Int. J. Adv. Appl. Math. and Mech. 6(3) (2019) 42 – 50 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

On the convolution of some analytic functionals via the Fourier transform

Research Article

Mirumbe Ismail*, **John Mango**

Department of Mathematics, Makerere University, P.O Box 7062, Kampala, Uganda

Received 20 January 2019; accepted (in revised version) 23 February 2019

Abstract: In this paper, we use the Fourier transform, inverse Fourier transform and the distribution ξ^s in the ξ-line with support $\{\xi \geq 0\}$ for s a complex number together with its meromorphic extension to prove the existence of the convolution of the distributions x_+^λ and x_-^μ denoted by $x_+^\lambda * x_-^\mu$ where μ and λ are complex parameters.

MSC: 42A85 • 42A38 • 46F10

Keywords: Fourier Transform • Convolution • Tempered distribution • Boundary value distribution

1. Introduction

The definition of products of distributions and their convolutions is a difficult task and not completely well understood for all distributions. However, there are many studies in this direction since L. Schwartz established the theory of distributions around 1950. This is due to their relevance in providing solutions to differential equations for example [1] - [3], and other uses in quantum mechanics and physics in general [4] and the references therein. Products of distributions, including powers of the δ-functions are very useful in the search for solutions of certain partial differential equations and for that reason they have gained interest from many different authors [5], [6], [7] and [8]. Products and convolutions of analytic functionals and/or distributions have been studied in [9] - [18] using the sequential approach and neutrix calculus. In the use of sequential approach the following definition of the commutative neutrix product have been used oftenly [4]. Let $\delta_n(x) = n\rho(nx)$ be Temple's δ-sequence for $n = 1, 2, 3, \ldots$ where $\rho(x)$ is fixed, infinitely differentiable function on the real line with the following properties:

(i) $\rho(x) \geq 0$

(ii) $\rho(x) = 0$ for $|x| \geq 1$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-\infty}^{\infty} \rho(x) = 1$.

* Corresponding author.
E-mail address(es): mirumbe@cns.mak.ac.ug (Mirumbe Ismail).

Definition 1.1.
Let f and g be distributions and $f_n = f * \delta_n$ and $g_n = g * \delta_n$. We say that the commutative neutrix product $f \circ g$ of f and g exists and is equal to h if

$$N - \lim_{n\to\infty} \frac{1}{2}\left\{(f_n g, \phi) + (f g_n, \phi)\right\} = (h, \phi)$$

for $\phi \in \mathscr{D}$, where N is the neutrix and $\mathscr{D}$ is the space of test functions. This neutrix is used to abandon unwanted infinite quantities from the asymptotic expressions having domain $N' = \{1,2,3,\ldots\}$ and the range, the real numbers, with negligible functions that are finite linear sums of the forms,

$$n^{\lambda} \ln^{r-1} n, \ln^r n \quad (\lambda > 0, r = 1,2,3,\ldots)$$

and all functions of n that converge to zero in the normal sense as $n \to \infty$. If the normal limit exists, then it is simply called the commutative product. Note that f_n and g_n are two infinitely differentiable functions and hence $(f_n g, \phi)$ as well as $(f g_n, \phi)$ are well defined.

The classical definition of the convolution of two functions f_1 and f_2 denoted by $f_1 * f_2$ exists when $f_1 * f_2 = f_2 * f_1$. If $(f_1 * f_2)'$ and either $(f_1' * f_2)$ or $(f_1 * f_2')$ exists then

$$(f_1 * f_2)' = (f_1' * f_2)((f_1 * f_2')) \tag{1}$$

This classical definition can be extended to define the convolution of the distributions $f, g \in \mathscr{D}'$ where $\mathscr{D}'$ is the space of distributions [19] and [20].

Definition 1.2.
Let f and g be distributions. Then their convolution $f * g$ is defined by

$$< f * g(x), \varphi(x) > \;=\; < f(y), < g(x), \varphi(x+y) >> \tag{2}$$

for arbitrary test function φ, provided f and g satisfy either of the following conditions:

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side.

Transforms (both Laplace and Fourier transforms) in mathematics are useful in solving differential equations and other equations. Laplace transforms have been used for example to solve the ground water model equation in [21] and by Salehbhai and Timol in obtaining a solution to some fractional differential equations in [22]. Li [4] defines and uses the Fourier transform in the space of tempered distributions together with Theorem 1.1 referred to as the Paley- Wiener-Schwartz Theorem and Theorem 1.2 to prove an exchange formula that is used to give meaning to products of distributions therein. Convolutions of distributions are discussed also in [23] and [24]. In this paper, we use the Fourier transform, the inverse Fourier transform and the product of boundary value distributions in the ξ-line to prove the existence of a normalised convolution $x_+^{\lambda} * x_-^{\mu}$ for λ and μ independent complex parameters.

2. Preliminaries

Definition 2.1.
Let $f(z)$ be an analytic function in the upper half-plane. Suppose that there exist non-negative integers N, M and a constant C such that

$$|f(z)| \le C(1+|x|+|y|)^N y^{-M} \tag{3}$$

for all $z = x + iy$ in the upper half-plane while x is allowed to vary in a bounded interval on the real axis then $f(z)$ is said to have moderate growth as $y \to 0$.

Analytic functions and operators have a vast relationship for example Salman and Joudah in [25] describe new classes of analytic functions defined by using a differintegral operator in the open unit disk therein referred to as star-like functions. In this paper, we shall consider all analytic functions that satisfy the moderate growth condition (3) as it is known that they have boundary value distributions $\mathfrak{b}(f(x \pm i0))$ well defined in the plane [26]. The boundary value distributions arising from analytic functions are in the class of tempered distributions ($\mathscr{S}'$).

Definition 2.2.
Let $f(x) \in \mathscr{S}$. The Fourier transform of $f(x)$ is denoted by $\widehat{f}(\xi)$ and defined by,

$$\widehat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)dx$$

and its inverse Fourier transform is given by

$$f(x) = \frac{1}{2\pi}\int e^{ix\cdot\xi}\widehat{f}(\xi)d\xi.$$

Lemma 2.1.
Let λ be a complex parameter. The Fourier transform of the tempered distribution $(x+i0)^{\lambda}$ is given by

$$\widehat{(x+i0)^{\lambda}} = \frac{2\pi\cdot i^{\lambda}\cdot\xi_{+}^{-\lambda-1}}{\Gamma(-\lambda)}. \tag{4}$$

Proof. For the real component of s denoted $\Re(s) > -1$ and $\epsilon > 0$, we consider the inverse Fourier transform of the distribution ξ_{+}^{s} in the ξ-line

$$\frac{1}{2\pi}\int_{0}^{\infty} e^{ix\cdot\xi-\epsilon\xi}\cdot\xi^{s}\varphi(\xi)d\xi \tag{5}$$

The variable substitution $(\epsilon - ix)\xi = t$ transforms the integral in (5) to

$$\frac{1}{2\pi}(\epsilon-ix)^{-s-1}\int_{0}^{\infty} e^{-t}\cdot t^{s}dt = \frac{1}{2\pi}(\epsilon-ix)^{-s-1}\Gamma(s+1) \tag{6}$$

Letting $\epsilon \to 0$ and substituting $\lambda = -s-1$ we get (4). □

Example 2.1.
Consider $\lambda \to 0$, on the LHS of (4) we have

$$\lim_{\lambda\to 0}\widehat{(x+i0)^{\lambda}} = \hat{1}$$

While the RHS of (4) with the fact that

$$\xi^{-\lambda-1} = -\frac{\partial_{\xi}(\xi^{-\lambda})}{\lambda}$$

and that λ is close to zero is

$$2\pi i^{\lambda}\frac{1}{-\lambda\Gamma(-\lambda)}\cdot\partial_{\xi}(\xi^{-\lambda}) \tag{7}$$

We note that $\lim_{\lambda\to 0}(\xi^{-\lambda}) = H_{+}(\xi)$ where $H_{+}(\xi)$ is the Heaviside distribution on the ξ-line with support $\{\xi \geq 0\}$, $(-i)^{-\lambda}\lambda\cdot\Gamma(-\lambda) \to 1$ as $\lambda \to 0$ thus (7) becomes $2\pi\partial_{\xi}(H_{+}(\xi)) = 2\pi\delta_0$. Thus $\hat{1} = 2\pi\delta_0$.

We state below a known result on the Fourier transforms of the distributions x_{+}^{λ} and x_{-}^{λ} whose proof can be found in [19]

Lemma 2.2.
Let λ and μ be complex parameters. We define the distribution valued functions $\gamma_{\lambda} = x_{+}^{\lambda}$ and $\gamma_{\mu} = x_{-}^{\mu}$ by

$$x_{+}^{\lambda} := \begin{cases} x^{\lambda}, & \textit{for } x > 0 \\ 0 & \textit{for } x < 0. \end{cases}$$

and

$$x_{-}^{\mu} := \begin{cases} 0, & \textit{for } x > 0 \\ x^{\mu} & \textit{for } x < 0. \end{cases}$$

whose Fourier transforms are respectively

$$ie^{i\lambda(\frac{\pi}{2})}\Gamma(\lambda+1)(\xi+i0)^{-\lambda-1}$$

and

$$-ie^{-i\mu(\frac{\pi}{2})}\Gamma(\mu+1)(\xi-i0)^{-\mu-1}.$$

Lemma 2.3.
The boundary value distribution $\mathfrak{b}(f)$ *defined from an analytic function with moderate growth* $f(z)$ *has a Fourier transform whose support is contained in* $[0,\infty)$ *that is* $supp(\widehat{\mathfrak{b}(f)}) \subset [0,\infty)$.

Proof. Consider a test function $g(\xi)$ with compact support contained in $[-b,-a]$ for $0<a<b$. The inverse Fourier transform

$$G(z) = \frac{1}{2\pi}\int e^{iz\cdot\xi} g(\xi)d\xi \tag{8}$$

extends to an entire function in the complex z-plane and repeated partial integration of (8) for a positive integer m gives,

$$(iz)^m G(z) = (-1)^m \frac{1}{2\pi}\int e^{iz\cdot\xi} g(\xi)d\xi \tag{9}$$

(9) implies that for every positive integer M there exists a constant C_M such that

$$|G(x+iy)| \le C_M(1+|z|)^{-M} e^{-ay}, \tag{10}$$

Let $f(z)$ be an analytic function that satisfies the moderate growth condition (3), choosing $m > M+2$. It follows via the Cauchy integral formula applied to the pair of analytic functions $f(z)$ and $G(z)$ in the upper half-plane that

$$\int_{-\infty}^{\infty} f(x+i\epsilon).G(x+i\epsilon)dx = \int_{-\infty}^{\infty} f(x+iB).G(x+iB)dx \tag{11}$$

for all $0<\epsilon<B$. The exponential decay with the factor e^{-aB} from (10) shows that the integral in (11) becomes arbitrarily small when B is large and hence vanishes. Taking the limit as $\epsilon \to 0$, the left hand side of (11) that evaluates the boundary value distribution $\mathfrak{b}(f)$ vanishes and Fourier's inversion formula entails that $\widehat{\mathfrak{b}(f)} = 0$ on $(-\infty,0)$. Since this vanishing hold for every test-function $g(\xi)$ then $supp(\widehat{\mathfrak{b}(f)}) \subset [0,\infty)$. □

We note that since every boundary value of an analytic function is a tempered distribution then there is a relationship between the space of analytic functions in the upper half-plane and the space $\mathscr{S}'$ [26] and [27]. We state and prove this relationship for completeness purposes.

Theorem 2.1.
There exists a one to one correspondence between the space of tempered distributions (boundary value distributions) on the x*-line whose Fourier transforms have support* $\{\xi \ge 0\}$ *and the family of analytic functions* $u(z)$ *in the upper half-plane satisfying the moderate growth condition* (3).

Proof. $(\Rightarrow)$ Let γ be a tempered distribution on the ξ-line with support $\{\xi \ge 0\}$. Such a distribution is represented by a derivative of some order N of a Riesz measure μ satisfying [26],

$$\int_0^{\infty} (1+|\xi|)^{-M} d|\mu(\xi)| < \infty \tag{12}$$

for some non-negative integer M. (12) yields an analytic function in the upper halfplane $(\mathfrak{Im}(z) > 0)$;

$$u(z) = \frac{1}{2\pi}\int_0^{\infty} e^{iz\xi} d\mu(\xi) \le \int_0^{\infty} e^{-y\xi} d\mu(\xi). \tag{13}$$

Consider the function $\psi_M(y) = \max_{\xi\ge 0}\left(e^{-y\xi}(1+\xi)^m\right)$. This function $\psi_M(y)$ has a maximum at $(1+\xi)y = M$ implying that

$$u(x+iy) \le CM^M y^{-M} \tag{14}$$

which is a modified form of (3) therefore the analytic function $u(z)$ in (14) has moderate growth. In a more general sense if we let γ be a tempered distribution on the ξ-line supported by $\xi \ge 0$. It is represented by a derivative of some order N of a Riesz measure γ satisfying (12) above and from which an analytic function $u(z)$ in the upper half-plane defined by $u(z) = \gamma(e^{iz\xi})$ and therefore there exists a pair of non-negative integers N and M and a constant C such that the moderate condition (3) is satisfied.

$(\Leftarrow)$ Let $f(z)$ be an analytic function in the upper half-plane that satisfy the moderate growth condition (3) then the boundary value distribution $\mathfrak{b}(f)$ exist, Theorem 3.1.12 [26]. We show that it is a tempered distribution. Given a positive integer N and some $g(x) \in C_o^{\infty}[-A,A]$ for $A>0$, we set

$$G_N(x+iy) = g(x) + \sum_{\nu=1}^{N} i^{\nu} \frac{g^{(\nu)}(x)y^{(\nu)}}{\nu!}$$

Since $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ then,

$$2\overline{\partial}(G_N(x+iy)) = \sum_{\nu=0}^{N} i^{\nu} \frac{g^{(\nu+1)}(x)y^{(\nu)}}{\nu!} + \sum_{\nu=1}^{N} i^{\nu+1} \frac{g^{(\nu)}(x)y^{(\nu-1)}}{(\nu-1)!} = i^N \frac{g^{(N+1)}(x)y^{(N)}}{N!}. \tag{15}$$

Given the moderate growth condition (3) and (15) there exist a constant K such that

$$|\mathfrak{b}(f)(g)| \leq K \int (1+|x|)^N |g^{M+1}(x)|) dx$$

for all test functions g implying that $\mathfrak{b}(f)$ is a tempered distribution and hence its Fourier transform exists [20], [27]. ☐

3. Main results

3.1. The boundary value distribution $(x+i0)^{\lambda}$

In the upper half-plane ($\Im\mathfrak{m} z > 0$), there exists a single-valued branch of $\log z$ whose argument stays in the interval $(0,\pi)$. For $x<0$ in this interval, we have for example

$$\lim_{\epsilon\to 0} \log(x+i\epsilon) = \log|x| + i\pi.$$

If $\lambda > 0$ is a complex number, then there exists an analytic function in the upper half-plane whose argument lies in $(0,\pi)$ and satisfies the moderate growth condition (3). This analytic function is given by

$$z^{\lambda} = e^{\lambda \log z}$$

and gives rise to boundary value distributions $(x \pm i0)^{\lambda}$ in the upper and lower half-planes given by,

$$(x \pm i0)^{\lambda} = \lim_{\epsilon\to 0} (x \pm i\epsilon)^{\lambda}. \tag{16}$$

The boundary value distributions $(x\pm i0)^{\lambda}$ in (16) are tempered distributions on the real x-line and therefore have Fourier transforms [19]. Below we state and prove a key Lemma that characterizes the distribution ξ^s in the ξ-plane and will be used to prove our main result.

Lemma 3.1.
Let λ be a complex number. The Fourier transform of the tempered distribution $(x+i0)^{\lambda}$ denoted by $\widehat{(x+i0)^{\lambda}}$ and the distribution

$$\xi_+^{-\lambda-1} := \begin{cases} \xi^{-\lambda-1}, & for\ \xi \geq 0 \\ 0 & for\ \xi < 0. \end{cases}$$

on the ξ-line are related by,

$$\widehat{(x+i0)^{\lambda}} = \begin{cases} \frac{2\pi\cdot(-i)^{\lambda}\cdot\xi_+^{-\lambda-1}}{\Gamma(-\lambda)}, & for\ \Re(\lambda) < 0 \\ \frac{2\pi\cdot(-i)^{\lambda}\cdot\xi_+^{-\lambda-1}}{\Gamma(-\lambda-1)} & for\ \Re(\lambda) \geq 0. \end{cases} \tag{17}$$

Proof. Let $\mu(\lambda)$ denote the Fourier transform of $(x+i0)^{\lambda}$ thus $\mu(\lambda) = \widehat{(x+i0)^{\lambda}}$. By Lemma 2.3 and Theorem 2.1, $\mu(\lambda)$ is an entire function of λ and belongs to the space of tempered distributions supported by $\{\xi \geq 0\}$. Consider the complex derivative of the analytic function z^{λ},

$$\frac{d}{dz}(z^{\lambda}) = \lambda \cdot z^{\lambda-1} \tag{18}$$

The derivative in (18) has a boundary value distribution,

$$x\partial_x((x+i0)^{\lambda}) = \lambda\cdot(x+i0)^{\lambda} \tag{19}$$

Taking the Fourier transform on both sides of (19) we get

$$(i\partial_{\xi} \circ i\xi)(\mu(\lambda)) = \lambda\cdot\mu(\lambda) \tag{20}$$

Since $i\partial_{\xi}\circ i\xi = -\xi\partial_{\xi} - 1$ then $\mu(\lambda)$ satisfies the differential equation

$$\xi\partial_{\xi}(\mu(\lambda)) = -(\lambda+1)\cdot\mu(\lambda) \tag{21}$$

Solving (21) gives,

$$\mu(\lambda) = A(\lambda) \cdot \xi^{-\lambda-1} \tag{22}$$

In order to find the constant $A(\lambda)$ in (22), we let s be a complex number such that $\Re(s) > -1$. This therefore means there exists a locally integrable function ξ^s with support $\{\xi \geq 0\}$ which is extended by zero on $\{\xi < 0\}$. This tempered distribution has an inverse Fourier transform as the boundary value distribution of the analytic function in the upper half-plane defined by,

$$f(z) = \frac{1}{2\pi} \int_0^\infty e^{iz\xi} \xi^s d\xi \tag{23}$$

Use the change of variable $t = -iz\xi$ in (23) to get

$$f(z) = \frac{1}{2\pi} (-iz)^{-s-1} \Gamma(s+1) \tag{24}$$

Let $\lambda = -s-1$ into (24) and take boundary values to get,

$$\frac{1}{2\pi} (-i)^{-\lambda} \widehat{(x+i0)^\lambda} \Gamma(-\lambda) = \xi_+^{-\lambda-1} \tag{25}$$

The constant $A(\lambda)$ in (22) is given by $A(\lambda) = \frac{(-i)^\lambda \cdot 2\pi}{\Gamma(-\lambda)}$ for $\Re(\lambda) < 0$ and therefore the tempered distribution $\frac{1}{\Gamma(-\lambda)} \xi_+^{-\lambda-1}$ is well defined in this range yielding the first part of (17). For a meromorphic extension to the case $\Re(\lambda) \geq 0$, we consider ξ_+^s as a distribution on the ξ-line and $\Re(s) > -1$ thus assuming $g(\xi)$ is a test function preferably in the Schwartz space,

$$\xi_+(g(\xi)) = \int_0^\infty \xi^s g(\xi) d\xi \tag{26}$$

Let m be a positive integer, performing m partial integrations on (26) gives

$$(s+m)(s+m-1)\dots(s+1)\xi_+(g(\xi)) = (-1)^m \int_0^\infty \xi^{s+m} g^m(\xi) d\xi \tag{27}$$

Equation (27) shows that ξ_+^s extends to a meromorphic function with at most simple poles at negative integers. Normalization of ξ_+^s yields the distribution $\frac{1}{\Gamma(s)} \cdot \xi_+^s$ that is entire since $\frac{1}{\Gamma(s)}$ has simple zeros at non-positive integers. In this case setting $-\lambda - 1 = s$ we get the second part of (17).

□

Example 3.1. (a) Let $\lambda = -1 < 0$ in (17). Here $\Gamma(-\lambda) = \Gamma(1) = 1$. The RHS of (17) becomes

$$i2\pi \cdot \xi_+^0 = 2\pi i \cdot H_+(\xi)$$

where $H_+(\xi)$ is the Heaviside distribution. The LHS is $\widehat{(x+i0)^{-1}}$ which is

$$\int_0^\infty e^{-\epsilon\xi + ix\xi} d\xi = \frac{i}{x+i\epsilon}.$$

(b) Let $\lambda = 0$ in (17). Here we start with $z^0 = 1$ in the upper half-plane thus yielding the identity boundary value distribution whose Fourier transform is δ_0 on the ξ-line. For the RHS of (17) we note that $\frac{1}{\Gamma(t)}$ has a simple zero at $\lambda = -1$ which therefore compensates the pole of the meromorphic extension ξ_+^s at $s = -1$. From equation (27) with a small λ we have,

$$\lambda \cdot \xi_+^{-1-\lambda} = \partial_\xi(\xi_+^\lambda)$$

As $\lambda \to 0$ then ξ_+^λ converges to $H_+(\xi)$ and we recall that $\partial_\xi(H_+^\xi)$ is the Dirac measure on the ξ-line.

3.2. A convolution of some analytic functionals

Let λ and μ be complex parameters. Let $\gamma_\lambda = x_+^\lambda$ and $\gamma_\mu = x_-^\mu$ be distributions as defined in Lemma 2.2 whose Fourier transforms are given in the same Lemma. We seek convolutions $x_+^\lambda * x_+^\mu$ and $x_+^\lambda * x_-^\mu$. On the ξ-line (the Fourier transform space), this corresponds to an analoguous problem of constructing products of distributions

$$(\xi + i0)^\alpha \cdot (\xi + i0)^\beta \text{ and } (\xi - i0)^\alpha \cdot (\xi + i0)^\beta \tag{28}$$

where $\beta = -\lambda - 1$ and $\alpha = -\mu - 1$. As mentioned in the introduction section, the products of two distributions is not always well defined. But if the real parts of λ and μ are both less than -1 then the products in (28) are well defined as a product of two continuous densities and Fourier's inversion formula gives respectively well defined distributions

$$\frac{1}{\Gamma(\lambda+1)\Gamma(\mu+1)} \cdot x_+^\lambda * x_+^\mu \text{ and } \frac{1}{\Gamma(\lambda+1)\Gamma(\mu+1)} \cdot x_+^\lambda * x_-^\mu. \tag{29}$$

This distributions in (29) can be extended to the whole complex plane. The convolution distribution $x_+^\lambda * x_+^\mu$ has been considered in [23] with applications in [24] and proved by the direct use of the definition of the convolution and as a product of distributions in the ξ-line in [10]. We use the product of distributions in the ξ-line and functional equation in Lemma 3.1 to prove the existence of $\frac{1}{\Gamma(\lambda+1)\Gamma(\mu+1)} \cdot x_+^\lambda * x_-^\mu$.

Theorem 3.1.
If x_+^λ and x_-^μ are locally integrable functions with two independent complex parameters λ and μ and defined as in Lemma 2.2 then there exists an entire distribution valued function $\gamma(\lambda,\mu)$ defined by

$$\gamma(\lambda,\mu) = \frac{1}{\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma(-\lambda-\mu-1)} \cdot x_+^\lambda * x_-^\mu \tag{30}$$

Proof. From the identities below adopted from [19],

$$(x+i0)^\lambda = x_+^\lambda + e^{i\pi\lambda} x_-^\lambda \;\;\&\;\; (x-i0)^\lambda = x_+^\lambda + e^{-i\pi\lambda} x_-^\lambda. \tag{31}$$

If α and β are two independent complex parameters one has the equation below derived from (31),

$$(\xi - i0)^\alpha = (\xi + i0)^\alpha - 2i \sin\pi\alpha \cdot H_-(\xi) \cdot |\xi|^\alpha \tag{32}$$

This means that using (32), we can write the product of distributions $(\xi - i0)^\alpha \cdot (\xi + i0)^\beta$ as

$$(\xi - i0)^\alpha \cdot (\xi + i0)^\beta = (\xi + i0)^{\alpha+\beta} - 2i \sin\pi\alpha \cdot H_-(\xi) \cdot |\xi|^\alpha \cdot (\xi + i0)^\beta \tag{33}$$

The product on the right hand side of (33) is well defined when $\Re(\alpha) > -1$ and $\Re(\beta) > 0$. Since then we can take a product of a continuous density and a locally integrable function. We next consider the remaining part on the right hand side of (33). Note that,

$$H_-(\xi) \cdot |\xi|^\alpha \cdot (\xi + i0)^\beta = e^{\pi i\beta} \cdot |\xi|^{\alpha+\beta} \cdot H_-(\xi) \tag{34}$$

By the analytic continuation considered in Lemma 3.1 and normalization by an appropriate Gamma function, there exists an entire distribution valued function for the right hand side of (34) given by

$$\gamma_{\alpha+\beta} = \frac{1}{\Gamma(\alpha+\beta+1)} e^{\pi i\beta} \cdot |\xi|^{\alpha+\beta} \cdot H_-(\xi). \tag{35}$$

Since the distribution valued functions $(\xi + i0)^{\alpha+\beta}$ and $H_-(\xi) \cdot |\xi|^\alpha \cdot (\xi + i0)^\beta$ in (33) where the later is transformed to $\gamma_{\alpha+\beta}$ and the former is considered with possible analytic continuations are both entire, then the product $\frac{1}{\Gamma(\alpha+\beta+1)} \cdot (\xi - i0)^\alpha \cdot (\xi + i0)^\beta$ is entire. Therefore the existence of the distribution valued function in (30) follows for $\beta = -\lambda - 1$ and $\alpha = -\mu - 1$ via the Fourier's inversion formula. □

Corollary 3.1.
The Fourier transform of the distribution $\gamma(\lambda,\mu)$ in (30) is

$$\frac{i^{-\lambda-\mu-2}}{\Gamma(-\lambda-\mu-1)} \cdot \left(\frac{1}{i(\xi - i0)}\right)^{-\lambda-\mu-2} - \frac{2i\sin\pi(-\mu-1)}{\Gamma(-\lambda-\mu-1)} \cdot e^{-i\pi(\lambda+1)} \cdot |\xi|^{-\lambda-\mu-2} \cdot H_-(\xi). \tag{36}$$

Proof. Let $\beta = -\lambda - 1$ and $\alpha = -\mu - 1$ then the proof follows from (33), (34), (35) and Lemma 2.1. □

Acknowledgement

This research was financially supported by SIDA project 316-2014 "Capacity building in Mathematics and its applications" under the SIDA bilateral program with Makerere University 2015-2020. The authors further extend their gratitude to the research support provided by the International Science Program(ISP).

References

[1] T. Mandai, Existence of Distribution (null)- solutions to every Fuchsian Partial Differential operator, Journal of Math. society University of Tokyo, Japan 5, (1998), 1–18.

[2] I. Opio, G.I. Mirumbe, J. Ssebuliba, J.M. Mango, On the solution space of ordinary differential equations with polynomial coefficients, Far East Journal of Mathematical Sciences, 101 issue 1 (2017) 103-118.

[3] G.I. Mirumbe, R. Bøgvad,V.A. Ssembatya J.E. Björk , On the distribution solutions to ordinary differential equations with polynomial coefficients on the real line, Journal of Pure and Applied Mathematics, Advances and Applications, 6 (2011) 135-155.

[4] C. Li, The convolution of Analytic Functionals, Journal of Analysis, (2018), https://doi.org/10.1007/s41478-018-0078-5, Springer.

[5] C. Li, Changpin Li, On defining the distributions δ^k and $(\delta')^k$ by fractional derivatives, Applied Mathematics and Computation, Elsevier, 246(2014), 502-513,

[6] C. Brouder, N, V. Dang, F. Helein, A smooth introduction to the wavefront set, Journal of Physics A: Mathematical and Theoretical, IOP Publishing, 47 (2014), 443001.

[7] E.L. Koh, C. Li, On the distributions δ^k and $(\delta')^k$, Math. Nachr. 157(1992), 243-248.

[8] P. Antosik, J. Mikusinski, R. Sikorski, Theory of distributions. The sequencial approach, PWN-Polish Scientific Publishers, (1973) Warsawa.

[9] M.A. Aguirre, C.K. Li, The distributional products of particular distributions, Applied Mathematics and Computation 187 (2007), 20 - 26.

[10] C. Li, The product and Fractional derivative of Analytic Functionals, International Journal of Mathematical Analysis, HIKARI Ltd, 11, (2017), issue no. 20, 955-972.

[11] C. Li, M.A. Aguirre, The Distributional products by the Laurent series, Thai Journal of Mathematics Vol. 4, (2006), no 2, 305-319.

[12] A. Kilicman, B. Fisher, Further results on the non-commutative neutrix product of distributions, Bulgaricae mathematicae publicationes, 19 (1993), 145-152.

[13] B. Fisher, A non-commutative neutrix product product of distributions, Math. Nachr., 108(1982),117-127.

[14] B. Fisher, The product of distributions, Quart. J. math, 22(1971), 291-298.

[15] B. Fisher, K. Tas, On the composition of distributions $x_+^{-1}\ln|x|$ and x_+^r, Integral transforms and special functions 7, (2005), 533-543.

[16] B. Fisher, A. Kananthai, G. Sritanatana, K.Nonlaopon, The composition of distributions $x_-^{-ms}\ln x_-$ and $x_+^{r-\frac{p}{m}}$, Integral transforms and special functions, 16 (2005), Issue 1, 13-19.

[17] B. Fisher, S. Orankitjaroen, T. Kraiweeradechachai, G. Sritanratana, K. Nonlaopon, On the composition of distributions $x_+^{\lambda}(\ln x)^m x_+$ and x_+^{μ}, East-West Journal of Mathematics, 9 (2007), Issue 1, 69-79.

[18] C. Li, Several results on the commutative neutrix product of distributions, Integral transforms and Special functions (2007), no 18, 559-568.

[19] I.M. Gel'fand, G.E. Shilov, Generalized functions Volume I - Properties and operations, Translated by Saletan. E, Academic Press, New York/ London, 1964.

[20] G. V. Dijk, Distribution theory; Convolution, Fourier transform and Laplace transform, De Gruyter Graduate lectures, Berlin, (2013).

[21] L.N.M. Tawfiq, A.K. Jabber, Solve the groundwater model equation using the Fourier transforms method, Int. J. Adv. App. Math. and Mech.5(1)(2017), 75-80.

[22] I.A. Salehbhai, M.G. Timol, On the solution of some fractional differential equations, Int. J. Adv. App. Math. and Mech.1(2)(2013),157-163.

[23] C. Li, Several results of fractional derivatives in $\mathscr{D}'(\mathbb{R}^+)$, Fractional Calculus and Applied Analysis (2015), no 18, 192-207.

[24] C. Li, Changpin Li, Kyle Clarkson, Several results of Fractional Differential and integral Equations in Distribution, www.mdpi.com/journal/mathematics (2018), no 6. doi: 10.3390/math6060097.

[25] J.H. Salman, A.S. Joudah, On new classes of analytic functions defined by using differintegral operator, Int. J. Adv. App. Math. and Mech. 2(1)(2014) 91-99.

[26] L. Hörmander, The Analysis of Linear Partial Differential Operators I, Distribution Theory and Fourier Analysis,

Reprint of the second edition 1990, Springer-Verlag, New York.
[27] R. Strichartz, A guide to distribution theory and Fourier transforms, Studies in advanced Mathematics, 1994.