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An Investor's Future Wealth by Fitting Bitcoin Data to Meixner-Exponential Distributional Relationship

Research Article

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Abstract: This work is aimed to enhance the understanding of which factors affect the price development of Bitcoin in order to uncover the characteristics of the complexity in many real world phenomena. The complexity of financial markets has always presented challenging issues and provided interesting findings, such as the inverse cubic law in the tail of stock price fluctuation-distributions. Motivated by the rise of novel digital assets based on block-chain technology, in this paper we derived a distribution which is related to meixner and exponential distribution, we call it distributional relationship, the meixner-exponential distribution" and fit it to Bitcoin data over various time intervals and multiple digital exchanges, in order to investigate the existence of universal scaling behaviour of an investors whose initial wealth is w_0 and whose returns is R_τ.

MSC: 62P05 • 97M30

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1. Introduction

Since the introduction of Bitcoin by [1], Bitcoin has received considerable attention among policy makers, investors and regulators. Due to its unique nature, Bitcoin can be traded any time online and exchanged into major currencies at a low cost [2]. The market for Bitcoin, was created in 2009 but active trading started in 2013. However, its innovative features, simplicity, transparency; and the late exponential growth in its value made Bitcoin more popular than ever.

Bitcoin has been constantly gaining popularity and Its price and volume traded have increased by several orders of magnitude. Being the so-called crypto-currency, i.e. an asset derived from mathematical cryptography, it is based on a new technology called the block-chain [3, 4]. Its other fundamentals are being decentralised, and having a fixed total number of coins: 21 million, with more than 16 million already in circulation. Bitcoin is still very young, especially considering the fact that the last coin is to be mined around year 2140 [5].

One of the key issues that are of interest to market participants is whether the pricing behaviour of Bitcoin is predictable, which would be inconsistent with the Efficient Market Hypothesis (EMH) [6]. The Bitcoin exchange rates

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have been already modelled and predicted by means of a non-causal autoregressive process [7]. On the other hand, financial time series, e.g. stocks, exchange rates, bonds or commodities, are characterised by the Hurst exponent [8, 9].

Bitcoin is a digital currency designed for the recent market scenario. The idea set out behind the creation of the coin is to use the white paper by one of the mysterious individual Satoshi Nakamoto, whose identity has not yet been recognized. The idea of the name Bitcoin is that a paper is termed as bit and the currency as the coin. The concept behind the creation of Bitcoin is the easy transfer of the money without paying a large amount of transaction fees. In the view point of [10], the traditional online payment charges some amount of transaction amount that is to be paid to the bank or other financial organization related to the transaction. As pointed out by [11], it must be a clear concept that there is no physical evidence of the Bitcoin. It is shown only as balance in the account of the user of the Bitcoin profile. The balance is maintained keeping all the accounting in mind such as the ledger or the balance sheet. At the time of transaction, a kind of verification is done as to undertake that the transaction of the money has been done without any kind of trouble or mishandling. The coin was available in various denominations such as millibitcoin and microbitcoin. There have been more that 21 million of Bitcoins issued till 2016 [12].

The Bitcoin operates like the ATM and includes string of numbers or letters attached in a particular fashion. A pin is often offered to the user of the Bitcoin account for carrying out any kind of monetary transaction with any person throughout the world [13]. However, in the year 2014, IRS (internal revenue service), declared that all digital currencies are to be charged with certain amount of transaction charges. Any gain or loss suffered in the Bitcoin will be considered as the gain or loss of the country as a whole. Thus, the transaction is to be included in the GDP (gross domestic product) of the country. This created an alarm among the users of the digital money system. The miners were the main users of this currency. These miners thought of getting a decentralized authority of the money without any intervention of the government [14]. For the same reason, a new Bitcoin system has been approach. The amount was fixed to the miners but periodically it has been seen that the value of this digital money was declining day by day. There was provision of blocking the payment to the person by the means of options available in the app itself.

The currency was recognised as the future currency because the transaction was very easy. One has to install an app in their smart phone and the account was automatically created following a limit steps of instructions. Thus, lending and borrowing or in fact making payment has become very easy for the users. From the view point of [15], this electronic transfer of money would increase the transparency of the economic condition and the transfer of money taking place. This would reduce the rate of corruption and the ill practices that became very relevant in the society. However, the same concept was not very true and in fact arguments were made by [16], that the concept of Bitcoin is increasing the chances of creation of more black money in the global market. One of the main advantages that were considered that the whole transaction could be easily recognized by the means of digital transaction became one major reason to worry about the monetary balance of the GDP of the country.

The miners using the currency find it very useful because it does not require any kind of specialized form to make it a currency. This could even save energy that is required for the production of the currency used in the market in the recent scenario. The minders were using efficient processors to keep a check of the things related to the transaction of the money through Bitcoin. The easy handling of money and the online transaction could be considered as the strength of the currency [17]. However, the fear of creation of the corrupt world and the malpractices arising due to the same is the major weaknesses of this new digitalized currency. Like any other money, Bitcoin also shows ups and downs in its value of share. It was based on the perception of the buyers and the sellers that the rate of the currency was fixed. No centralized banks would make any approaches towards the acceptance of the money [18]. In fact there have been situations when the rise and fall of the value of the currency has been evident.

The Meixner distribution is a type of generalized $z-$distributions. Its properties made it potentials very useful in modelling short-term financial returns, it is well known that financial returns measured on short time intervals, that is, daily or weekly are non-Gaussian. In fact, numerous empirical evidences show that their distribution has heavier than Gaussian tails and is often skewed.

We derived a distribution which is related to Meixner and Exponential distribution. We call this distributional relationship 'the Meixner-Exponential distribution' and fit it to Bitcoin data to show how good it fits, we apply the distribution to determine the expected value (wealth) of an investor whose initial wealth is W_0 and whose return is R_τ.

Contributors on the study of Bitcoin include [19] who used many test to analyse the efficiency of Bitcoin market and a few academics have produced significant results on this subject by testing on the weak and semi-strong form of Efficient Market Hypothesis (EMH) concludes that it becomes more efficient in the latter time sample.

[20] utilize eight different tests on an odd integer power transformation of Bitcoin returns and showed that returns are weakly efficient.

[21] studied the dynamics of long-range dependence properties of the Bitcoin price and finds a trend toward efficiency via Hurst exponent analysis. Similar results are found by [22].

On the other hand using daily data with Hurst exponent analysis, [23] showed that the Bitcoin market does not become more efficient over time.

From a different perspective, [24] compared the efficiency of Bitcoin with other assets such as gold, stock and currency. They show that the Bitcoin market is the least efficient.

Whether it is a static or dynamic analysis, the above-mentioned studies use daily data to perform their empirical test. However, in today's modern financial markets, computers have taken over the trading scene through algorithmic (especially high frequency) trading activities.

All these created a situation of dilemma among the users of the currency and that situation motivated the derivation of a distributional relationship which we call meixner exponential distribution which has the following characteristics function

$$\varnothing_{GX}(X,\ a,\ b_{1,}b_2,\ d,\ m)=\left(\frac{B(b_1+\frac{iax}{2\pi},b_2-\frac{iax}{2\pi}}{B(b_1,b_2)}\right)^{2d}\exp(imx)\ ,$$

where $a, b_1, b_2, d > 0$ and $m \in R$.

The Meixner distribution is a special case of generalized $z-$distributions. Its properties made it potentially very useful in modelling short-term financial returns defined through the characteristic function.

In this work we apply this meixner distribution which has an exponential property to model the price of Bitcoin and to obtain the future wealth of the investor.

2. Theory of Meixner distribution

A random variable X is said to follow a Meixner distribution having parameters m, a, b, d, with $a > 0, -\pi < b < \pi, d > 0$, and $m \in IR$, in symbols $X \sim MXN(m, a, b, d)$, if its density function, for $x \in IR$, is

$$f_{MXN}(x; m, a, b, d) = a^{-1} f_{MXN}\left(\frac{x-m}{a}; 0, 1, b, d\right),$$

where

$$f_{MXN}(z; 0, 1, b, d) = \frac{(2\cos(b/2))^{2d}}{2\pi\Gamma(2d)} e^{bz}|\Gamma(d+iz)|^2,$$

with $i = \sqrt{-1}$ and $(\cdot)$ being the gamma function. It should be noted that m and a are location and scale parameters, while b and d decide the shape of the distribution. In other terms, if $X \sim MXN(m,\ a,\ b,\ d)$, then $Z = (X-m)/a$ is its standardized form, with distribution $MXN(0,\ 1,\ b,\ d)$.

The characteristic function of X is [25]

$$\varnothing_{MXN}(u) = e^{imu}\left(\frac{2cosb/2)}{\cosh\left(\frac{au-ib}{2}\right)}\right)^{2d}$$

so that the Meixner distribution is clearly infinitely divisible. From this definition the mean, variance, skewness and kurtosis of a $MXN(m,\ a,\ b,\ d)$ distribution can also be immediately derived:

$$E[X] =_{MXN} \mu = m + ad\tan(b/2),$$

$$\mathrm{Var}[X] =_{MXN\sigma^2} = \frac{a^2 d}{cosb+1},$$

$$\mathrm{Skew}[X] =_{MXN} k_1 = sinb\sqrt{\frac{1}{d(cosb+1)}},$$

$$\mathrm{Kurt}[X] =_{MXN} k_2 = 3 - \frac{\cos b - 2}{d}.$$

Here,

$$\mathrm{Skew}] = E[(X-E[X])^3]/[Var(X)]^{\frac{3}{2}} \text{and } \mathrm{Kurt}[X] = E[(X-E[X])^4]/[Var(X)]^2.$$

It follows that when $b = 0$ the distribution is symmetric; besides, its kurtosis is always larger than 3, the Gaussian kurtosis.

An important property of the Meixner distribution, that makes it potentially useful in financial applications, is that it has semi-heavy tails [25]. Formally, this implies that, for a $MXN(m,\ a,\ b,\ d)$ distribution, we have

$$f_{MXN}(m, a, b, d) \sim C_-|x|^\rho e^{-\sigma_-|x|}\quad as\ \ x \to -8,$$

and

$$f_{MXN}(m, a, b, d) \sim C_+|x|^\rho e^{-\sigma_+|x|}\quad as\ \ x \to +8$$

The Bitcoin network uses cryptography to validate transactions during the payment processing and create transaction blocks. In particular, Bitcoin relies on two cryptographic schemes: 1) digital signatures and 2) a cryptographic hash function. The first scheme allows the exchange of payment instructions between the involved parties, while the second is used to maintain the discipline when recording transactions to the public ledger (known as Blockchain). It should be noted that none of these schemes is unique to Bitcoin, since they are widely used to protect commercial and government communications. A short description of how the Bitcoin network works is reported below, while more details can be found in [26], [27], [28], [29]. Digital signatures are used to authenticate digital messages between a sender and a recipient, and they provide:

i. Authentication: the receiver can verify that the message came from the sender;

ii. Non-repudiation: the sender cannot deny having sent the message;

iii. Integrity: the message was not altered in transit.

The use of digital signatures includes public key cryptography, where a pair of keys (open and private) is generated with certain desirable properties. A digital signature is used for signing messages: the transaction is signed using a private key, and then transferred to the Bitcoin network. All the members of the network can verify that the transaction came from the owner of the public key, by taking the message, the signature, and the public key by running a test algorithm.
A cryptographic hash function takes as input a string of arbitrary length (the message m), and returns the string with predetermined length (the hash h). The function is deterministic, which means that the same input m will always give the same output h. In addition,the function must also have the following properties:

i. Pre-image resistance: for a given hash h, it is difficult to find a message m such that hash $(m) = h$.

ii. Collision resistance: for a given message $m1$, it is hard to find another message $m2$ such that hash $(m1) = hash\ (m2)$. In other words, a change in the message leads to a change in the hash.

At the time of writing this project, more than 15 million Bitcoins have been mined. The price of one Bitcoin in first four months of 2016 was in the range of 400 – 460 USD Therefore, the total value of all issued bitcoins was close to 7 billion U.S. dollars. The market capitalization of Bitcoin is approximately 10 times higher than the second largest crypto currency, so that Bitcoin is (currently) the undisputed leader. The value of a digital currency is highly dependent on the number of participants, which in turn attracts more participants, powering a network effect. Therefore, Bitcoin enjoys a significant first-mover advantage which has three aspects:

1. The more users it has, the more useful Bitcoin becomes: there are more places where you can spend Bitcoins, and business partners with whom you can exchange Bitcoins, which in turns attracts more users.

2. Currencies require trust, but it can only be obtained over time, so that, ceteris paribus, the oldest currency has a natural advantage over competitors.

3. The greater the volume, the higher the transaction fees, which attracts more miners and makes the network more secure, which in turn again attracts more users and traded funds.

3. Meixner-Pollaczek-Exponential-Relationship

In this section, we obtain the distributional relationship between the Meixner distribution and the exponential distribution in which we shall call the Meixner-Exponential distribution.
Describe for each $d > 0$ the Symmetric Meixner-Pollaczek polynomials $P_n^{(d)}(x)$ by the following recurrence relation.

$$P_{-1}^{(d)}(x) = 0, \quad P_0^{(d)}(x) = 1$$

and

$$(n+1)\,P_{n+1}^{(d)}(x) - xP_n^{(d)}(x) + (n-1+2d)\,P_{n-1}^{(d)}(x) = 0, \quad n = 1, 2, \ldots \tag{1}$$

This sequence of polynomials has the generating function

$$G_d(x, t) = \frac{e^{xarctant}}{(1+t^2)^d} = \sum_{n=0}^{8} P_n^{(d)}(x)\, t^n\,, \tag{2}$$

With a hyper geometric representation given by

$$P_n^{(d)}(x) = \frac{(2d)^n}{n!} i^n 2F_1\left(\begin{matrix} -n,\ d+\ ix/2 \\ 2d \end{matrix} \middle| 2 \right), \tag{3}$$

and a weight function.

$$W_d(x) = \frac{\left|G\left(d + \frac{ix}{2}\right)\right|^2}{2\pi} \tag{4}$$

The density of the meixner distribution related to the Symmetric Meixner-Pollaczek polynomials $P_n^{(d)}(x)$ is given by [30] as

$$f(x; a, b, d, m) = \frac{\left(2\cos\left(\frac{b}{2}\right)\right)^{2d}}{2a\pi G(2d)} exp\left(\frac{b(x-m)}{a}\right) G\left(d + \frac{i(x-m)}{a}\right)|^2, \tag{5}$$

where $a > 0, -\pi < b < \pi, d > 0$ and $m\pi \in R$, and weight function

$$W_d(x) = \left|G\left(d + i\left(\frac{x-m}{a}\right)\right)\right|^2 \tag{6}$$

The moments of this distribution are outlined in [31]. To this end we state;

Theorem 3.1.
Given (5), the Meixner-Exponential distribution is

$$f(x; a, b, d, m) = \frac{2^{(2-(a+m))d}\left(\cos\left(\frac{b}{2}\right)\right)^{2d}}{2cosh^{(a+m)d}(t_n)} e^{-\frac{b}{a}(m-x)} \tag{7}$$

The proof of theorem 3.1 is a consequence of the proposition below.

Proposition 3.1.
Let $f(t) = e^{i\omega_0 t}$ in the sense of distribution with known facts

1. $f(\omega) = \int_{-8}^{8} f(t) e^{-i\omega_0 t} dt = 2\pi\delta(\omega - \omega_0) = 2\pi\delta_{\omega_0}(\omega.)$.
2. *From* $G(z) = \int_0^8 e^{-s} S^z \frac{ds}{s}$, *put* $s = e^u$, *then,* $G(z) = \int_0^8 e^{-(e^u)} e^{zu} du$.
3. $\cosh x = \frac{1+e^{-2x}}{2e^{-x}}$.

Then the Fourier transform of the weight function (6) is given by:

$$I(W_d(x)(t)) = \frac{a\pi 2^{-(a+m)d} G(2d)}{cosh^{(a+m)d}(t)} \tag{8}$$

Proof. By definition

$$\begin{aligned} I(W_d(x)(t)) &= \int_{-8}^{8} \left|G\left(d + i\left(\frac{x-m}{a}\right)\right)\right|^2 e^{-itx} dx \\ &= \int_{-8}^{8} e^{-itx} G\left(d + i\left(\frac{x-m}{a}\right)\right) G\left(d - i\left(\frac{x-m}{a}\right)\right) dx, \end{aligned}$$

and by fact 1,

$$I(W_d(x)(t)) = \int_{-8}^{8} e^{-itx} \left(\int_{-8}^{8} e^{-e^u} e^{\left(d+i\left(\frac{x-m}{a}\right)\right)} du\right) \left(\int_{-8}^{8} e^{-e^v} e^{\left(d-i\left(\frac{x-m}{a}\right)\right)} dv\right) dx.$$

Let $y = \frac{x-m}{a}$ so that $dx = a dy$, then

$$I(W_d(x)(t)) = a \int_{-8}^{8} e^{-(e^u)} e^{du} \left(e^{-(e^v)} e^{dv} \left(\int_{-8}^{8} e^{-i(-mt+(at-u+v)y)} dy\right) dv\right) du.$$

By fact 2;

$$\begin{aligned} I(W_d(x)(t)) &= 2a\pi \int_{-8}^{8} e^{-(e^u)} e^{du} \left(\int_{-8}^{8} e^{-(e^v)} e^{dv} \left(\delta_{u-(a+m)t}(V)\right) dv\right) du \\ &= 2a\pi \int_{-8}^{8} e^{-(e^u)} e^{du} e^{-\left(e^{u-(a+m)t}\right)} e^{d(u-(a+m)t)} du. \end{aligned}$$

Substituting back the variable $s = e^u$, gives

$$\begin{aligned} I(W_d(x)(t)) &= 2ap \int_0^8 e^{-s} s^d e^{-se^{-(a+m)t}} s^d e^{-(a+m)dt} \frac{ds}{s} \\ &= 2a\pi e^{-(a+m)dt} \int_0^8 s^{2d} e^{-\left(1+e^{-(a+m)t}\right)s} \frac{ds}{s} \end{aligned}$$

Again let $Z = \left(1 + e^{-(a+m)t}\right)s \Rightarrow dZ = \left(1 + e^{-(a+m)t}\right)ds$,
such that

$$\begin{aligned} I(W_d(x)(t)) &= \frac{2a\pi e^{-(a+m)dt}}{\left(1+e^{-(a+m)t}\right)2d}\int_0^8 e^{-Z}Z^{2d}\frac{dz}{z} \\ &= \frac{2a\pi e^{-(a+m)dt}}{\left(1+e^{-(a+m)t}\right)^{2d}}G(2d). \end{aligned}$$

By fact 3, we have;

$$\begin{aligned} I(W_d(x)(t)) &= a\pi\left(\frac{2^{-1}2e^{-t}}{1+e^{-2t}}\right)^{(a+m)d}G(2d) \\ &= \frac{a\pi 2^{-(a+m)d}G(2d)}{cosh^{(a+m)d}(t)} \quad \text{(as required)}. \end{aligned}$$

□

Substituting the Fourier transformed weight function (8) into (5) gives (7)

$$f(x;a,\ b,\ d,\ m) = \frac{\left(2\cos\left(\frac{b}{2}\right)\right)^{2d}}{2cos^{(a+m)d}(\omega)}exp\left(\frac{b(x-m)}{a}\right), \tag{9}$$

The moments of this distribution are straight forward and may be derived as follows:
$\mathrm{E}(X) = \left(\frac{a}{b}\right)^2K, E\left(X^2\right) = \left(\frac{a}{b}\right)^4 2K$ and $Var(X) = (2-K)\left(\frac{a}{b}\right)^2K$,
with

$$K = \frac{2^{(2-(a+m\ ))d}\left(Cos\left(\frac{b}{2}\right)\right)^{2d}}{2Cosh^{(a+m)d}(t_n)}.$$

Notice that K=1, $\mathrm{E}(X) = Var(X)$.

Corollary 3.1.
Using the moment generating function, we obtain the Meixner-exponential distribution for $\omega = \frac{b}{2}$ and $a+m=2$ as

$$f(x,a,b,d,m) = \frac{1}{2}e^{-\frac{b}{a}(m-x)}. \tag{10}$$

Proof: Replacing $t\ by\ \omega\ (where\ \omega = \sum_{v\in(r,s,t)}$ arctan v [32]
we get

$$\int_{-8}^{8} e^{\omega x}Wd(x)\,dx = \frac{a\pi 2^{(2-(a+m))d}\Gamma(2d)}{(1-ts-tr-sr)^{2d}}u,$$

where

$$u = \{\left(1+t^2\right)\left(1+s^2\right)\left(1+r^2\right)\}^d.$$

By proposition 1,

$$\int_{-8}^{8} e^{\omega x}W_d(x)\,dx = \frac{a\pi 2^{(2-(a+m))^d}\Gamma(2d)}{cos^{(a+m)d}(arctant + arctans + arctanr)}.$$

But

$$\cos(arctant + arctans + arctanr) = \frac{1-ts-tr-sr}{\left(1+t^2\right)^{0.5}\left(1+s^2\right)^{0.5}\left(1+r^2\right)^{0.5}},$$

so that

$$\int_{-8}^{8} e^{\omega x}W_d(x)\,dx = \frac{a\pi 2^{(2-(a+m))d}\Gamma(2d)}{cos^{(a+m)d}(\omega)}. \tag{11}$$

Submitting the Fourier transformed weight function (11) into (5) gives

$$f(x;a,b,d,m) = \frac{2\cos\left(\frac{b}{2}\right)^{2d}}{2cos^{(a+m)d}(\omega)} exp\left(\frac{b(x-m)}{a}\right) \tag{12}$$

which gives (10) for $a+m=2$ and $\omega = \frac{b}{2}$.
It follows, therefore that the graph of the curve $x = \ln f(x,a,b,m)$ (or equivalently) $f(x,a,b,m) = \frac{b}{a}e^{(x-m)}$ is the reflection of the curve $f(x,a,b,m)\ln x$, about the line $f = x$ as seen in Fig. 1 below.
The following properties are evident from the graph of

$$f(x;a,b,m) = \frac{b}{a}e^{(x-m)}: \quad e^{(x-m)} > 0 \;\; for\ all\ x,$$

$$\lim_{x\to -8} e^{(x-m)} = +8,$$

$$\lim_{x\to -8} e^{(x-m)} = 0.$$

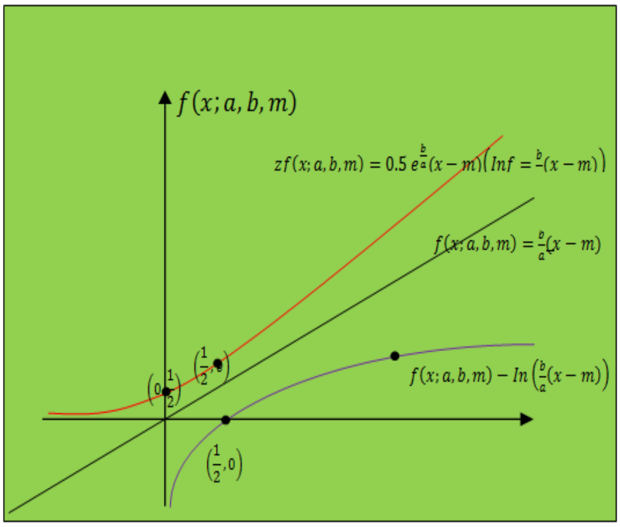

Fig. 1. The graph of the curve $x = In\ f(x,a,b,m)$ (or equivalently)

$f(x,a,b,m) = \frac{b}{a}e^{(x-m)}$ for $a+m=2$ and $\omega = \frac{b}{2}$.

Lemma 3.1.
Suppose that $i = \frac{a}{2(x-m)}$, *then our Meixner-Exponential distributional relationship becomes*

$$f(x,a,b,m) = \frac{1}{2}\sum_{d=0}^{8} \frac{(-1)^{2d}\left(\frac{b}{2}\right)^{2d}}{ad!} \exp\left\{b\left(\frac{x-m}{a}\right)\right\} \tag{13}$$

Proof. Given $i = \frac{a}{2(x-m)}$, the weight function (4) becomes;

$$W_d(x) = \left|\Gamma\left(d+\frac{1}{2}\right)\right|^2.$$

By the Legendre duplication formula, we have

$$\left(\Gamma\left(d+\frac{1}{2}\right)\right)^2 = \left(\frac{\Gamma(2d)}{2^{k-1}\Gamma(d)}\right)^2 = \pi\left(\frac{(2d!)}{2^{2d}d!}\right)^2 \tag{14}$$

Substituting (11) into (5) we have;

$$f(x,a,b,m) = \frac{\left(cos\left(\frac{d}{2}\right)\right)^{2d}(2d!)}{2a(d!)^2} exp\left\{b\left(\frac{x-m}{a}\right)\right\}$$

$$= \frac{1}{2}\sum_{d=0}^{8}\frac{(-1)^{2d}\left(\frac{b}{2}\right)^{2d}}{2a(d!)^2}exp\left\{b\left(\frac{x-m}{a}\right)\right\},$$

given that

$$cos\left(\frac{d}{2}\right)=\sum_{d=0}^{8}\frac{(-1)^{d}\left(\frac{b}{2}\right)^{2d}}{(2d!)}$$

□

4. Bitcoin Data Fitting to Meixner Distribution

A portfolio composition is predicted to change with the investment time horizon (i.e., the time scale) in a way that can be fully determined once an adequate measure of risk is chosen. The portfolio optimization problem consists in finding the optimal diversification on a set of possibly dependent assets in order to maximize return and minimal risk. Let $x_i(t)$ be the price of the bitcoin asset i at time t, where time is counted for trading days in multiples of a fundamental unit (days, say).

With the notation

$$\mathfrak{z}_i(t,\ \tau)=\ln\left(\frac{x_i(t)}{x_i(t-\tau)}\right), \tag{15}$$

the return $\mathfrak{r}_i(t,\ \tau)$ between time $t-\tau$ and t of bitcoin asset i is defined as

$$\mathfrak{r}_i(t,\tau)=\frac{x_i(t)-x_i(t-\tau)}{x_i(t-\tau)}e^{\mathfrak{z}i(t-\tau)}-1, \tag{16}$$

while (15) is the continuous return, (16) is the discrete return.

Consider a portfolio with n_i shares of bitcoin asset i whose initial wealth is

$$W(0)=\sum_{i=1}^{n}n_1x_i(0) \tag{17}$$

at time τ later, the wealth has become $W(\tau)=\sum_{i=1}^{n}n_ix_i(\tau)$ and the wealth variation is [33]

$$\delta_\tau W=W(\tau)-W(0)=\sum_{i=1}^{n}n_ip_i(0)\frac{x_i(\tau)-x_i(0)}{x_i(0)}=W(0)\sum_{i=1}^{N}w_i\left(e^{\mathfrak{z}i(\tau)}-1\right), \tag{18}$$

where

$$wi{=}\frac{nixi\ (0)}{\Sigma_{I=1}^{N}w_i\mathfrak{r}_i(0)} \tag{19}$$

Is the fraction in capital invested in the ith asset at time 0. By (15), it is justifiable to write the R_τ of the portfolio over a time T as the weight sum of the returns $\mathfrak{r}_i\ (\tau)$ of the assets $i=1,2\ldots,\ N$ over the time interval τ:

$$\mathrm{R}_\tau=\frac{\delta_\tau w}{w\ (0)}=\dot{\Sigma}_{i=1}^{N}w_i\mathfrak{r}_{i\ (\tau)} \tag{20}$$

Now consider R_τ as an autonomous non-stationary $\tau-$dimensional Bessel process govern by the scalar equation

$$dR_\tau\left(\mathfrak{r}_i(t)\right)=\frac{\delta-1}{2R_\tau\left(\mathfrak{z}_i\right)}d\mathfrak{r}_i(t)+dB(\mathfrak{r}_i(t)) \tag{21}$$

with

$$R_\tau(0)=r_0=0.$$

The Bessel process R_τ is transformed into an autonomous process with finite speed measure (i.e. a process that possesses a limiting distribution) to which the Motoo's theorem can be applied. More precisely, if

$$x_\tau\left(\mathfrak{r}_i(t)\right)=e^{-\mathfrak{r}_i(t)}(t)R_\tau^2(e^{\mathfrak{r}_i(t)}-1), \tag{22}$$

then

$$dx_{\mathfrak{r}}\left(\mathfrak{r}_i(t)\right) = \left(\tau - x_\tau\left(\mathfrak{r}_i(t)\right) = d\mathfrak{r}_i(t) + 2\sqrt{Sx_\tau(\mathfrak{r}_i, t}\right) dB\left(\mathfrak{r}_i(t)\right) \tag{23}$$

[34].
On the other hand, let π represent the investment policy. Under this policy the portfolio process $\left\{R_\tau^\pi\left(\mathfrak{r}_i(t)\right)\right\}$ of the investor over a time τ evolves according to the stochastic differential equation

$$dR_\tau^\pi = \left[\left(\mathfrak{r} + \pi\mu\right) R_\tau^\pi - R_\tau^2\right] d\tau + \pi\sigma R_\tau^\pi dB(\tau) \tag{24}$$

This simplifies that $\left\{R_\tau^\pi\left(\mathfrak{r}_i(t)\right)\right\}$ is a temporally homogenous diffusion process with drift function

$$b(x) = \left(\mathfrak{r} + \pi\mu\right) x - x^2$$

and diffusion function
$a^2(x) = \pi^2\sigma^2 x^2$, [35].
μ and σ are constants and $B(t)$ denotes the standard Brownian motion process.

As a first application of the Meixner-Exponential process, we try to find how good our model fits to empirical entrepreneurial bitcoin data. The data sets obtained consist of the closing prices for the shares denoted by $(x_i)_{1=i<n}$.

The series of log returns is obtained by (2). We measure the portfolio for two sets data, namely, Small Scale Investment (SSI) and Large Scale Investment (LSI) from December, 2008 and 13 trading months ahead. Figs. 2 and 3 below show the performance of (7) with the SSI and LSI respectively. To estimate the Meixner distribution we assume independent observations as in the parent Meixner distribution and use moments. In particular case of the SSI and LSI bitcoin data the result of the estimation procedure is given by $a = 3, w = 0.6$, $m = 0.8e^{-1}$,$d = 0.34, x = 0.56$

$$f(x; a, b, d, m) = \frac{2^{(2-(a+m))d}\left(\cos\left(\frac{b}{2}\right)\right)^{2d}}{2cosh^{(a+m)d}(t_n)} e^{-\frac{b}{a}(m-x)}$$

$$\frac{0.2256357986 e^{-1.8+0.02666666667b} \cos\left(\frac{b}{2}\right)^2 e^{\frac{b}{3}-3}}{\cosh(\tau)^{3.08}(b-9)}$$

$$a = 3, W = 0.6, m = 0.8e^{-1}, d = 0.34, x = 0.56$$

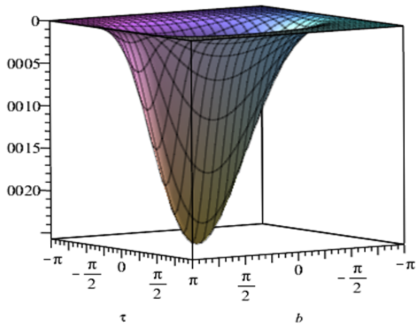

Fig. 2. The Small Scale Investment (SSI) Bitcoin Data Image using equation (7)

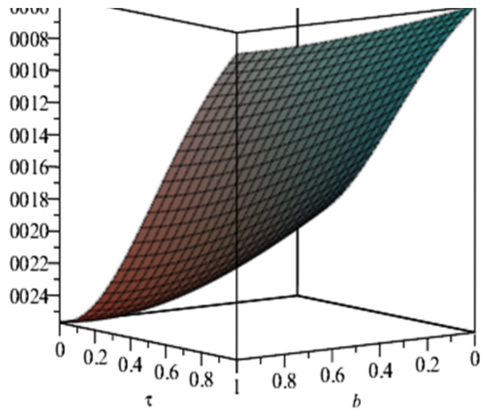

Fig. 3. The Large Scale Investment (LSI) Bitcoin Data Image using equation (7)

Also for

$$f(x; a, b, d, m) = \frac{\left(2\cos\left(\frac{b}{2}\right)\right)^{2d}}{2cos^{(a+m)d}(\omega)} exp\left(\frac{b(x-m)}{a}\right), \tag{25}$$

and

$$a = 3, W = 0.6, m = 0.8e^{-1}, d = 0.34, x = 0.56$$

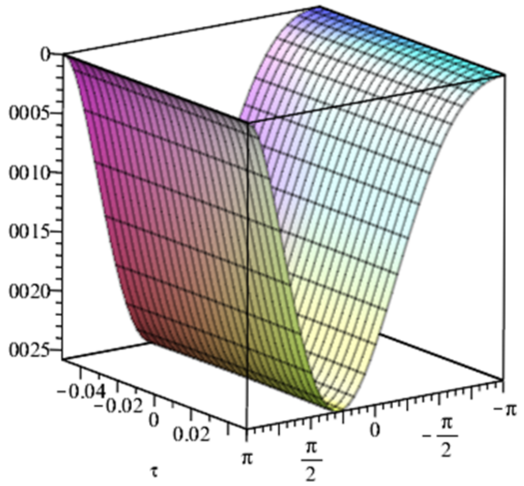

Fig. 4. The Small Scale Investment (SSI) Bitcoin Data Image using equation (25)

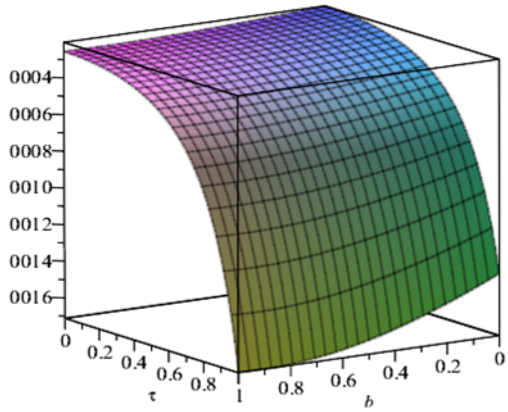

Fig. 5. The Large Scale Investment (LSI) Bitcoin Data Image using equation (25)

$$\frac{0.2256357986e^{-1.8+0.02666666667b}\cos\left(\frac{b}{2}\right)^2 e^{\frac{b}{3}-3}}{\cosh(\tau)^{3.08}(b-9)}$$

we obtain the following Figs. 4 and 5. The expected utility of the Bitcoin investor's wealth (future wealth) is given by

$$E^{\tau}\left[u\left(\mathscr{W}_0+\ x_{\tau}\right)\right]=\frac{e^{-\left(a\mathscr{W}_0-\ \frac{b}{a}m\right)}2^{(2-(a+d))d}\left(\cos\left(\frac{b}{2}\right)\right)^{2d}}{\cosh^{(a+m)d}\left(\tau_n\right)}\left[\left(\frac{e^{-\left(\frac{a^2-b}{a}\right)}}{b-\ a^2}\right)ax_{\tau}\right]$$

$$=e^{-\left(a\mathscr{W}_0-\ \frac{b}{a}m\right)}\sum_{d=0}^{8}\frac{(-1)^d\left(\frac{b}{2}\right)^{2d}}{(2d!)}\ \frac{2^{(2-(a+d))d}}{\cosh^{(a+m)d}\left(\tau_n\right)}\left[\left(\frac{e^{-\left(\frac{a^2-b}{a}\right)}}{b-\ a^2}\right)ax_{\tau}\right]$$

$$f(x,a,b,d,m)=\frac{2\cos\left(\frac{b}{2}\right)^2 e^{\frac{b(x-m)}{a}}}{\cos(\omega)^{a+m}} \tag{26}$$

For

$$a=-0.003, W=0.6,\ \ m=0.08,\ d=0.34,\ x=0.56,$$

we have the following Figs. 6 and 7 for Small Scale Investment (SSI) and Large Scale Investment Bitcoin Data fitting.

In Fig. 2, the plot shows that the function has its minimum at $(tau,\ b)=(0,0)$ and rises steeply from zero in both the positive and negative side of b and tau in the interval 0 to $\pm Pi$.

In Fig. 3, within the interval 0 to 1, the function rises as tau goes from 0 to 1 but decreases slightly as b goes from 0 to 1.

Fig. 4 shows that the value of the function falls as b moves from $-Pi$ to 0 and rises as b goes from 0 to $\pm Pi$. The function is not affected by changes in the value of tau in the interval $(-0.06,\ 0.06)$

Fig. 5 shows that there is a fall in the value of the function as tau goes from 0 to 1, also there is a slow but steady decline as b goes from 0 to 1.

It is indicated in Fig. 6 that there is a slide for the function as b goes from 1 to 1.01, while there is just a little fall generally as tau goes from 1 to 1.01.

Fig. 7 indicates that the function attains its maximum at $tau = 0$, and decreases as tau goes from 0 to ± 4. It also attains its maximum at $b = 1$ and decreases as b increases in the interval $(1, 1.001)$.

5. The Expected Bitcoin trading Approach

Starting the period with initial capital $\mathscr{W}_0 > 0$, the investor is assumed to have preferences that are rational with respect to the end-of-period distribution of wealth $\mathscr{W}_0+\ x_{\tau}$. The preferences are therefore represented by a utility function

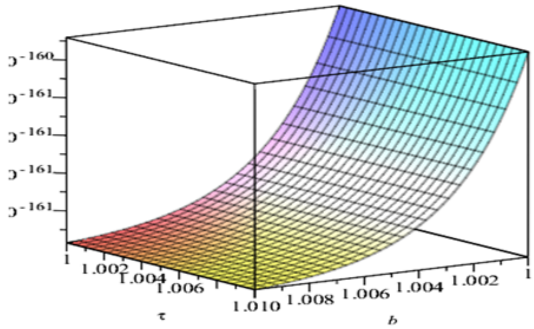

Fig. 6. The Small Scale Investment (SSI) Bitcoin Data Image using equation (26)

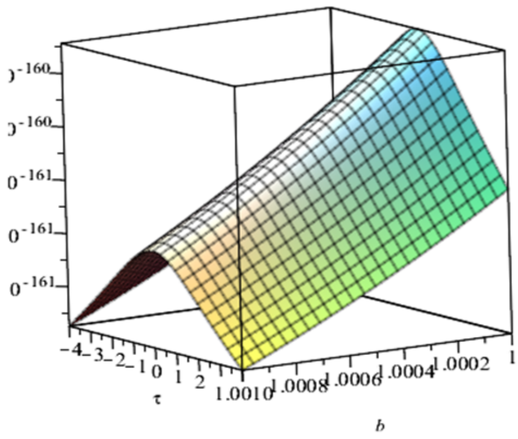

Fig. 7. The Large Scale Investment (LSI) Bitcoin Data Image using equation (26)

$u(\mathscr{W}_0 + x_\tau)$ determined by the wealth variation $x_\tau = \delta x$ at the end-of-period τup. The expected utility theorem states that the investor's problem is to
Maximize $E^\tau[u(\mathscr{W}_0 + x_\tau)]$,
where $E(.)$ denotes the expectation operator:

$$E^\tau[u(\mathscr{W}_0 + x_\tau)] = \int_{\mathscr{W}_0}^{8} dx u(\mathscr{W}_0 + x_\tau) G^\tau_{x_t}(x_\tau). \tag{27}$$

The utility function $u(W)$ has a positive first derivative (wealth) and a negative second derivative (risk aversion)[11]. Let $\frac{u''}{u'} = a$, (a constant risk measure of risk aversion). It is easy to see that

$$u(\mathscr{W}) = -\exp\{-\mathscr{W}\}. \tag{28}$$

For large initial wealth, we have that

$$E^\tau[u(\mathscr{W}_0 + x_\tau)] = e^{-a\mathscr{W}_0}\int_{-8}^{8} e^{-ax_\tau} dx G^\tau_{x_\tau}(x_\tau) \tag{29}$$

According to the fundamental theorem of asset pricing, the arbitrage free price V_t of the derivative at time $t \in [0, T]$ is given by

$$V_t = E_Q\left[e^{-r(T-t)} G(\{x_\tau, 0 = \tau = T\} \mid \mathscr{F}_t)\right],$$

where Q is an equivalent Martingale measure, $\{f_t\}_t$ is the natural filteration of , $\{X_t\}_t$.
Let $G^\tau_{x_\tau}(x_\tau)$ be as in (7), then

$$\begin{aligned} V_\tau &= E^\tau[u(\mathscr{W}_0 + x_\tau)] = \frac{e^{-a\mathscr{W}_0} 2^{(2-(a+d))d}\left(\cos\left(\frac{b}{2}\right)\right)^{2d}}{cosh^{(a+m)d}(\tau_n)} \int_0^{+8} e^{-ax_\tau} e^{-\frac{b}{a}(m-x_\tau)} dx \\ &= \frac{e^{-\left(a\mathscr{W}_0 - \frac{b}{a}m\right)} 2^{(2-(a+d))d}\left(\cos\left(\frac{b}{2}\right)\right)^{2d}}{cosh^{(a+m)d}(\tau_n)} \end{aligned}$$

$$= e^{-\left(a\mathscr{W}_0 - \frac{b}{a}m\right)} \sum_{d=0}^{8} \frac{(-1)^d\left(\frac{b}{2}\right)^{2d}}{(2d!)} \frac{2^{(2-(a+d))d}}{cosh^{(a+m)d}(\tau_n)} \tag{30}$$

in particular. An indefinite solution is given as

$$\begin{aligned} E^\tau[u(\mathscr{W}_0 + x_\tau)] &= \frac{e^{-\left(a\mathscr{W}_0 - \frac{b}{a}m\right)} 2^{(2-(a+d))d}\left(\cos\left(\frac{b}{2}\right)\right)^{2d}}{cosh^{(a+m)d}(\tau_n)} \left[\left(\frac{e^{-\left(\frac{a^2-b}{a}\right)}}{b - a^2}\right) ax_\tau\right] \\ &= e^{-\left(a\mathscr{W}_0 - \frac{b}{a}m\right)} \sum_{d=0}^{8} \frac{(-1)^d\left(\frac{b}{2}\right)^{2d}}{(2d!)} \frac{2^{(2-(a+d))d}}{cosh^{(a+m)d}(\tau_n)} \left[\left(\frac{e^{-\left(\frac{a^2-b}{a}\right)}}{b - a^2}\right) ax_\tau\right] \end{aligned} \tag{31}$$

6. Conclusion

Equation(4) shows that Bitcoin price is continuously rebalanced so as to always keep a fixed constant proportion of wealth investor at each point in time. The fixed trading policy π is the process $\mathfrak{r}_{\pi}(\tau), \tau \geq 0$. For such strategic trading in continuous time, the rate of return on Bitcoin trading is defined as the net gain in wealth divided by the cumulative trading. The average return on trading is maximized by the same strategy that maximizes logarithmic utility, which is also known to maximize the exponential rate at which Bitcoin appreciates.

Notice also that in (7), $f(x, a, b, d, m)$ will optimally decrease over time if $m > x$, but increase over time if $m < x$.

In this dissertation, we derived a distribution which is related to meixner and exponential distributions, we call it distributional relationship, the meixner-exponential distribution" was fit into Bitcoin data over various time intervals and multiple digital exchanges,in order to investigate the existence of universal scaling behaviour of an investors whose initial wealth is w_0 and whose returns is R_τ.

7. Recommendation

The analysis above in the modelling of Bitcoin price with Meixner distribution shows that there is a persistence increase in the price of Bitcoin within a specific period of time and investors have a more profitable trading environment at the intraday level and can meet up their period of solvency which holds if $m < x$ over time.

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