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Structured singular values for Bernoulli matrices

Research Article

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Abstract: In this article we present the computation of lower bounds of Structured Singular Values (SSV) for a class of Bernoulli matrices. The comparison of lower bounds with the well-known MATLAB routine mussv is studied and investigated. SSV provides important tools to analyze, synthesize the robustness, performance, stability and instability analysis of feedback systems in the linear control theory as well as in structured eigenvalue perturbation theory.

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Keywords: Structured Singular Value • Spectral value set • Block diagonal • Uncertainties • Spectral radius • Low-rank matrix manifold

1. Introduction

The Structured Singular Value (SSV) [1] is an important and versatile tool in control, as it allows to address a central problem in the analysis and synthesis of control systems. To quantify the stability of a closed-loop linear time-invariant systems subject to structured perturbations. The class of structures addressed by the SSV is very general and allows to cover all types of parametric uncertainties that can be incorporated into the control system via real or complex linear fractional transformations. We refer to [2–9] and the references therein for examples and applications of the SSV.

The versatility of the SSV comes at the expense of being notoriously hard, in fact Non deterministic Polynomial time that is NP hard [10], to compute. Algorithms used in practice thus aim at providing upper and lower bounds, often resulting in a coarse estimate of the exact value. An upper bound of the SSV provides sufficient conditions to guarantee robust stability, while a lower bound provides sufficient conditions for instability and often also allows determining structured perturbations that destabilize the closed loop linear system.

The widely used function mussv in the MATLAB Control Toolbox computes an upper bound of the SSV using diagonal balancing / LMI techniques [11, 12]. The lower bound is computed by a generalization of the power method developed in [13, 14]. This algorithm resembles a mixture of the power methods for computing the spectral radius and the largest singular value, since the SSV can be viewed as a generalization of both. When the algorithm converges, a lower bound of the SSV results which is always an equilibrium point of the iteration. However, in contrast to the standard power method, there are, in general, several stable equilibrium points and not all of them correspond to the SSV. In turn, one cannot guarantee convergence to the exact value but only to a lower bound. We remark that, despite this drawback, mussv is a very reliable and powerful routine, which reflects the state of the art in the approximation of the SSV.

In this paper, we present numerical computation to compute a lower bound of the SSV for Bernoulli matrices associated with pure complex, mixed real and complex perturbations. We construct a gradient system that evolves

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perturbations on a certain matrix manifold towards critical perturbations. Among the theoretical properties established for this gradient system, we prove a monotonicity property that indicates robustness and can also be exploited in the numerical discretization.

Overview of the article: Section 2 provides the basic framework for the proposed problem. In particular, we explain how the computation of the SSV can be addressed by an inner-outer algorithm, where the outer algorithm determines the perturbation level ε and the inner algorithm determines a (local) extremizer of the structured spectral value set. In Section 3, we develop the inner algorithm for the case of pure complex structured perturbations. An important characterization of extremizers shows that we can restrict ourselves to a manifold of structured perturbations with normalized and low-rank blocks. A gradient system for determining extremizers on this manifold is established and analyzed. The outer algorithm is addressed in Section 4, where a fast Newton method for determining the correct perturbation level is developed. Finally, in Section 5, we present a range of numerical experiments to compare the lower bounds obtained with our algorithm [17] to those obtained with mussv.

2. Framework

We consider a matrix $M \in \mathbb{C}^{n,n}$ (or $\mathbb{R}^{n,n}$) and an underlying perturbation set $\mathscr{B}$ with prescribed block diagonal structure,

$$\mathscr{B} = \left\{ diag\left(\delta_1 I_{r_1}, ..., \delta_S I_{r_S}; \Delta_1 ..., \Delta_F\right) : \delta_i \in \mathbb{C}\,(\mathbb{R}), \Delta_i \in \mathbb{C}^{m_j, m_j}\left(\text{ or } \mathbb{R}^{m_j, m_j}\right)\right\} \tag{1}$$

where I_{r_i} denotes the $r_i \times r_i$ identity matrix. Each of the scalars δ_i and the m_j, m_j matrices Δ_j may be constrained to stay real in the definition of $\mathscr{B}$. The integer S denotes the number of repeated scalar blocks that is scalar multiples of the identity matrix and F denotes the number of full blocks.

In order to distinguish complex and real scalar blocks, we assume that the first $S' \le S$ blocks are complex while the remaining $S - S'$ blocks are real. Similarly, we assume that the first $F' \le F$ full blocks are complex and the remaining $F - F'$ blocks are real. The literature (see e.g. [1]) usually does not consider real full blocks, that is, $F' = F$. In fact, in control theory, full blocks arise from uncertainties associated to the frequency response of a system, which is complex-valued.

For simplicity, we assume that all full blocks are square. Similarly, the chosen ordering of blocks should not be viewed as a limiting assumption; it merely simplifies notation. The following definition is given in [1], $\|\cdot\|_2$ denotes matrix 2-norm and I is the (n, n) identity matrix.

Definition 2.1.
Let $M \in \mathbb{C}^{(n,n)}$ and consider a set $\mathscr{B}$ of the form (1) and $\Delta \in \mathscr{B}$, an admissible perturbation. Then, SSV is denoted by $\mu_{\mathscr{B}}(M)$ and is defined as

$$\mu_{\mathscr{B}}(M) = \begin{cases} 0 \quad \text{if} \quad \det(I - M\Delta) \neq 0 \\ \min\{\|\Delta\|_2 : \det(I - M\Delta) = 0\}^{-1}, \quad \text{otherwise} \end{cases} \tag{2}$$

In Definition 2.1 and in the following, we use the convention that minimum over an empty set is $+\infty$.

For $\mathscr{B} = \mathbb{C}^{(n,n)}$, , it follows directly from the Definition 2.1 that $\mu_{\mathscr{B}}(M) = \|M\|$. For a general $\mathscr{B}$, the SSV become smaller and thus we have an upper bound $\mu_{\mathscr{B}}(M) \le \|M\|$. The important special case, when set $\mathscr{B}$ only allows the pure complex perturbations, that is $S = S'$ and $F = F'$, deserve the particular attention. For this case, we denote the set of block diagonal matrices with $\mathscr{B}^*$ instead of $\mathscr{B}$.

For $\Delta \in \mathscr{B}^*$ implies that $e^{i\theta}\Delta \in \mathscr{B}^*$ for any $\theta \in \mathbb{R}$. In turn, there is $\Delta \in \mathscr{B}^*$ such that $\rho(M\Delta) = 1$ if and only if there is $\hat{\Delta} \in \mathscr{B}^*$, with the same norm such that $M\hat{\Delta}$ has an eigenvalue λ which is exactly equal to 1 and furthermore $\left(I - M\hat{\Delta}\right)$ is a singular matrix. This gives us following definition of SSV,

$$\mu_{\mathscr{B}^*}(M) = \begin{cases} 0 \quad \text{if} \quad \rho(M\Delta) \neq 1 \\ \min\left(\left\{\|\Delta\|_2 : \rho(M\Delta) = 1\right\}\right)^{-1}, \quad \text{otherwise} \end{cases} \tag{3}$$

In Eq. (3), $\rho(\cdot)$ denotes the spectral radius of a matrix. For any non-zero eigenvalue λ of M, the matrix $\Delta = \dfrac{1}{\lambda} I$. This gives the lower bound $\rho(M) \le \mu_{\mathscr{B}^*}(M)$. For the case when $\mathscr{B} = \{\delta I : \delta \in \mathbb{C}\}$, we have that $\rho(M) = \mu_{\mathscr{B}^*}(M)$.

2.1. Reformulation of the definitions of SSV [17]

We give the reformulation based on the structured spectral value set. The structured spectral value set of $M \in \mathbb{C}^{n,n}$ with respect to a perturbation level $\varepsilon > 0$ defined as

$$\Lambda_\varepsilon^{\mathscr{B}}(M) = \{\lambda \in \Lambda(\varepsilon M\Delta) : \Delta \in \mathscr{B}, \|\Delta\|_2 \le 1\} \tag{4}$$

where $\Lambda(\cdot)$ denotes the spectrum of the matrix.

For $\mathscr{B}^*$, the pure complex perturbations, Eq. (4) is simply a disk centered at origin. Thus, the Eq. (3) for pure complex perturbations can be reformulated as

$$\mu_{\mathscr{B}^*}(M) = \frac{1}{\arg\min_{\varepsilon>0}\left\{\max_{\lambda\in\Lambda(M)_\varepsilon^{\mathscr{B}}}|\lambda| = 1\right\}} \tag{5}$$

we have that $\Lambda(M)_\varepsilon^{\mathscr{B}}$ is the open complex unit disk, if and only if $\mu_{\mathscr{B}^*}(M) < \frac{1}{\varepsilon}$.

2.2. Overview of the methodology

We need to solve the maximization problem

$$\lambda(\varepsilon) = \arg\max_{\lambda\in\Lambda(M)_\varepsilon^{\mathscr{B}}}|\lambda| \tag{6}$$

for some fixed $\varepsilon > 0$. By the discussion above, $\mu_{\mathscr{B}^*}(M)$ is the reciprocal of the smallest value of ε for which $\lambda(\varepsilon) = 1$. This suggests a two level algorithm. In the inner algorithm, we need to solve the problem addressed in Eq. (6). In the outer algorithm, we vary ε by using the fast Newton's iteration and this exploits the knowledge of the computation of the exact derivative of the extremizers. We address Eq. (6) by solving a system of ordinary differential equations. In general, this only yields a local optima of Eq. (6), which in turn, gives an upper bound of ε and hence a lower bound for $\mu_{\mathscr{B}^*}(M)$.

3. Computation of local extremizers

In this section, we consider the solution of problem addressed in the Eq. (6) by using the inner algorithm. Now, we use the following standard eigenvalue perturbation result by T. Kato [15].

Lemma 3.1 ([15]).
Consider a smooth matrix family $A:\mathbb{R}\to\mathbb{C}^{(n,n)}$ and let $\lambda(t)$ be an eigenvalue of $A(t)$. The eigenvalue $\lambda(t)$ converges to a simple eigenvalue $\lambda_0 = \lambda(0)$ of $A_0 = A(0)$ as $t\to 0$. Then, $\lambda(t)$ is analytic near $t = 0$ with

$$\dot{\lambda}(0) = \frac{X_0^* \dot{A}(0) X_0}{X_0^* X_0}, \quad X_0^* X_0 \neq 0, X_0^* X_0 = 1$$

where X_0 and X_0^* are the right and left eigenvectors of $A_0 = A(0)$ associated with simple eigenvalue $\lambda_0 = \lambda(0)$ that is $(A_0 - \lambda_0 I) X_0 = 0$ and $X_0^*(A_0 - \lambda_0 I) = 0$.

Definition 3.1 ([17]).
A matrix $\Delta \in \mathscr{B}^*$ such that $\|\Delta\|_2 \leq 1$ and the matrix $(\varepsilon M\Delta)$ for some fixed $\varepsilon > 0$ has the largest eigenvalue $\lambda_{\max}$, which maximizes (locally) the modulus of the $\Lambda_\varepsilon^{\mathscr{B}^*}(M)$, is called a local maximizer.

We give following theorem in order to compute an important characterization of the local extremizer.

Theorem 3.1 ([17]).
Let

$$\hat{\Delta} = \left\{diag\left(\delta_1 I_{r_1},\ldots,\delta_S I_{r_S};\Delta_1\ldots,\Delta_F\right) : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j,m_j}\,\forall\, i = 1:S, j = 1:F\right\} \tag{7}$$

with $\left\|\hat{\Delta}\right\|_2 = 1$ be a local extremizer of $\Lambda_\varepsilon^{\mathscr{B}^*}(M)$.

The matrix $(\varepsilon M\Delta)$ has a simple eigenvalue $\lambda = |\lambda| e^{i\theta}, \theta \in \mathbb{R}$ with x and y being right and left eigenvectors, which are scaled such that $s = e^{i\theta} y^* x > 0$. Partitioning x and y according to size and structure of $\hat{\Delta}$.

$$x = \left(x_1^T,\ldots,x_S^T;x_{S+1}^T,\ldots,x_{S+F}^T\right)^T, y = \left(y_1^T,\ldots,y_S^T;y_{S+1}^T,\ldots,y_{S+F}^T\right)^T \tag{8}$$

Let $z = M^* y = \left(z_1^T,\ldots,z_S^T;z_{S+1}^T,\ldots,z_{S+F}^T\right)^T$, such that x_k and y_k components have the same size as the kth block of $\hat{\Delta}$. We assume the non-degeneracy conditions

$$z_k^* x_k \neq 0, \quad \forall\, k = 1:S \tag{9}$$

$$\|z_{s+h}\|_2 \cdot \|x_{s+h}\|_2 \neq 0, \quad \forall\, h = 1:F. \tag{10}$$

Then $|\delta_k| = 1, \quad \forall\, k = 1:S$ and $\|\Delta_h\|_2 = 1, \quad \forall\, h = 1:F$.

We give next theorem to replace full blocks in a local extremizer of the rank-1 matrices.

Theorem 3.2 ([17]).
Let

$$\hat{\Delta} = \{diag(\delta_1 I_{r_1}, \ldots, \delta_S I_{r_S}; \Delta_1 \ldots, \Delta_F) : \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j, m_j} \forall i = 1:S, j = 1:F\}$$

with $\|\hat{\Delta}\|_2 = 1$ *be a local extremizer of* $\Lambda_\varepsilon^{\mathscr{B}^*}(M)$. *Let* λ, x, z *be defined and partitioned as in Theorem 3.1. Assume that the non-degeneracy of Eq. (10) holds and every block* Δ_h *has a singular value which is exactly equal to 1 associated to singular vectors* $u_h = \gamma_h \frac{z_{S+h}}{\|z_{S+h}\|_2}$ *and* $v_h = \gamma_h \frac{x_{S+h}}{\|x_{S+h}\|_2}$ *for some* $|\gamma_h| = 1$.

Moreover, the matrix

$$\hat{\Delta} = \{diag(\delta_1 I_{r_1}, \ldots, \delta_S I_{r_S}; u_1 v_1^*, \ldots, u_F v_F^*)\} \tag{11}$$

is a local extremizer that is $\rho(\varepsilon M \hat{\Delta}) = \rho(\varepsilon M \hat{\Delta})$.

Remark 3.1.
Theorem 3.2 allows us to restrict the perturbations in the structured spectral value set given in Eq.(4) to those with rank-1 blocks. Since the Frobenius and the matrix 2-norms of a rank-1 matrix are equal, we can equivalently search for extremizers within the submanifold

$$\mathscr{B}_1^* = \{diag(\delta_1 I_{r_1}, \ldots, \delta_S I_{r_S}; \Delta_1 \ldots, \Delta_F) : \delta_i \in \mathbb{C}, |\delta_i| = 1, \Delta_j \in \mathbb{C}^{m_j, m_j}, \|\Delta_j\|_F = 1\} \tag{12}$$

3.1. A system of ODEs to compute extremal points of $\Lambda_\varepsilon^{\mathscr{B}^*}(M)$

In order to compute the local maximizer for $|\lambda|$, $\lambda \in \Lambda_\varepsilon^{\mathscr{B}^*}(M)$. We will first construct a matrix valued function $\Delta(t)$, where $\Delta(t) \in \mathscr{B}_1^*$ such that a largest eigenvalue $\lambda(t)$ of the matrix $(\varepsilon M \Delta(t))$ has the maximum (local) growth. We then derive a system of ordinary differential equations which satisfies the choice of the initial matrix $\Delta(t)$.

3.1.1. Orthogonal projection onto $\mathscr{B}^*$

Lemma 3.2 ([17]).
For $C \in \mathbb{C}^{n,n}$

$$C \odot \rrbracket_{\mathscr{B}^*} = diag(C_1, \ldots, C_S; C_{S+1}, \ldots, C_{S+F}), \tag{13}$$

denote the block diagonal matrix obtained by the entry wise multiplication of C *with the pattern matrix* $\rrbracket_{\mathscr{B}^*}$. *The pattern matrix* $\rrbracket_{\mathscr{B}^*}$ *is defined as*

$$\rrbracket_{\mathscr{B}^*} = diag(\rrbracket, \ldots, \rrbracket; \rrbracket, \ldots, \rrbracket)$$

where, $\rrbracket_{\mathscr{B}^*} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}_{d \times d}$ is $d \times d$-matrix of all ones. Then the orthogonal projection of C on $\mathscr{B}$ is given by

$$C_{\mathscr{B}^*} = P_{\mathscr{B}^*}(C) = diag(\gamma_1 I_{r_1}, \ldots, \gamma_S I_{r_S}; \Gamma_1, \ldots, \Gamma_F), \tag{14}$$

where, $\gamma_i = \frac{trace(C_i)}{r_i}$, $\forall i = 1:S$ and $\Gamma_1 = C_{S+1}, \ldots, \Gamma_F = C_{S+F}$.

3.1.2. The local optimization problem

We consider that $\lambda = |\lambda| e^{i\theta}$ is the simple eigenvalue with the eigen vectors x, y normalized such that

$$\|x\| = \|y\| = 1, \quad y^* x = |y^* x| e^{-i\theta} \tag{15}$$

As a consequence of Lemma 3.1, we have

$$\frac{d}{dt}|\lambda(t)|^2\Big]_{t=0} = 2\frac{|\lambda|}{|y^* x|}\mathrm{Re}(z^* \dot{\Delta} x) \quad z = M^* y \tag{16}$$

The eigenvectors x and y are defined and normalized as in Theorem 3.1. The dependent on t is intentionally omitted. Letting $\Delta \in \mathscr{B}_1^*$, with $\Delta \in \mathscr{B}_1^*$ in Eq. (12), we now aim at determining a direction $\dot{\Delta} = Z$ that (locally) maximizes the increase of the modulus of λ This amounts to determining Z as given in the following equation,

$$Z = \{diag(w_1 I_{r_1}, \ldots, w_S I_{r_S}; \Omega_1 \ldots, \Omega_F)\} \tag{17}$$

which is the solution of the following optimization problem $Z_* = \arg\max\{Re(z^* Z x)\}$ subject to

$$\left.\begin{array}{l} \mathrm{Re}\left(\bar{\delta}_i w_i\right) = 0, \quad \forall i = 1:S \\ \mathrm{Re}\left\langle \Delta_j, \Omega_j \right\rangle = 0, \quad \forall j = 1:F \end{array}\right\} \tag{18}$$

The constraints in optimization problem in Eq. (18) ensure that Z lies in the tangent space of $\mathscr{B}_1{}^*$ at $\Delta(t)$. In particular, Eq. (18) implies that the norm of the blocks of the $\Delta(t)$ are conserved. Note that Eq. (18) becomes well-posed after imposing an additional normalization on the norm of Z. The scaling chosen in the following lemma aims at $Z \in \mathscr{B}_1{}^*$.

Lemma 3.3 ([17]).
The solution of the optimization problem proposed in Eq. (18) is given by $Z_* = \{\mathrm{diag}(w_1 I_{r_1}, \ldots, w_S I_{r_S}; \Omega_1 \ldots, \Omega_F)\}$ *with* $w_i = \nu_i \left(x_i{}^* z_i - \mathrm{Re}\left(x_i{}^* z_i \bar{\delta}_i\right)\delta_i\right), \forall i = 1:S$ *and* $\Omega_j = \xi_j \left(z_{S+j} x^*{}_{S+j} - \mathrm{Re}\left\langle \Delta_j, z_{S+j} x^*{}_{S+j} \right\rangle \Delta_j\right), \forall j = 1:F$.

Here, $\hat{\mathrm{I}}_{|i} > 0$ *is the reciprocal of the absolute value of the right-hand side of the expression for* w_i*, if this is different from zero, and* $\hat{\mathrm{I}}_{|i} = 1$ *otherwise. Similarly,* $\xi_j > 0$ *is the reciprocal of the Frobenius norm of the right-hand side of the expression for* Ω_j*, if this is different from zero, and* $\xi_j = 1$ *otherwise. If all right-hand sides are different from zero then* $Z \in \mathscr{B}_1{}^*$.

Corollary 3.1 ([17]).
The result of Lemma 3.3 can be written as

$$Z_* = D_1 P_{\mathscr{B}^*}\left(z x^*\right) - D_2 \Delta, \tag{19}$$

where $P_{\mathscr{B}^*}(\cdot)$ *is the orthogonal projection from the Lemma 3.2 and* $D_1, D_2 \in \mathscr{B}^*$ *are orthogonal matrices with* D_1 *positive.*

3.1.3. The system of ordinary differential equations

Lemma 3.3 and Corollary 3.1 suggests to consider the following differential equations on the manifold $\mathscr{B}_1{}^*$.

$$\dot{\Delta} = D_1 P_{\mathscr{B}^*}\left(z x^*\right) - D_2 \Delta, \tag{20}$$

where $x(t)$ is an eigenvector of the unit norm, associated to a simple eigenvalue $\lambda(t)$ of the matrix $(\varepsilon M \Delta(t))$ for some fixed $\varepsilon > 0$. Note that $z(t), D_1, D_2$ depends on $\Delta(t)$ as well. The differential Eq.(20) is a gradient system of ODEs because, by definition, the right-hand side is the projected gradient of $Z \mapsto \mathrm{Re}(z^* Z x)$.

The following result follows directly from the Lemma 3.1 and Lemma 3.3.

Theorem 3.3 ([17]).
Let $\Delta(t) \in \mathscr{B}_1{}^*$ *satisfying the differential Eq. (20). If* $\lambda(t)$ *is a non-zero simple eigenvalue of the matrix* $(\varepsilon M \Delta(t))$ *then* $|\lambda(t)|$ *increases monotonically.*

3.1.4. Choice of initial value matrix Δ_0 and ε_0

In our two level algorithm for the computation of ε we use the perturbation Δ obtained for the previous value ε as the initial value matrix for the system of ODEs in Eq. (20). However, it remains to discuss a suitable choice of the initial values $\Delta(0) = 0$ and ε_0 in the very beginning of the algorithm. For the moment, let us assume that M is invertible and write

$$I - \varepsilon_0 M \Delta_0 = M(M^{-1} - \varepsilon_0 \Delta_0),$$

which we aim to have as close as possible to the singularity. To determine Δ_0, we perform an asymptotic analysis around $\varepsilon_0 \approx 0$. For this purpose, let us consider that the matrix valued function

$$G(\tau) = M^{-1} - \tau \Delta_0,$$

and let us denote $\eta(\tau)$ as eigenvalue of $G(\tau)$ with the smallest modulus. Letting x, y denote the right and left eigen vectors corresponding to $\eta(0) = \eta_0 = |\eta_0| e^{i\theta}$, scaled such that $e^{i\theta} y^* x > 0$, Lemma 3.1 implies that

$$\frac{d}{dt}|\eta(\tau)|^2{}_{\tau=0} = 2Re\left(\bar{\eta}\dot{\eta}\right) = -2Re\left(\bar{\eta}\frac{y^*\Delta_0 x}{y^* x}\right) = -2|\eta_0| Re\left(\frac{y^*\Delta_0 x}{e^{i\theta} y^* x}\right) = -2\frac{|\eta|}{|y^* x|} Re\left\langle y x^*, \Delta_0 \right\rangle$$

In order to have the local maximal decrease of $|\eta(\tau)|^2$ as $\tau = 0$, we choose

$$\Delta_0 = D P_{\mathscr{B}}\left(y x^*\right), \tag{21}$$

where the positive diagonal matrix D is chosen such that $\Delta_0 \in \mathscr{B}_1{}^*$. The very natural choice of the ε_0 is given by

$$\varepsilon_0 = \frac{1}{\mu^{upper}_{\mathscr{B}}} \tag{22}$$

where $\mu^{upper}_{\mathscr{B}}$ is the upper bound for the SSV computed by the MATLAB function **mussv**.

4. Outer algorithm

In the following, we let $\lambda(\varepsilon)$ continuous branch of (local) maximizers for $Re\left\langle_{\lambda\in\Lambda_\varepsilon^{\mathscr{B}^*}(M)}|\lambda|\right\rangle$ computed by determining the stationary points $\Delta(\varepsilon)$ of the system of ODEs in Eq. (20). The computation of SSV is equivalent to the smallest solution ε of $|\lambda(\varepsilon)|=1$.

In order to approximate this solution, we aim at determining ε^* such that the boundary of ε^*-spectral value set is locally contained in the unit disk and its boundary $\partial\Lambda_{\varepsilon^*}^{\mathscr{B}^*}(M)$ is the tangential to the unit circle. This provides a lower bound $\dfrac{1}{\varepsilon^*}$ for $\mu_{\mathscr{B}^*}(M)$.

In order to apply fast Newton's method for the solution of $|\lambda(\varepsilon)|=1$ or $|\lambda(\varepsilon)|-1=0$, we need to compute the derivative of $|\lambda(\varepsilon)|-1$ with respect to ε. For this purpose, we make use of the following generic assumption.

Assumption 4.1 ([17]).
For a local extremizer $\Delta(\varepsilon)$ or $\Lambda_{\varepsilon^*}^{\mathscr{B}^*}(M)$ with corresponding largest eigenvalue $\lambda(\varepsilon)$, we assume that $\lambda(\varepsilon)$ is simple and that $\Delta(\cdot)$ and $\lambda(\cdot)$ smooth in a neighborhood of ε.

The following theorem gives an explicit and easily computable expression for the derivative of $\lambda(\varepsilon)$.

Theorem 4.1.
].Suppose that Assumption 4.1 holds for $\Delta(\varepsilon)\in{\mathscr{B}_1}^*$ and $\lambda(\varepsilon)$. Let $x(\varepsilon)$ and $y(\varepsilon)$ be the corresponding right and left eigenvectors of the matrix $(\varepsilon M\Delta(\varepsilon))$, scaled as $s=e^{i\theta}y^*x>0$. Consider the partitioning of $x(\varepsilon)$ and $z(\varepsilon)$ as discussed in the Theorem 3.1 and suppose that assumptions in Eq. (9) and in Eq.(10) holds. Then,

$$\frac{d}{d\varepsilon}|\lambda(\varepsilon)|=\frac{1}{\left|y^*(\varepsilon)\,x(\varepsilon)\right|}\left[\sum_{i=1}^{S}\left|{z_i}^*(\varepsilon)\,x_i(\varepsilon)\right|+\sum_{j=1}^{F}\left\|z_{S+j}(\varepsilon)\right\|\cdot\left\|y_{S+j}(\varepsilon)\right\|\right]>0. \tag{23}$$

Theorem 4.1 allows us to easily realize the fast Newton's method

$$\varepsilon^{(k+1)}=\varepsilon^{(k)}-\frac{\left|\lambda^{(k)}\right|-1}{d\left|\lambda^{(k)}\right|} \tag{24}$$

where $\lambda^{(k)}=\lambda\left(\varepsilon^{(k)}\right)$, and $d\left|\lambda^{(k)}\right|$is the derivative of the $|\lambda(\varepsilon)|$ at $\varepsilon=\varepsilon^{(k)}$ given by Eq. (23). Note that Theorem 4.1 implies local quadratic convergence of Eq. (24) to ε^*, provided that the assumption of the theorem hold for $\varepsilon=\varepsilon^*$.

4.1. Summary

For the numerical integration of the ODEs we have made use of the forward Euler method with the step size controlled by the monotonicity of the extremal eigenvalue. The stopping rule is based on two criteria, the ïňĄrst is the condition for the step size to not decrease under a prescribed minimal value and the second relies on the difference of the extremal eigenvalues in two subsequent steps, which should not decrease under a given tolerance.

5. Numerical experimentation

In the following examples, we consider pure complex perturbations, mixed real and complex perturbations and real perturbationsand do not impose a particular order of appearance of repeated scalar blocks and full blocks, which has been done for notational convenience only.

Example 5.1.

Consider the following three dimensional Bernoulli matrix $M=\begin{pmatrix}0&0&1\\1&0&0\\1&0&0\end{pmatrix}$, along with the perturbation set $\mathscr{B}=\{diag(\delta_1 I_1,\delta_2 I_1,\delta_3 I_1):\delta_1,\delta_2,\delta_3\in\mathbb{R}\}$. Applying the MATLAB function **mussv**, we obtain the perturbation set $\hat{\Delta}$ as

$$\hat{\Delta}=1.0e+050\begin{pmatrix}5.0000&0.0000&0.0000\\0.0000&5.0000&0.0000\\0.0000&0.0000&5.0000\end{pmatrix}$$

with norm$\left(\hat{\Delta}\right)=5.0000\text{e}+050$ and the lower bound ${\mu^l}_{PD}(M)=0.0000$ and the upper bound ${\mu^u}_{PD}(M)=1.0000$.

Applying the **algorithm** presented in article [17], we obtain the perturbation $\varepsilon\breve{\Delta}$ with $\varepsilon = 1.0000$ and $\breve{\Delta}$ with

$$\breve{\Delta} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with norm$\left(\breve{\Delta}\right) = 1$ while and the lower bound $\mu^l{}_{PD}(M) = 1.0000$.

Example 5.2.
Consider the following five dimensional Bernoulli matrix'

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

along with the perturbation set $\mathscr{B} = \{diag(\delta_1 I_1, \Delta_1, \Delta_2) : \delta_1 \in \mathbb{R}, \Delta_1, \Delta_2 \in \mathbb{C}^{2,2}\}$.

Applying the MATLAB function mussv, we obtain the perturbation set $\breve{\Delta}$ as

$$\breve{\Delta} = \begin{pmatrix} 0.3593 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.2002 & 0.1360 & 0.0000 & 0.0000 \\ 0.0000 & 0.2197 & 0.1492 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.3021 & 0.1699 \\ 0.0000 & 0.0000 & 0.0000 & 0.0825 & 0.0464 \end{pmatrix}$$

with norm $\breve{\Delta} = 0.3593$ while and the lower bound $\mu^l{}_{PD}(M) = 2.7831$ and the upper bound $\mu^u{}_{PD}(M) = 2.7841$.

Applying the algorithm presented in article [17], we obtain the perturbation $\varepsilon\breve{\Delta}$ with $\varepsilon = 1.0000$ and $\breve{\Delta}$ with

$$\breve{\Delta} = \begin{pmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.5573 & 0.3785 & 0.0000 & 0.0000 \\ 0.0000 & 0.6114 & 0.4152 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.8409 & 0.4728 \\ 0.0000 & 0.0000 & 0.0000 & 0.2297 & 0.1291 \end{pmatrix}$$

with norm $\left(\breve{\Delta}\right) = 1\ 1$ while and the lower bound $\mu^l{}_{PD}(M) = 2.7831$.

Example 5.3.
Consider the following six dimensional Bernoulli matrix,

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

along with the perturbation set $\mathscr{B} = \{\text{diag}\,(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1, \Delta_1, \delta_4 I_1) : \delta_1, \delta_2, \delta_3 \in \mathbb{C}, \Delta_1 \in \mathbb{C}^{2,2}, \delta_4 \in \mathbb{C}\}$.

Applying the MATLAB function **mussv**, we obtain the perturbation set $\breve{\Delta}$ as

$$\breve{\Delta} = \begin{pmatrix} 0.2635 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.2635 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.2635 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.1329 & 0.1166 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.1469 & 0.1289 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.2635 \end{pmatrix}$$

with norm $\left(\breve{\Delta}\right) = 0.2635$ while and the lower bound $\mu^l{}_{PD}(M) = 3.7947$ and the upper bound $\mu^u{}_{PD}(M) = 3.7956$.

Applying the algorithm presented in article [17], we obtain the perturbation $\varepsilon\breve{\Delta}$ with $\varepsilon = 0.2635$ and $\breve{\Delta}$ with

$$\breve{\Delta} = \begin{pmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.5044 & 0.4426 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.5573 & 0.4891 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

with norm $\left(\breve{\Delta}\right) = 1$ while and the lower bound ${\mu^l}_{PD}(M) = 2.7947$.

In the following Table 1, we present the comparison of lower bounds obtained by **mussv** and **our algorithm** presented in the article [17] for mixed real and complex perturbations. In the first column, it is presented the dimension of each Bernoulli matrix. In the second column, it is presented the Block Structure (BLK) of the uncertainties. If Block Structure $(i,:) = [-r\ 0]$, then the i-th block is an r-by-r repeated, diagonal real scalar perturbation. If Block Structure $(i,:) = [r\ 0]$, then the i-th block is an r-by-r repeated, diagonal complex scalar perturbation. If Block Structure$(i,:) = [r\ c]$, then the i-th block is an r-by-c complex full-block perturbation.In the third column, it is presented the upper bound μ^u_{PD} by **mussv** while in the fourth column; it is presented the lower bound μ^l_{PD} by **mussv**. In the fifth and last column, it is presented the lower bound μ^l_{New} obtain by our **new algorithm** [17].

Table 1. Bounds of SSV by **mussv** and **new algorithm** for mixed real and complex perturbations.

n	BLK	$\mu^u_{PD}(M)$	$\mu^l_{PD}(M)$	$\mu^l_{New}(M)$
05	$[-10,-10,-10,22]$	1.6679	1.0000	1.6621
10	$[-10,55,44]$	5.7761	5.7729	5.7729
15	$[-10,1010,44]$	8.0384	8.0294	8.0294
20	$[-10,1010,55,44]$	11.0763	11.0497	11.0497

In the following Table 2, we present the comparison of lower bounds obtained by **mussv** and **our algorithm** presented in the article [17] for mixed pure complex perturbations.

Table 2. Bounds of SSV by **mussv** and **new algorithm** for pure complex perturbations.

n	BLK	$\mu^u_{PD}(M)$	$\mu^l_{PD}(M)$	$\mu^l_{New}(M)$
05	$[10,10,33]$	2.4548	2.4548	2.4548
10	$[10,55,44]$	5.3243	5.3236	5.3235
15	$[10,1010,44]$	7.5853	7.5671	7.5671
20	$[10,1010,55,44]$	10.9018	10.8991	10.8988

6. Conclusions

In this article we have considered the problem of approximating structured singular values, which play an important role in robust control. Our main results provide a characterization of extremizers and gradient system of the ordinary differential equations, which can be integrated numerically in order to provide approximations from below to the structured singular value of a matrix subject to pure complex block perturbations. The experimental results show the comparison of the lower bounds computed by algorithm proposed by [17] when compared to some classical algorithms proposed in the literature and implemented in the Matlab Robust Control Toolbox.

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