized. The analysis of the result permit some concluding remarks.

- The significant differences of the thermophysical quantities predicated by GN theory of type II and III are remarkable.
- For a thermoelastic exponentially varying functionally graded material, the values of the thermoelastic constants decreases exponentially for a fixed value of x with the increase of nonhomogeneity parameter n. Also, the values of these constants decrease with the increasing value of x for constant n. Thus, the solidification bonding among the ions become lesser and predict a larger value of the thermophysical quantities with increase of x or n. For these types of materials, the values of Lame' constants also decrease exponentially as well as the Young's modulus, and it is expected that for a constant stress, material feels more strain as n increases.
- It is seen that the increase of the nonhomogeneity parameter will also increase the magnitude of the thermophysical quantities. Thus, linear opening of Mode-I crack should be taken into consideration while designing any FGM.
- Here, all the results for $n = 0$ complies with the existing literature [43].

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References

[1] M.A Biot, Thermoelasticity and irreversible thermodynamics, Joural of Applied Physics 27 (1956) 240–253.

[2] P. Chadwick, Thermoelasticity: the dynamic theory. In: Sneddon, I.N., Hill, R. (eds.) Progress in Solid Mechanics, North-Holland, Amsterdam 1 (1960) 265.
[3] H.W Lord, Y. Shulman, A generalized dynamical theory of thermoelasticity, Journal of Mechanics Physics Solids 15 (1967) 299–309.
[4] A.E Green, K.A Lindsay, Thermoelasticity, Journal of Elasticity 2 (1972) 1–7.
[5] A.E Green, P.M Naghdi, A re-examination of the basic postulates of thermomechanics, Proceedings of Royal Society London A 432 (1991) 171–194.
[6] A.E Green, P.M Naghdi, On undamped heat waves in an elastic solid, Journal of Thermal Stresses 15 (1992) 252–264.
[7] A.E Green, P.M Naghdi, Thermoelasticity without energy dissipation, Journal of Elasticity 31 (1993) 189–208.
[8] D.Y Tzou, Experimental support for the lagging behavior in heat propagation, Journal of Thermophysics and Heat Transfer 9 (1995) 686–693.
[9] K. Mitra, S. Kumar, A. Vedaverg, Experimental evidence of hyperbolic heat conduction in processed meat, Journal of Heat Transfer (ASME) 117 (1995) 568–573.
[10] K.J Chauhan, D.M Patel, The similarity solutions of concentration dependent diffusion equation, International Journal of Advances in Applied Mathematics and Mechanics, 1(2) (2013) 80–85.
[11] A.L Florence, J.N Goodier, The Linear Thermoelastic Problem of Uniform Heat Flow Disturbed by a Penny-Shaped Insulated Crack, International Journal of Engineering Science 1 (1963) 533–540.
[12] G.C Sih, On the Singular Character of Thermal Stress Near a Crack Tip, Journal of Applied Mechanics 29 (1962) 587–589.
[13] M.K Kassir, A.M Bergman, Thermal Stress in a Solid Containing Parallel Circular Crack, Applied Science Research 25 262–280.
[14] N.N.V Prasad, M.H Aliabadi, The Dual BEM for Transient Thermoelastic Crack Problems, International Journal of Numerical Methods and Engineering 35 (1996) 169–187.
[15] P. Hosseini-Tehrani, M.R Eslami, H.R Daghyani, Dynamic Crack Analysis under Coupled Thermoelastic Assumption, Journal of Applied Mechanics 38 (2001) 584–588.
[16] A.A Abdel-Halim, A. Elfalaky, An Internal Penny-Shaped Crack Problem in an Infinite Thermoelastic Solid, Journal of Applied Science Research 1(2) (2005) 325–334.
[17] P.K Chaudhuri, S. Ray, Thermal Stress in a Nonhomogeneous Transversely Isotropic Medium Containing a Penny-Shaped Crack, Bulletin of Calcutta Mathematical Society 98(6) (2006) 547–570.
[18] L.C Guo, L.Z Wu, T. Zeng, L. Ma, Mode I crack problem for a functionally graded orthotropic strip, European Journal of Mechanics A/ Solids 23(2) (2004) 219–234.
[19] L.C Guo, N. Noda, Modelling method for a crack problem of functionally graded materials with arbitrary properties-piecewise exponential model, International Journal of Solids and Structures 44 (2007) 6768–6790.
[20] A. Elfalaky, A.A Abdel-Halim, A mode-I crack problem for an infinite space in thermoelasticity,Journal of Applied Science 6 (2006) 598–606.
[21] S. Ueda, Thermally induced fracture of a piezoelectric laminate with a crack normal to interfaces, Journal of Thermal Stresses 26 (2003) 311–323.
[22] H.H Sherief, N.M El-Maghraby, A mode I crack problem for an infinite space in generalized thermoelasticity, Journal of thermal stresses 28 (2005) 465–484.
[23] S.P Barik, M. Kanoria, P.K Chaudhuri, Steady state thermoelastic problem in an infinite functionally graded solid crack, International Journal of Applied Mathematics and Mechanics 6 (2010) 44–66.
[24] S.H Mallik, M. Kanoria, A unified generalized thermoelasticity formulation: Application to penny shaped crack analysis, Journal of Thermal stresses 32 (2009) 945–965.
[25] R. Patra, S.P Barik, P.K Chaudhuri, An internal crack problem in an infinite transversely isotropic elastic layer, International Journal of Advances in Applied Mathematics and Mechanics 3(1) (2015) 62–70.
[26] S. Suresh, A. Mortensen, Fundamentals of Functionally Graded Materials, Institute of Materials Communications Ltd., London (1988).
[27] J. Aboudi, M.J Pindera, S.M Arnold, Thermo-elastic response of functionally graded composites, International Journal of Solids and Structures 32 (1995) 1675–1710.
[28] S.S Vel, R.C Batra, Three-dimensional exact solution for the vibration of functionally graded rectangular plates, Journal of Sound and Vibration 272 (2004) 703–730.
[29] S.S Vel, R.C Batra, Exact solution for thermoelastic deformations of functionally graded thick rectangular plates, AIAA Journal 40 (2002) 1421–1433.
[30] S. Shaw, B. Mukhopadhyay, Periodically varying heat source response in a functionally graded microelongated medium, Applied Mathematics and Computation 218 (2012) 6304–6313.
[31] G. Giunta, D. Crisafulli, S. Belouettar, E. Carrera, A Thermo-Mechanical analysis of functionally graded beams via hierarchical modelling, Composite Structures 95(1) (2012) 676–690.
[32] M.I.A Othman, I.A Abbas, Generalized thermoelasticity of thermal shock problem in a non-homogeneous isotropic hollow cylinder with energy dissipation, Interantional Journal of Thermophysics 33(5) (2012) 913–923.
[33] Y. Kiani, M.R Eslami, An exact solution for thermal buckling of annular plate on an elastic medium, Composites B 45(1) (2013) 101–110.

[34] Y. Kiani, M.R Eslami, Nonlinear thermo-inertial stability of thin circular FGM plates, Journal of the Franklin Institute 351(2) (2014) 1057–1073.

[35] A. Behravan Rad, Thermo-elastic analysis of functionally graded circular plates resting on a gradient hybrid foundation, Applied Mathematics and Computation 256 (2015) 276–298.

[36] F. Barati, H. Farahani, A. Nezamabadi, M. Veiskarami, Buckling analysis of functionally graded beams with imperfectly integrated surface piezoelectric layers under low velocity, International Journal of Advances in Applied Mathematics and Mechanics 2(1) (2014) 64–73.

[37] S.H Mallik, M. Kanoria, Generalized thermoelastic functionally graded infinite solid with a peridically varying heat source, International Journal of Solids and Structures 44 (2007) 7633–7645.

[38] S. Banik, M. Kanoria, Generalized thermoelastic interaction in a functionally graded isotropic unbounded medium due to varying heat sources with three–phase lag effect, Mathematics and Mechanics of Solids, 18(3) (2013) 231–245.

[39] A. Sur, M. Kanoria, Fractional order generalized thermoelastic functionally graded solid with variable material properties, Journal of Solid Mechanics 6 (2014) 54–69.

[40] A. Sur, M. Kanoria, Thermoelastic interaction in a viscoelastic functionally graded half-space under three phase lag model, European Journal of Computational Mechanics 23(5-6) (2014) 179–198.

[41] A. Sur, M. Kanoria, Propagation of thermal waves in a functionally graded thick plate, Mathematics and Mechanics of Solids (2015) DOI: 10.1177/1081286515609652.

[42] K. Lotfy, M.I.A Othman, The effect of magnetic field on 2-D problem for a Mode-I crack of a fiber-reinforced in generalized thermoelasticity, International Journal of Thermophysics 35 (2014) 154–174.

[43] M.I.A Othman, S.Y Atwa, 2-D problem of a Mode-I crack for a generalized thermoelasticity under Green-Naghdi theory, Meccanica 48 (2013) 1543–1551.