Int. J. Adv. Appl. Math. and Mech. 6(3) (2019) 21 – 26 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

Usage modified of homotopy perturbation and Adomian decomposition techniques for solving Fredholm integral equations

Research Article

Mohammed S. Bani issa[a,*], Ahmed A. Hamoud[b], Giniswamy[a], Kirtiwant P. Ghadle[b]

[a] *Department of Mathematics, University of mysore, P. E. T. Research Foundation, 570401, Mandya, India*

[b] *Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, 431004, Aurangabad, India*

Received 06 January 2019; accepted (in revised version) 28 February 2019

Abstract: In this paper, we present a modification to the homotopy perturbation method and the Adomain decomposition method for solving Fredholm integral equations. The results reveal that the proposed methods are very effective and simple and give the exact solution.

MSC: 45B05 • 65H20

Keywords: Fredholm integral equation • Modified Adomain decomposition method • Modified homotopy perturbation method

1. Introduction

In recent years, many works have been focusing on the developing and applying of advanced and efficient methods for integral equations such as Adomain decomposition method [2, 6, 11–16], homotopy perturbation method [1, 4], product integration method [10], variational iteration method [5, 20], collocation method [10] and homotopy analysis method [26].

In this work, we investigate the performance of modified techniques of Adomian decomposition method and homotopy perturbation method applied to Fredholm integral equations of the second kind. This type of integral equations has the following form:

$$u(x) = f(x) + \lambda \int_a^b K(x,t) G(u(t))\, dt, \quad a \le x \le b, \tag{1}$$

where $u(x)$ is the unknown function that will be determined, $K(x,t)$ is the kernel of the equation, $f(x)$ is an analytic function, G is nonlinear function of u, and a, b, and λ are real finite constants.

2. Description of the Methods

Some powerful methods have been focusing on the development of more advanced and efficient methods for Fredholm integral equations such as the Modified Adomian Decomposition Method (MADM) [8, 9, 27, 28] and Modified Homotopy Perturbation Method (MHPM) [29, 30].

* Corresponding author.

E-mail address(es): moh.smarh@yahoo.com (Mohammed S. Bani issa), drahmed985@yahoo.com (Ahmed A. Hamoud).

2.1. Modified Adomian Decomposition Method (MADM)

The standard technique for the Fredholm integral Equation (1), starts by Adomian decomposition method:

$$G(u(x)) = \sum_{n=0}^{\infty} A_n, \tag{2}$$

where A_n; $n \geq 0$ are the Adomian polynomials determined formally as follows:

$$A_n = \frac{1}{n!}\Big[\frac{d^n}{d\mu^n} G(\sum_{i=0}^{\infty} \mu^i u_i)\Big]\Big|_{\mu=0}. \tag{3}$$

The Adomian polynomials were introduced in [3, 7, 10] as:

$$\begin{aligned}
A_0 &= G(u_0);\\
A_1 &= u_1 G^{'}(u_0);\\
A_2 &= u_2 G^{'}(u_0) + \frac{1}{2!} u_1^2 G^{''}(u_0);\\
A_3 &= u_3 G^{'}(u_0) + u_1 u_2 G^{''}(u_0) + \frac{1}{3!} u_1^3 G^{'''}(u_0), \cdots .
\end{aligned}$$

The standard decomposition technique represents the solution of u as the following series [17–19]:

$$u(x) = \sum_{i=0}^{\infty} u_i(x). \tag{4}$$

By substituting (2) and (4) in Eq. (1) we have

$$\sum_{i=0}^{\infty} u_i(x) = f(x) + \lambda \sum_{i=0}^{\infty} \Big(\int_a^b K(x,t) A_i(t) dt\Big).$$

The components $u_0, u_1, u_2, \cdots$ are usually determined recursively by

$$\begin{aligned}
u_0 &= f(x)\\
u_1 &= \lambda \int_a^b K(x,t) A_0(t) dt,\\
u_n &= \lambda \int_a^b K(x,t) A_{n-1}(t) dt, \ n \geq 1.
\end{aligned} \tag{5}$$

Then, $u(x) = \sum_{i=0}^{n} u_i(x)$ as the approximate solution.

The MADM was introduced by Wazwaz [31]. This method is based on the assumption that the function $f(x)$ can be divided into two parts, namely $f_1(x)$ and $f_2(x)$. Under this assumption we set

$$f(x) = f_1(x) + f_2(x). \tag{6}$$

We apply this decomposition when the function f consists of several parts and can be decomposed into two different parts. In this case, f is usually a summation of a polynomial and trigonometric or transcendental functions. A proper choice for the part $f_1(x)$ is important. For the method to be more efficient, we select $f_1(x)$ as one term of f or at least a number of terms if possible and $f_2(x)$ consists of the remaining terms of f.

By using the MADM, from (6), we can write Eq. (1) in the form

$$u(x) = f_1(x) + f_2(x) + \lambda \int_a^b K(x,t) G(u(t)) dt.$$

The components $u_0, u_1, u_2, \cdots$ are usually determined recursively by

$$\begin{aligned}
u_0 &= f_1(x)\\
u_1 &= f_2(x) + \lambda \int_a^b K(x,t) A_0(t) dt,\\
u_n &= \lambda \int_a^b K(x,t) A_{n-1}(t) dt, \ n \geq 1.
\end{aligned} \tag{7}$$

Then, $u(x) = \sum_{i=0}^{n} u_i(x)$ as the approximate solution.

2.2. Modified Homtopy Perturbation Method (MHPM)

The homotopy perturbation method first proposed by He [21–24]. To illustrate the basic idea of this method, we consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{8}$$

under the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{9}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω.

In general, the operator A can be divided into two parts L and N, where L is linear, while N is nonlinear. Eq. (8) therefore can be rewritten as follows [25]:

$$L(u) + N(u) - f(r) = 0. \tag{10}$$

By the homotopy technique (Liao 1992, 1997) [26].

We construct a homotopy $v(r,p) : \omega \times [0,1] \longrightarrow R$ which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, p \in [0,1]. \tag{11}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0)] + p[N(v) - f(r)] = 0, \tag{12}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq.(8) which satisfies the boundary conditions. From Eqs.(11), (12) we have

$$\begin{aligned} H(v,0) &= L(v) - L(u_0) = 0, \\ H(v,1) &= A(v) - f(r) = 0. \end{aligned} \tag{13}$$

The changing in the process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology this is called deformation and $L(v) - L(u_0)$, and $A(v) - f(r)$ are called homotopic.

Now, assume that the solution of Eqs. (11), (12) can be expressed as

$$v = v_0 + pv_1 + p^2 v_2 + \cdots. \tag{14}$$

The approximate solution of Eq.(8) can be obtained by setting $p = 1$.

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{15}$$

Now, we apply MHPM for solving the of Fredholm integral equations.

$$f(x) = f_1(x) + f_2(x) \tag{16}$$

We define a convex homotopy by

$$H(u,p) = (1-p)F(u) + pL(u),$$

$$F(u) = u(x) - f_l(x),$$

we get

$$\begin{aligned} p^0 &: v_0(x) = f_1(x), \\ p^1 &: v_1(x) = \int_0^1 k(x,t) G(v_0(t))\, dt, \\ &\vdots \end{aligned}$$

3. Illustrative Examples

In this section, we present the semi-analytical techniques based on MADM and MHPM to solve Fredholm integral equations.

Example 3.1.
Example 1. Consider the following Fredholm integral equation:

$$u(x) = e^{3x} - \frac{1}{9}(2e^3+1)x + \int_0^1 xtu(t)dt. \tag{17}$$

1. The Modified Decomposition Method (MADM):

$$f(x) = f_1(x) + f_2(x) = e^{3x} - \frac{1}{9}(2e^3+1)x$$
$$f_1(x) = e^{3x}$$
$$f_2(x) = -\frac{1}{9}(2e^3+1)x$$

Then,

$$u_0(x) = e^{3x}$$
$$u_1(x) = f_2(x) + \int_0^1 xte^{3t}dt$$
$$u_1(x) = -\frac{1}{9}\left[2e^3+1\right]x + \frac{x}{9}\left[2e^3+1\right] = 0$$
$$u_n(x) = 0, \quad n \geq 1.$$

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = e^{3x},$$

which is the same as the exact solution.

2. The Modified Homtopy Perturbation Method (MHPM):

$$F(u) = u(x) - f_1(x)$$

we get

$$p^0 : v_0(x) = f_1(x) \Longrightarrow v_0(x) = e^{3x},$$
$$p^1 : v_1(x) - \int_0^1 k(x,t)v_0(t)dt = 0 \Longrightarrow v_1(x) = 0$$

thus, $v_i(x) = 0, \ \forall\ i \geq 1$, and

$$f_1(x) = \sum_{n=0}^{\infty} v_n(x) = e^{3x},$$

which is the same as the exact solution.

Example 3.2.
Consider the following Fredholm integral equation:

$$u(x) = xe^x - x + \int_0^1 xu(t)dt, \tag{18}$$

1. The Modified Decomposition Method (MADM):

$$f(x) = f_1(x) + f_2(x) = xe^x - x,$$
$$f_1(x) = xe^x,$$
$$f_2(x) = -x.$$

Then,

$$u_0(x) = f_1(x) = xe^x,$$
$$u_1(x) = f_2(x) + \int_0^1 xte^t dt,$$
$$u_1(x) = -x + x\big[e - (e-1)\big] = 0$$
$$u_n(x) = 0, \quad n \geq 1.$$

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = xe^x,$$

which is the same as the exact solution.

2. The Modified Homtopy Perturbation Method (MHPM):

$$F(u) = u(x) - f_1(x)$$

we get

$$p^0 : v_0(x) = f_1(x) \Longrightarrow v_0(x) = xe^x,$$
$$p^1 : v_1(x) - \int_0^1 k(x,t)v_0(t)dt = 0 \Longrightarrow v_1(x) = 0$$

thus, $v_i(x) = 0, \ \forall \ i \geq 1$, and

$$f_1(x) = \sum_{n=0}^{\infty} v_n(x) = xe^x,$$

which is the same as the exact solution.

4. Conclusion

We present a comparative study between the Modified Adomain decomposition and Modified homotopy perturbation methods for solving Fredholm integral equations. From the computational viewpoint, the methods are efficient, convenient and easy to use. The methods are very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and nonlinear Fredholm or Volterra integral equations. The numerical results establish the precision and efficiency of the proposed techniques.

References

[1] S. Abbasbandy, Iterated He's homotopy perturbation method for quadratic Riccati differential equation, *Appl. Math. Comput.* 175 (1) (2006) 581-589.

[2] G. Adomian, A review of the decomposition in applied Mathematics, J. Math.Anal. Appl. 135 (1988) 501-544.

[3] G. Adomian, Solving frontier problems of Physics: The decomposition method, Kluwer Academic Publishers, Boston, 1994.

[4] P. Ariel, T. Hayat, S. Asghar, Homotopy perturbation method and axisymmetric flow over a stretching sheet, Int. J. Nonlinear Sci. Numer. Simul. 7 (4) (2006) 399-406.

[5] D.D. Ganji, M. Nourollahi, M. Rostamin, A comparison of variational iteration method and Adomian decomposition method in some highly nonlinear equations, International Journal of Science and Technology, 2 (2), (2007), 179-188.

[6] A. Hamoud, K. Ghadle, The reliable modified of Laplace Adomian decomposition method to solve nonlinear interval Volterra-Fredholm integral equations, Korean J. Math., 25 (3) (2017), 323-334.

[7] A. Hamoud, K. Ghadle, On the numerical solution of nonlinear Volterra-Fredholm integral equations by variational iteration method, Int. J. Adv. Sci. Tech. Research, 3 (2016), 45-51.

[8] A. Hamoud, K. Ghadle, The combined modified Laplace with Adomian decomposition method for solving the nonlinear Volterra-Fredholm integro-differential equations, J. Korean Soc. Ind. Appl. Math., 21 (2017), 17-28.

[9] A. Hamoud, K. Ghadle, Modified Adomian decomposition method for solving fuzzy Volterra-Fredholm integral equations, J. Indian Math. Soc. 85 (1-2) (2018), 52-69.

[10] A. Hamoud, K. Ghadle, Recent advances on reliable methods for solving Volterra-Fredholm integral and integro-differential equations, Asian Journal of Mathematics and Computer Research, 24, No 3 (2018), 128-157.

[11] A. Hamoud, K. Ghadle, Modified Laplace decomposition method for fractional Volterra-Fredholm integro-differential equations, *J. Math. Model.*, 6 (1) (2018), 91-104.

[12] A. Hamoud, K. Ghadle, Usage of the homotopy analysis method for solving fractional Volterra-Fredholm integro-differential equation of the second kind, *Tamkang Journal of Mathematics*, 49 (4) (2018), 301-315.

[13] A.A. Hamoud, K.P. Ghadle, The approximate solutions of fractional Volterra-Fredholm integro-differential equations by using analytical techniques, *Probl. Anal. Issues Anal.*, 7 (25) (1), (2018), 41-58.

[14] A.A. Hamoud, M.SH. Bani Issa, K.P. Ghadle, Existence and uniqueness results for nonlinear Volterra-Fredholm integro-differential equations, *Nonlinear Functional Analysis and Applications*, 23 (4), (2018), 797-805.

[15] A.A. Hamoud, K.P. Ghadle, Existence and uniqueness of solutions for fractional mixed Volterra-Fredholm integro-differential equations, *Indian J. Math.*, 60 (3), (2018), 375-395.

[16] A.A. Hamoud, K.P. Ghadle, M.SH. Bani Issa, Giniswamy, Existence and uniqueness theorems for fractional Volterra-Fredholm integro-differential equations, *Int. J. Appl. Math.*, 31 (3), (2018), 333-348.

[17] A.A. Hamoud, K.P. Ghadle, Some new existence, uniqueness and convergence results for fractional Volterra-Fredholm integro-differential equations, *J. Appl. Comput. Mech.* 5 (1), (2019), 58-69.

[18] A.A. Hamoud, K.P. Ghadle, S.M. Atshan, The approximate solutions of fractional integro-differential equations by using modified Adomian decomposition method, *Khayyam J. Math.* 5 (1), (2019), 21-39.

[19] A.A. Hamoud, K.P. Ghadle, Existence and uniqueness of the solution for Volterra-Fredholm integro-differential equations, *Journal of Siberian Federal University. Mathematics & Physics*, 11 (6), (2018), 692-701.

[20] J.H. He, A variational approach to nonlinear problems and its application, Mech. Applic. 20 (1), (1998), 30-34.

[21] J.H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, Int. J. Non-Linear Mech. 35 (1) (2000) 37-43.

[22] J.H. He, The homotopy perturbation method for non-linear oscillators with discontinuities, Appl. Math. Comput. 151 (1) (2004) 287-292.

[23] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos Solitons Fract. 26 (3) (2005) 695-700.

[24] J.H. He, Asymptotology by homotopy perturbation method, Appl. Math. Comput. 156 (3) (2004) 591-596.

[25] J.H. He, Homotopy perturbation method for solving boundary problems, Phys. Lett. A, 350 (1-2) (2006) 87-88.

[26] S. Liao, The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems, Ph.D. Thesis, Shanghai Jiao Tong University, 1992.

[27] X. Lie-jun, A new modification of Adomian decomposition method for Volterra integral equations of the second kind, Journal of Applied Mathematics, 2013 (2013), 1-7.

[28] H. Ali, F. Abdelwahid, Modified Adomian techniques applied to non-linear Volterra integral equations, Open Journal of Applied Sciences, 3 (2013), 202-207.

[29] J. Saberi-Nadjafi, M. Tamamgar, Modified homotopy perturbation method for solving integral equations, International Journal of Modern Physics B, 24 (24) (2010), 4741-4746.

[30] J. Saberi-Nadjafi, M. Tamamgar, Modified homotopy perturbation method for solving the system of Volterra integral equations, International Journal of Nonlinear Sciences and Numerical Simulation, 9 (4) (2008), 409-413.

[31] A.M. Wazwaz, Linear and Nonlinear Integral Equations Methods and Applications, Springer Heidelberg Dordrecht London New York, 2011.